Online Appendix: Information arrival, delay, and clustering in financial markets with dynamic freeriding

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A Distributions without a density

In Section 3, we have assumed that the type distribution $F(\pi_i)$ has a density for ease of exposition. We analyze the general case for Section 4, where pathological distributions may emerge when we endogenize the type distribution. Here, we generalize our analyses to more general distributions that potentially have atoms.

We simplify our analysis by introducing artificial types so that we can represent equilibrium strategies that naturally involve mixing as well-behaving pure strategies. The introduction of such types is standard for games with incomplete information. Also, the equilibrium with artificial types is observationally equivalent to that in the original setting (see Appendix C.1 for details).

Specifically, we consider the following scenario. Each agent privately observes a *quantile* q_i, instead of π_i, and signal s_i. The quantiles are independently and uniformly distributed between 0 and 1. The precision associated with quantile q_i is determined by a *quantile function* $\pi(q_i)$. The function $\pi(q_i)$ is non-negative and non-decreasing. Note that a quantile function uniquely induces the distribution of precision by $F(\pi_i) = \text{Prob}\{\pi(q_i) \leq \pi_i\}$. In turn, any distribution $F(\pi_i)$ induces a quantile function $\pi(q_i) = \inf\{\pi_i : F(\pi_i) \geq q_i\}$.[48]

We can construct a unique equilibrium for this situation without modifications except that we need to replace $\pi^{(k)}$ with $q^{(k)}$, the k-th highest quantile.

Theorem 5. *A symmetric separating equilibrium uniquely exists. The waiting time of the* $(k+1)$*-st mover is given by*

$$t_{k+1}(q^{(k+1)}|q^{(1,\dots,k)}) = \frac{n-k-1}{r} \int_{q^{(k+1)}}^{q^{(k)}} q^{-1}\Gamma(\pi(q)|\Pi_k)\,\mathrm{d}q. \tag{24}$$

B Additional Results

B.1 Properties of the equilibrium waiting time

In this appendix, we present some properties of the equilibrium waiting time presented in Section 3. We first observe that the marginal waiting time

$$-\frac{\mathrm{d}t_{k+1}}{\mathrm{d}\pi_i} = \frac{n-k-1}{r} \cdot h(\pi_i)\Gamma(\pi_i, \Pi_k). \tag{25}$$

[48]Even though this quantile function is not a unique function that induces the original distribution $F(\pi_i)$, the indeterminacy of $\pi(q_i)$ occurs only on a set with measure 0.

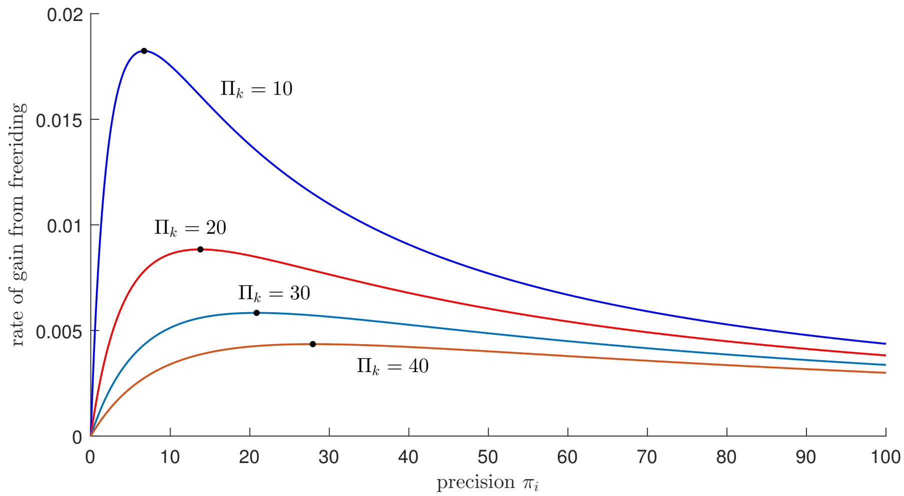

Figure 15: The value of $\Gamma(\pi_i, \Pi_k)$ as a function of precision π_i with $u_0 = 1$. The dots represent peaks.

is decreasing in the discount rate r and increasing in the number $n-k-1$ of remaining agents, hazard rate $h(\pi)$, and the freeriding gain $\Gamma(\pi_i, \Pi_k)$. When the discount rate is large, agents are discouraged to wait and thus promptly move. As the number of remaining agents decreases, agents face fewer opportunities for freeriding, which lowers the incentive to delay. Similarly, if the hazard rate for a particular precision is high, an agent with this precision level expects a higher chance of freeriding on an agent with the same precision, which stagnates the flow of actions. Finally, with a narrower gain from freeriding, agents are compelled to act more quickly.

Also, the marginal waiting time (25) indirectly depends on the individual precision π_i and accumulated public precision Π_k. The marginal waiting time is decreasing in Π_k. An improvement in Π_k lowers the freeriding gain $\Gamma(\pi_i, \Pi_k)$, as we see below, and consequently the marginal waiting time decreases. It is less transparent how the marginal waiting time depends on π_i. The relationship is naturally non-monotonic when the hazard rate $h(\pi_i)$ is nonmonotone. However, even with monotone $h(\pi_i)$, the marginal waiting time may still exhibit non-monotonicity because $\Gamma(\pi_i, \Pi_k)$ is nonmonotonic in π, as shown in Figure 15.

The following proposition describes how the freeriding gain $\Gamma(\pi_i, \Pi_k)$ changes as a function of π_i and Π_k:

Proposition 8. *The rate of gain from freeriding, defined in equation* (8)*, has the following expression:*

$$\Gamma(\pi_i, \Pi_k) = \frac{\pi_i}{(\Pi_k + 2\pi_i)(u_0\Pi_k + u_0\pi_i - 1)} = \left(2u_0\pi_i + (3u_0\Pi_k - 2) + \frac{\Pi_k^2 u(\Pi_k)}{\pi_i}\right)^{-1}. \quad (26)$$

This function and thus the marginal waiting time (25) *are decreasing in the accumulated public precision* Π_k*. With* Π_k *fixed,* $\Gamma(\pi_i, \Pi_k)$ *is single-peaked in the individual precision* π_i*. The peak is achieved at* $\pi_i = \Pi_k \cdot (u(\Pi_k)/2u_0)^{1/2}$*. If* $\overline{\pi}_i \leq \pi_0 \cdot (u(\pi_0)/2u_0)^{1/2}$*, then* $\Gamma(\pi_i, \Pi_k)$ *is increasing in* π_i *on the entire domain* $[\underline{\pi}, \overline{\pi}]$*.*

The non-monotonicity of $\Gamma(\pi_i, \Pi_k)$ in π_i is a natural consequence of the dual roles of π_i

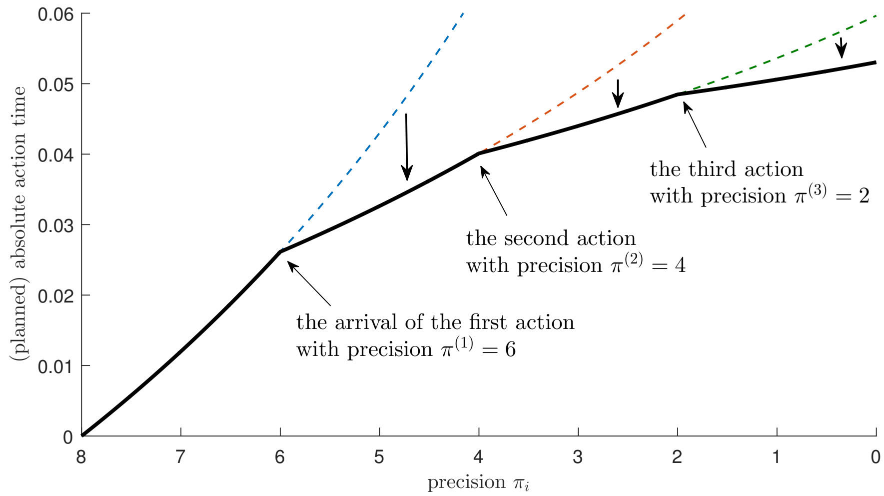

Figure 16: Equilibrium action time. The solid line represents the actually realized action time and dashed lines represent the planned action time $T_k(\pi^{(1,\dots,k)}) + t_{k+1}(\pi_i|\pi^{(1,\dots,k)})$. Agents reduce remaining waiting time when a new piece of information arrives.

in $\Gamma(\pi_i, \Pi_k)$: the variable π_i is not only the precision of the freerider but also that of the freeridden side. To illustrate this point, consider two extreme cases, $\pi_i \to 0$ and $\pi_i \to \infty$. On one hand, when almost all of the types above $\pi = 0$ have disappeared from the game, the remaining agents have virtually no informational impact and thus the freeriding gain $\Gamma(\pi_i, \Pi)$ naturally shrinks to 0. On the other hand, an agent i with an extremely large π_i almost precisely knows the state v through her signal. Thus, agent i gains little from freeriding in *payoffs* even when the informational gain will be large in *quantity*. Hence, as π_i increases, the freeriding gain $\Gamma(\pi_i, \Pi_k)$, again, decreases to 0.

The monotonicity of $\Gamma(\pi_i, \Pi_k)$ in Π_k implies two notable consequences. First, the arrival of new information (i.e., an increase in k) shortens the marginal waiting time and thus the absolute action time by increasing the public precision Π_k and decreasing the number of remaining agents. Figure 16 depicts how agents lower their action time $T_k(\pi^{(k)}|\pi^{(1,\dots,k-1)}) = \sum_{\ell=1}^{k} t_k(\pi^{(\ell)}|\pi^{(1,\dots,\ell-1)})$ after observing new pieces of information with precisions $\pi^{(1)} = 6$, $\pi^{(2)} = 4$, and $\pi^{(3)} = 2$.

Corollary 2. *The marginal waiting time* (25) *is decreasing in k. Consequently, the arrival of new information shortens the absolute action time:*

$$T_{k+1}(\pi^{(k+1)}|\pi^{(1,\dots,k)}) \leq T_k(\pi^{(k+1)}|\pi^{(1,\dots,k-1)})$$

for all $k \in \{1, \dots, n-1\}$. The inequality is strict if $\pi^{(k)} > \pi^{(k+1)}$.

We also find that an improvement in individual precision can never decelerate any agent's absolute waiting time. An increase in the precision of the first mover shortens not only the first mover's own absolute waiting time, but also that of subsequent movers' through an improvement in the public precision. The corollary below describes more general relationships.

Corollary 3. *The action time of the k-th mover is decreasing in the precision $\pi^{(\ell)}$ of the ℓ-th mover for all $\ell \in \{1, \dots, k\}$. This action time is also decreasing in the initial precision π_0 when the distribution $F(\pi_i)$ is exogenously given.*

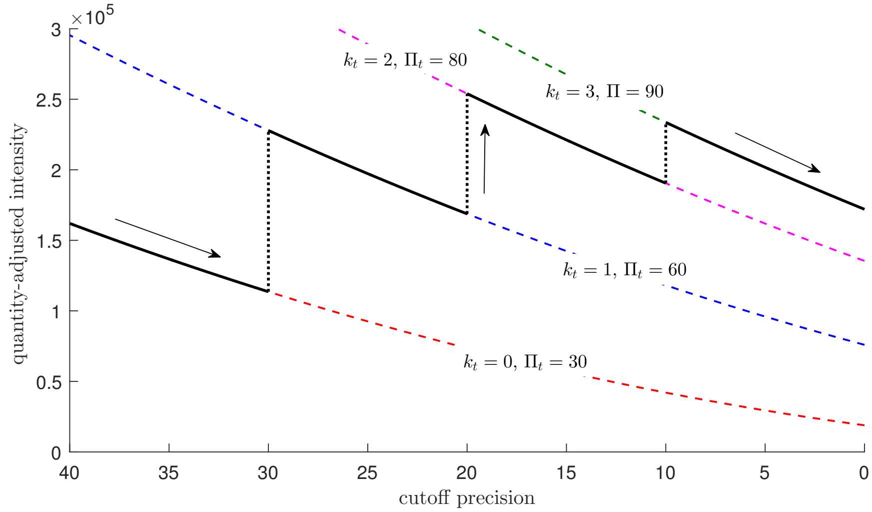

Figure 17: Evolution of the quantity-adjusted intensity $\mu_t = \pi_t^c \lambda_t^*$ (the expected informational improvement). Each smooth curve represents the adjusted intensity as a function of cutoff π_t^c with k_t and Π_t fixed. The adjusted intensity jumps upward when new information arrives (at $\pi_t^c = 30, 20, 10$).

B.2 Additional results on the dynamics of information arrival

In Section 6, we are mostly silent about the magnitude of the impact that new information brings. The counting process k_t and its intensity λ_t^* describe the speed and number of actions, but they do not capture informational quantities.

To take informational quantities into account, we investigate how the *quantity-adjusted intensity* $\mu_t = \pi_t^c \lambda_t^*$ changes over time. The product $\pi_t^c \lambda_t^*$ represents the *instantaneous expected impact on public precision*: within a time interval $[t, t + \mathrm{d}t]$, the probability of information arrival is $\lambda_t^* \, \mathrm{d}t$ while the informational gain is the current cutoff precision π_t^c. In other words, $\mu_t \, \mathrm{d}t$ represents the flow of public information in expectation.

Moreover, the quantity-adjusted intensity μ_t is related to the public precision process Π_t in the language of stochastic processes. The stochastic process μ_t is a *compensator* of Π_t in the sense that $\Pi_t - \int_0^t \mu_s \, \mathrm{d}s$ is a martingale. In other words, the compensator μ_t is a concept analogous to the concept of drift coefficients for diffusion processes. Indeed, the process Π_t follows the stochastic differential equation $\mathrm{d}\Pi_t = \mu_t \, \mathrm{d}t - \pi_t^c \, \mathrm{d}M_t$ because of $\mathrm{d}\Pi_t = \pi_t^c \, \mathrm{d}k_t$, where the process $\mathrm{d}M_t = \mathrm{d}k_t - \lambda_t^* \, \mathrm{d}t$ is a martingale.

The next proposition reports how the quantity-adjusted intensity μ_t, which represents the expected informational contribution to the public precision, depends on k_t, Π_t, and π_t^c.

Proposition 9. *The quantity-adjusted intensity*

$$\mu_t = \pi_t^c \lambda_t^* = \frac{n - k_t}{n - k_t - 1} \cdot r \Pi_t (\Pi_t + 2\pi_t^c) u(\Pi_t + \pi_t^c). \tag{27}$$

is increasing in k_t, Π_t, and π_t^c. The quantity-adjusted intensity decreases over time until new information arrives (i.e., k_t and thus Π_t increases) and jumps upward when new information arrives (and both k_t and Π_t increases).

Figure 17 illustrates the evolution of the quantity-adjusted intensity μ_t, the instantaneous expected impact on the public precision. Unlike the unadjusted intensity, λ_t^*, this adjusted intensity monotonically decreases over time during a quiet period of no additional information. Once a new piece of information arrives, the expected impact jumps upward but then gradually declines until more information arrives.

Notably, *the quantity-adjusted intensity never shrinks to zero* even when π_t^c decreases to 0. The four curves in Figure 17 converge to positive values as $\pi_t^c \to 0$; moreover, these values are increasing in k_t. Indeed, as $\pi_t^c \to 0$, the instantaneous impact $\pi_t^c \lambda_t^*$ converges to a positive value, $\frac{n-k_t}{n-k_t-1} r \Pi_t^2 u(\Pi_t)$, that is increasing in k_t and Π_t. This fact implies that, even when the informational size π_t^c itself is vanishingly small, the intensity of information arrival is extremely large and offsets the decrease in the informational quantity.

B.3 Additional results on delay

When the worst precision $\underline{\pi}$ is bounded away from 0, a nearly-ending game becomes almost identical to the model with homogeneous precision $\pi = \underline{\pi}$. At a terminal phase of the game, every remaining analyst turns out to have precision $\underline{\pi}$ or slightly higher and this fact becomes common knowledge. As a result, the intensity of information arrival approaches

$$\lambda_t^* = \frac{n-k_t}{n-k_t-1} \cdot \frac{r}{\Gamma(\underline{\pi}, \Pi_t)}. \tag{28}$$

If the game terminates by a certain time, the intensity λ_t^* must be infinitely large by then. However, it goes to infinite only when only one analyst is left, i.e., $k_t = n-1$. Indeed, the maximum delay $t_{k+1}(\underline{\pi}|\pi^{(1,\dots,k)})$ goes to infinite because

$$\begin{aligned} t_{k+1}(\underline{\pi}|\pi^{(1,\dots,k)}) &= t_{k+1}(\underline{\pi}+\varepsilon|\pi^{(1,\dots,k)}) + \frac{n-k-1}{r}\int_{\underline{\pi}}^{\underline{\pi}+\varepsilon} \Gamma(\pi, \Pi_k) h(\pi)\,\mathrm{d}\pi \\ &\simeq t_{k+1}(\underline{\pi}+\varepsilon|\pi^{(1,\dots,k)}) + \frac{n-k-1}{r}\Gamma(\underline{\pi}, \Pi_k)\int_{\underline{\pi}}^{\underline{\pi}+\varepsilon} h(\pi)\,\mathrm{d}\pi, \end{aligned} \tag{29}$$

with sufficiently small $\varepsilon > 0$, and the integral of the hazard rate, by definition, goes to infinity:

$$\int_{\underline{\pi}}^{\underline{\pi}+\varepsilon} h(\pi)\,\mathrm{d}\pi = \int_{\underline{\pi}}^{\underline{\pi}+\varepsilon} \frac{\mathrm{d}\log F(\pi)}{\mathrm{d}\pi}\,\mathrm{d}\pi = \log F(\underline{\pi}+\varepsilon) - \log 0 = \infty.$$

The above argument fails when $\underline{\pi} = 0$. In this case, the freeriding gain $\Gamma(\pi, \Pi_k)$ disappears as the cutoff type π goes down to 0. Consequently, the aggregate intensity (28) goes to infinity and the integral in equation (29) may have a finite value. As characterized below, delay persists when the hazard rate $h(\pi)$ grows as fast as π^{-2} or slower as π decreases to 0. For example, delay persists with a uniform distribution on $[0, \overline{\pi}]$ because the hazard rate explodes with the speed of π^{-1}, slower than π^{-2}.

Proposition 10. *Suppose $\underline{\pi} = 0$. Delay persists in the unique symmetric separating equilibrium if $\pi^\alpha h(\pi)$ converges to a constant as $\pi \to 0$ with some $\alpha < 2$. If $\pi^2 h(\pi)$ converges to a positive value or diverges to infinity, then time $t_1(0)$ is finite and the game ends by then* *with probability 1.*

We now consider delay in a large population when the distribution $F(\pi)$ is *exogenously given*. We are tempted to expect the initial delay to diminish and tend to zero as $n \to \infty$, because the probability of an agent drawing precision π_i sufficiently close to $\overline{\pi}$ approaches 1. However, we find that the limit case exhibits delay in the initial action, as agents have an incentive to wait to observe the action of the agent with near-perfect information. In short, the first stage of the game converges to the homogeneous case with $\pi = \overline{\pi}$ because a large number of agents have nearly the best precision $\overline{\pi}$.

Proposition 11. *With an exogenous precision distribution, the distribution of the first action time converges to an exponential distribution with intensity* $\lambda = r/\Gamma(\overline{\pi}, \pi_0)$ *as* $n \to \infty$.

B.4 Additional results on clustering

We are also interested in how the number of actions at an absolute time T changes as a function of $\pi^{(1,\ldots,n)}$. An improvement in precision—when the distribution of individual precision is kept fixed—never has a negative effect on the number of actions. This result is a straightforward corollary of Proposition 5—improved precision in an action accelerates subsequent action in absolute time.

Proposition 12. *Let* $T > 0$. *Conditional on* $\pi^{(1,\ldots,n)}$, *the number of actions already taken at absolute time* T *is non-decreasing in each of* $\pi^{(1)}, \ldots, \pi^{(n)}$.

Note, in contrast, that such simple monotonicity collapses once we alternatively improve the precision distribution rather than the realized precision, as we observe in Appendix B.5.

We now consider how long a cluster needs to consume all the types above a certain quantile. We define the action for types above $\hat{\pi}$ by

$$T(\hat{\pi}) = \sup_{\pi^{(1,\ldots,n)}} \sum_{\ell=1}^{n} \mathbf{1}\{\pi^{(\ell)} > \hat{\pi}\} t_\ell(\pi^{(\ell)} | \pi^{(1,\ldots,\ell-1)}).$$

This time represents the minimum time by which all agents with types better than $\hat{\pi}$ have acted for sure. In calculating the action time, we include the additional waiting time for the ℓ-th agent only when the precision of the ℓ-th action is greater than the target precision $\hat{\pi}$. It does not matter whether the inequality in the indicator function is strict or weak because the distribution $F(\pi)$ is atomless.

Proposition 13. *For all* $\hat{\pi} > \underline{\pi}$, *the action time* $T(\hat{\pi})$ *for types above* $\hat{\pi}$ *is finite and equal to* $t_1(\hat{\pi}) = r^{-1} \int_{\underline{\pi}}^{\overline{\pi}} h(\hat{\pi})\Gamma(\pi|\pi_0)\, \mathrm{d}\pi$. *The action time is decreasing in the threshold* $\hat{\pi}$, *the initial precision* π_0, *and becomes arbitrarily close to 0 as indefinitely increasing the initial precision. The action time goes to infinity as* $\hat{\pi}$ *approaches* $\underline{\pi}$ *if and only if the equilibrium exhibits persistent delay, defined in Section 6.*

B.5 Uniform improvement in precision

We now consider how an improvement in the distribution of $F(\pi)$ affects the speed of information supply. This question is not straightforward for a reason similar to the non-monotonicity of Γ: the improvement of π not only reduces the incentive of freeriding by increasing the precision of the freeriding agent, but also increases the stake of freeriding by increasing the precision of the freeridden agents. Surprisingly, we see that an improvement in public information can rather further *delay* the forecast times rather than speed them up, even though the quality of information provided is higher.

Specifically, we investigate how the rate λ_t^* of information arrival changes after all the agents' precisions are marginally improved by Δ. That is, we marginally compare the baseline distribution $F(\pi)$ with the improved distribution $F_\Delta(\pi+\Delta) = F(\pi)$ in the difference in the rate λ_t^* of information arrival.

To make two outcomes comparable, we evaluate the values of the intensity with the same quantile realizations. We first fix the realizations $q^{(1)}, \ldots, q^{(k)}$ of anterior moves in quantiles and generate the sequences of precision levels for the baseline and improved distributions: $\pi(q^{(1)}), \ldots, \pi(q^{(k)})$ and $\pi(q^{(1)}) + \Delta, \ldots, \pi(q^{(k)}) + \Delta$, respectively. Here, $\pi(q)$ denotes the quantile function for the baseline distribution; the corresponding function for the improved distribution is $\pi_\Delta(q) = \pi(q) + \Delta$. Consequently, if the accumulated public precision for the baseline distribution is Π_k, then that for the improved distribution is $\Pi_k + k\Delta$. Given $q^{(1)}, \ldots, q^{(k)}$, we compare the value of the intensity at the same level of cutoff quantile q, or at the two different levels of cutoffs $\pi(q)$ for the baseline and $\pi(q)+\Delta$ for the improved case.

The above argument justifies the use of the following function:

$$\lambda_k^*(\Delta; \pi, \Pi) = \frac{n-k}{n-k-1} \cdot \frac{r}{\Gamma(\pi+\Delta, \Pi+k\Delta)},$$

which represents the intensity of information arrival with k moves, cutoff precision $\pi+\Delta$, and public precision $\Pi + k\Delta$. We first characterize in which direction λ_k^* moves after marginally increasing Δ:

Proposition 14. *The marginal change $\partial\lambda_k^*/\partial\Delta$ evaluated at $\Delta = 0$ is given by*

$$\frac{\partial}{\partial\Delta}\lambda_k^*(0; \pi, \Pi) = \frac{n-k}{n-k-1} \cdot \frac{r}{\pi^2} \cdot \Omega(k, \pi, \Pi), \tag{30}$$

where

$$\Omega(k, \pi, \Pi) = 2u_0\pi^2 + \Pi - u_0\Pi^2 + k\pi(2u_0\Pi + 3u_0\pi - 1).$$

The numerator Ω is increasing in k and π. There is a positive-valued function $\pi_\Omega(k, \Pi)$ such that the sign of $\Omega(k, \pi, \Pi)$ is identical to $\pi - \pi_\Omega(k, \Pi)$. Information arrival increases both Ω and $\partial\lambda_k^*/\partial\Delta$.[49]

According to Proposition 14, the uniform improvement of precision tends to accelerate information arrival when the cutoff π remains high enough or after new pieces of information arrive, whereas deceleration is likely to occur after a long quiet period without new information. When the cutoff precision π (or cutoff quantile q) is high enough, the distributional improvement accelerates information arrival as long as $\Omega > 0$. As π_t declines, Ω gradually and monotonically approaches 0, while the change in $\partial\lambda_k^*/\partial\Delta$ itself may not be monotonic.

[49] That is, $\Omega(k+1, \pi, \Pi+\pi) > \Omega(k, \pi, \Pi)$ and $\partial\lambda_{k+1}^*(0; \pi, \Pi+\pi)/\partial\Delta > \partial\lambda_k^*(0; \pi, \Pi)/\partial\Delta$ for all $k \in \{1, \ldots, n-3\}$, $\pi \in [\underline{\pi}, \overline{\pi}]$, and $\Pi \in (u_0^{-1}, \infty)$.

Once Ω enters the negative region, the uniform improvement of precision conversely decelerates information arrival due to enhanced freeriding incentives. The improvement keeps the process slower unless the arrival of new information raises the value of Ω to the positive region again. Information arrival always has a positive effect on both Ω and $\partial\lambda_k^*/\partial\Delta$, but if Ω is deeply sunk, the precision improvement still decelerates the subsequent moves.

Although the effect of the distributional shift is not very transparent with the general model with heterogeneous types, the effect becomes more straightforward when precision levels are homogeneous across agents. In this case, we can simply define

$$\lambda_k^{**}(\pi, \pi_0) = \frac{n-k}{n-k-1} \cdot \frac{r}{\Gamma(\pi, \pi_0 + k\pi)} \tag{31}$$

to see how the uniform improvement (i.e., increase in π) changes the speed λ_k^{**} of information arrival.

Proposition 15. *When precision is homogeneous, improvement in π marginally changes the intensity $\lambda_k^{**}(\pi, \pi_0)$ defined by* (31) *by*

$$\frac{\partial\lambda_k^{**}}{\partial\pi} = r \cdot \frac{n-k}{n-k-1} \cdot \left[(k^2 + 3k + 2)u_0 - \left(\frac{\pi_0}{\pi}\right)^2 u(\pi_0)\right], \tag{32}$$

which is decreasing in π_0 and increasing in k and π. The derivative (32) *is positive for all $k \in \{0, 1, \ldots, n-2\}$ if and only if $2u_0 \cdot \pi^2 > \pi_0^2 \cdot u(\pi_0)$. The derivative* (32) *is negative for all $k \in \{0, 1, \ldots, n-2\}$ if and only if $n(n-1)u_0 \cdot \pi^2 < \pi_0^2 \cdot u_0(\pi_0)$.*

With homogeneous precision, improvement in individual precision π tends to accelerate late stages more than early stages. The marginal difference (32) starts with a relatively low value, either negative or positive, when $k = 0$, but monotonically increases at a quadratic speed as new information arrives. When π or $1/\pi_0$ is large enough, the marginal change (32) starts with a positive value at $k = 0$ and never falls below 0. That is, the improvement of precision accelerates the entire information arrival process. In contrast, when π or $1/\pi_0$ is extremely small, the marginal difference (32) never reaches the positive region—the improvement decelerates the whole process.

Even with heterogeneous types, we can attain monotonic results when π_i is always extremely large or small.

Corollary 4. *There exist functions $\underline{\pi}_*(\pi_0; \overline{\pi}, u_0)$ and $\overline{\pi}^*(\pi_0; \underline{\pi}, u_0)$ such that*

(i) *the change in λ_t^* is always positive until the game ends (almost surely) if and only if $\underline{\pi} \geq \underline{\pi}_*(\pi_0; \overline{\pi}, u_0)$, and*

(ii) *the change in λ_t^* is always negative until the game ends (almost surely) if and only if $\overline{\pi} \leq \overline{\pi}^*(\pi_0; \overline{\pi}, u_0)$.*

Both functions are increasing in π_0.

Corollary 4 considers two scenarios extreme enough to obtain monotonic results. On the one hand, the change accelerates the process when the worst precision $\underline{\pi}$ is good enough and the initial public precision is low enough. In this scenario, even the worst agent is knowledgeable enough so that no agent thirsts for the opportunity of freeriding. The improvement in precision further accelerates this tendency. On the other hand, when even the best type $\overline{\pi}$ has little information, an improvement in precision increases the incentive of freeriding and slows the entire process. A decrease in π_0 further exacerbates the slowdown.

B.6 Additional results on boldness

The sell-side security analyst industry has long been an area of research interest to financial economists. A number of these papers employ measures of forecast "boldness" that are meant to capture the degree of herding among security analysts (e.g., Gleason and Lee (2003), Clement and Tse (2005), Jegadeesh and Kim (2010), Keskek et al. (2014)). We build from the theory developed in the previous sections to provide underpinnings for measures of boldness and herding that can be used in estimation.

Recall that agents choose the conditional mean $a^{(k)} = \mathbb{E}[v|s^{(1,\dots,k)}, \pi^{(1,\dots,k)}]$ as their estimates. Note that $a^{(k)}$ is not only the k-th estimate but also, in this model, the best public estimate until the next estimate is publicized. We hence set $a^{(0)} = \mu$. Given $\pi^{(1,\dots,n)} = (\pi^{(1)}, \dots, \pi^{(n)})$, these estimates are normally distributed and thus fully characterized by their means and covariances, summarized below. Note that the conditional variance of $a^{(k)}$ differs from $\Pi_k^{-1} = \mathrm{Var}(a^{(k)} - v|\pi^{(1,\dots,k)})$, the conditional variance of the estimation error.[50]

Lemma 1. *The mean and covariance of $a^{(1,\dots,n)}$ conditional on $\pi^{(1,\dots,n)}$ are given by*

$$\mathrm{Cov}(a^{(k)}, a^{(\ell)}|\pi^{(1,\dots,n)}) = \mathrm{Var}(a^{(k)}|\pi^{(1,\dots,n)}) = \frac{1}{\pi_0} - \frac{1}{\Pi_k},$$

for $k \le \ell$, and $\mathbb{E}[a^{(k)}|\pi^{(1,\dots,n)}] = \mu$.

This calculation provides the correlation between forecasts as a measure of herding behavior that early forecasts induce by influencing subsequent estimates. For example, the correlation between the first and second estimates is

$$\mathrm{Corr}(a^{(1)}, a^{(2)}|\pi^{(1,2)}) = \sqrt{\frac{\pi_0 + \pi^{(1)} + \pi^{(2)}}{\pi_0 + \pi^{(1)}} \cdot \frac{\pi^{(1)}}{\pi^{(1)} + \pi^{(2)}}}.$$

Naturally, the correlation is increasing in $\pi^{(1)}$ and decreasing in $\pi^{(2)}$. The increase in $\pi^{(1)}$ allows the second mover to rely more heavily on the first forecast, and the decrease in $\pi^{(2)}$ further exacerbates this reliance. Perhaps counter-intuitively, we find that the initial precision π_0 has a negative effect on the correlation. An increase in π_0 makes the first two estimates less volatile by relatively weakening their informational quantities. According to the calculation above, the relative decline in $\pi^{(1)}$ is more severe than that in $\pi^{(2)}$, so that π_0 ultimately reduces the correlation. We can generalize the above analysis, as summarized in the following proposition.

[50] These two variances are related as follows: $\mathrm{Var}(a^{(k)} - v|\pi^{(1,\dots,k)}) = \mathrm{Var}(a^{(k)}|\pi^{(1,\dots,k)}) - 2\,\mathrm{Cov}(a^{(k)}, v|\pi^{(1,\dots,k)}) + \mathrm{Var}(v|\pi^{(1,\dots,k)})$. It is easy to verify that the covariance is $\Pi_k^{-1} - \pi_0^{-1} = (\Pi_k - \pi_0)/\Pi_k$.

Proposition 16. *Conditional on $\pi^{(1,\dots,n)}$, the correlation between the k-th and ℓ-th estimates ($k < \ell$) is given by*

$$\mathrm{Corr}(a^{(k)}, a^{(\ell)} | \pi^{(1,\dots,n)}) = \sqrt{\frac{\Pi_\ell}{\Pi_k} \cdot \frac{\Pi_k - \pi_0}{\Pi_\ell - \pi_0}},$$

which is increasing in $\pi^{(1)}, \dots, \pi^{(k)}$ and decreasing in $\pi^{(k+1)}, \dots, \pi^{(\ell)}$, and π_0.

Lemma 1 is also useful in evaluating the boldness of relatively earlier forecasts as compared to forecasts made subsequently. Our model predicts that the earliest mover, who has the highest precision, tends to make a bolder forecast than subsequent movers with lower precision. In this appendix, we define the *boldness* of the k-th forecast as the squared difference $(a^{(k)} - a^{(k-1)})^2$. The $(k-1)$-st estimate $a^{(k-1)}$ is the best publicly available estimate before the k-th and thus we regard it as the most appropriate benchmark in our model.[51]

The next proposition gauges how bold each estimate tends to be by calculating the expected boldness. Most notably, the expected boldness reaches the highest value with the first mover and decreases as time proceeds. That is, although such decline may not occur in a single realization, earlier movers tend to be bolder than subsequent movers, at least in expectation.

Proposition 17. *Conditional on $\pi^{(1,\dots,n)}$, the expected boldness of the k-th estimate is*

$$B_k = \frac{1}{\Pi_{k-1}} - \frac{1}{\Pi_k},$$

increasing in $\pi^{(k)}$ and decreasing in $\pi^{(1)}, \dots, \pi^{(k-1)}$, and π_0. In particular, the expected boldness B_k decreases as k increases, so long as the worst precision $\pi^{(n)}$ is positive.

The above proposition also determines conditions under which the expected boldness remains unchanged with different values of k—the position of the estimate. The expected boldness only depends on the public precision Π_{k-1} at that point and the private precision $\pi^{(k)}$. Hence, when the precision of an agent is fixed, the expected boldness of her estimate is solely determined by the accumulated public precision before her move. The role of k is rather indirect—an increase in k decreases the expected boldness by elevating the public precision Π_k and lowering $\pi^{(k)}$.

Correspondingly, the following proposition reports how fast the expected boldness decreases as k increases.

Proposition 18. *Let $k, \ell \in \{1, \dots, n\}$ with $k < \ell$ and assume $\pi^{(\ell)} > 0$. Both the difference $B_k - B_\ell$ and the ratio $(B_k - B_\ell)/B_k$ increases in $\pi^{(k)}, \dots, \pi^{(\ell-1)}$ and decreases in $\pi^{(\ell)}$ and Π_{k-1}.*

This proposition gauges the speed of decrease in the expected boldness by the absolute difference $B_k - B_\ell$ as well as the rate $(B_k - B_\ell)/B_k$ of the change and obtains the same

[51] Keskek et al. (2014) also employ the last estimate to define their concept of boldness and provide a justification. Their boldness measure is a dummy variable that becomes one if the new estimate does not stay between the most recent estimate and the previous estimate of the analyst in question.

result with these two different measures. Notably, the decline in boldness decelerates in both measures after raising the initial public precision π_0, or more generally, the accumulated public precision Π_{k-1} before the k-th move. An increase in Π_{k-1} clearly decreases both B_k and B_ℓ, but it is less clear *a priori* how the two measures of declining boldness change.

B.7 Additional results on comparative statics

In this appendix, we examine robustness of some comparative statics results presented in Section 7. We first study comparative statics with respect to the *cost spent for information acquisition*, $C(\pi_i) = c\pi_i^2/2$, the counterpart of the analyses in Figures 5 and 9. The results reported in Figure 18 are in the same direction as in Figures 5 and 9, except for the top-right panel. Figure 18 reports that the average cost under the social planner is increasing in c, whereas the average precision under the planner is decreasing in c according to Figure 9. This is not surprising because an increase in c naturally increases the cost $C(\pi_i) = c\pi_i^2/2$. What is rather surprising is the fact that the average cost *decreases* in the equilibrium despite an increase in c. It is also noteworthy that the undersupply ratio in cost, reported in the lower half of Figure 18, is *worse* than in precision. In other words, the undersupply of effort appears more severe when gauged by the actual cost spent than by the amount of information acquired.

We then compare the boldness concept considered in Section 7, namely, *boldness against the previous action*, with a related boldness concept, *boldness against consensus*. Formally, we define the k-th mover's boldness against consensus as the absolute value of the difference between the k-th action and the average of all of the prior actions. In the case of the first action, we replace the average with the prior mean. Figure 19 reports the results. Although this concept of boldness tends to overestimate boldness compared to the one employed in the main text, the effects are in the same direction as the results for the boldness concept in the text (Figure 14).

Lastly, we extend the analysis regarding the sensitivity of followers to the first movers. As in Figure 12, we consider a linear regression and examine the coefficient on the non-constant explanatory variable as well as the associated correlation coefficient. In the regression, we employ a constant and the average action time of the first k movers ($k = 1, 3, 5$). We consider three dependent variables: the average action time *excluding* the first k movers, the average action time *including* the first k movers, and the action time of the last mover. Figure 20 shows that the results are robust to the choice of k or the three independent variables. The only exception is that the anomaly where the sensitivity coefficient exhibits non-monotonicity for small n disappears with $k = 3$ and 5.

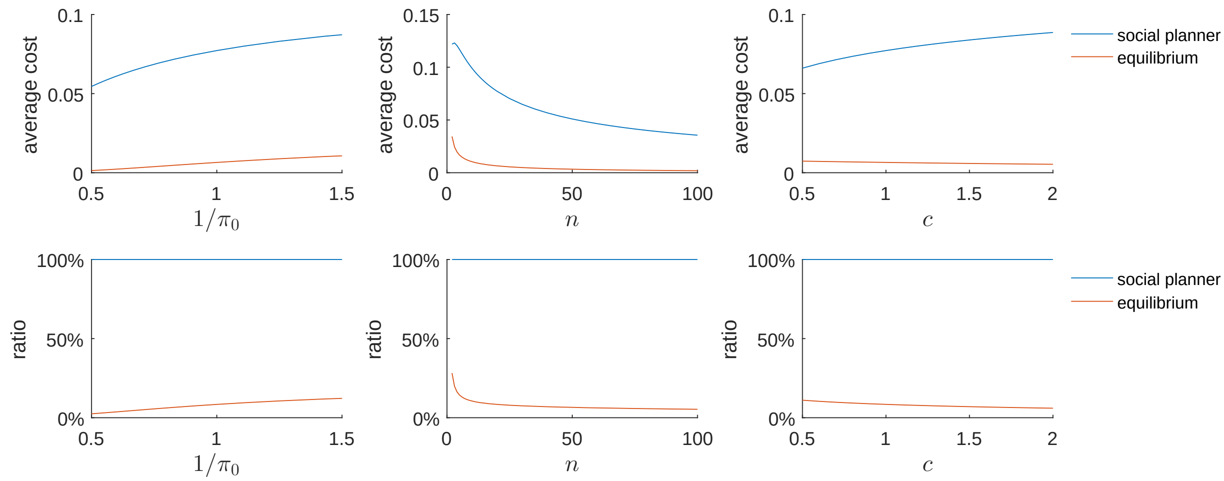

Figure 18: Changes in the cost spent for information acquisition, $C(\pi_i) = c\pi_i^2/2$. The bottom row displays the ratio of the average cost spent relative to that under the social planner. The displayed values are in expected value.

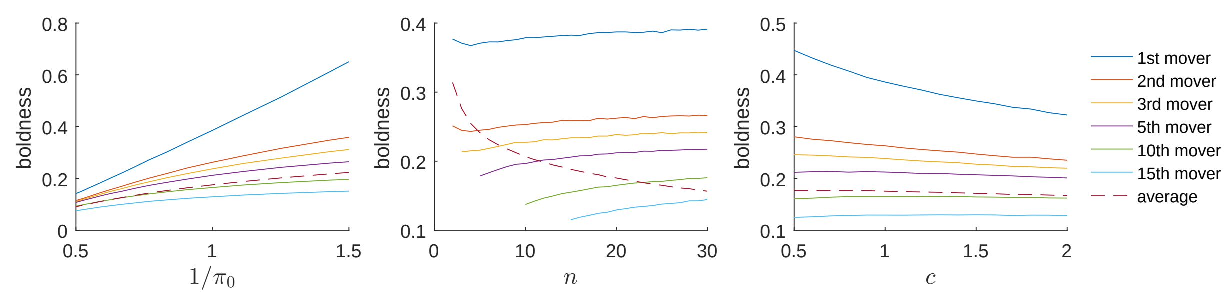

Figure 19: Changes in *boldness against consensus*, defined as the absolute difference between the current action and the average of the previous actions. The displayed values are in expected value.

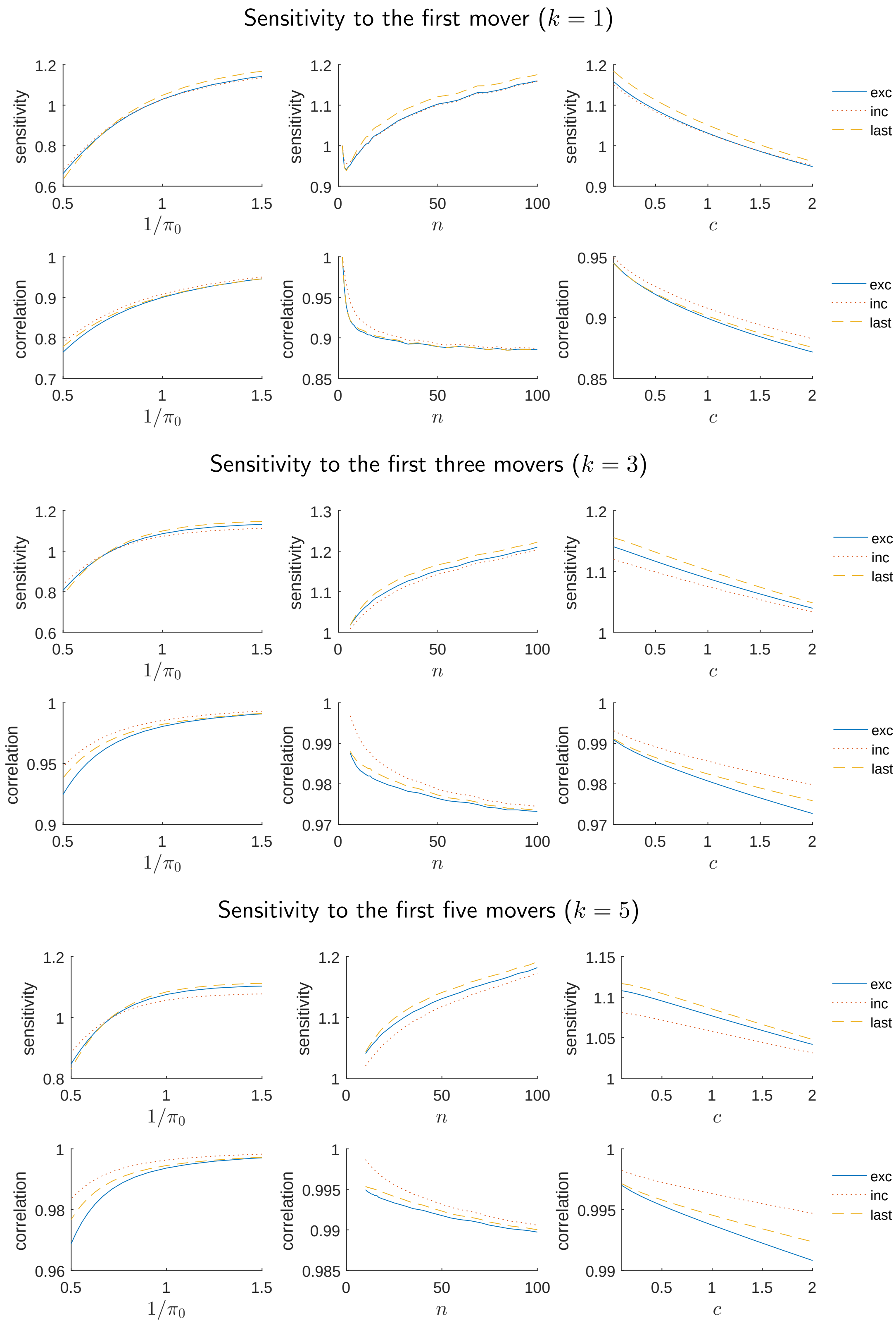

Figure 20: Changes in the sensitivity coefficient and the corresponding correlation coefficient (cf. Figure 12). The independent variables are a constant and the average action time of the first k movers ($k = 1, 3, 5$). The dependent variable is either the average action time *excluding* the first k movers ("exc"), the average action time of all agents *including* the first k movers ("inc"), or the action time of the last mover ("last"). Only the sensitivity and correlation coefficients for $n \geq 2k$ are reported.

C Proofs

C.1 Proofs of Theorems 1, 2, and 5

We first show Theorem 2 and then discuss how to modify the proof to show Theorem 5. Theorem 1 is a special case of Theorem 2.

Consider the decision of agent j after history $h_k = (i^{(1)}, t^{(1)}, a^{(1)}, \ldots, i^{(k)}, t^{(k)}, a^{(k)})$. As noted in the text, once her action time is reached without observing additional actions, agent j truthfully chooses her current best estimate $\mathbb{E}[v|s^{(1,\ldots,k)}, s_j, \pi^{(1,\ldots,k)}, \pi_j]$ as a_j. We focus only on the additional waiting time $t_k(\pi_j, s_j|h_k)$ after history h_k.

We introduce notations to define the objective function. Let $V_{k+1}(\pi_j, s_i|h_k, i^{(k+1)}, t^{(k+1)}, a^{(k+1)})$ denote the equilibrium payoff after observing $(i^{(k+1)}, t^{(k+1)}, a^{(k+1)})$ in addition to h_k. Also, let $\bar{V}_{k+1}(\pi_j, s_j|h_k, t^{(k+1)}, a^{(k+1)})$ be the average of $V_{k+1}(\pi_j, s_j|h_k, i^{(k+1)}, t^{(k+1)}, a^{(k+1)})$ over $i^{(k+1)} \in I_j(h_{k-1})$, where $I_j(h_k) = \{1, \ldots, n\} \setminus \{j, i^{(1)}, \ldots, i^{(k-1)}\}$ is the set of the remaining agents excluding agent j.

Now we can express the payoff of choosing waiting time τ:

$$\begin{aligned}U_k(\tau|\pi_j, s_j, h_k) = &\int_{\{t^{(k+1)}<\tau\}} e^{-rt^{(k+1)}} \bar{V}_{k+1}\, \mathrm{d}\beta_k(h_k) + \int_{\{t^{(k+1)}>\tau\}} e^{-r\tau} u(\Pi_k + \pi_j)\, \mathrm{d}\beta_k(h_k) \\ &+ \int_{\{t^{(k+1)}=\tau\}} e^{-r\tau} \cdot p\left[\bar{V}_{k+1} - u(\Pi_k + \pi_j)\right] \mathrm{d}\beta_k(h_k). \qquad (33)\end{aligned}$$

Here, $t^{(k+1)} = \min_{i \in I_j(h_k)} t_{k+1}(\pi_i, s_i|h_k)$ is the shortest waiting time and $p = (1 + \#\{i \in I_j(h_k) : t_k(\pi_i, s_i|h_k) = \tau\})^{-1}$ is the probability that agent j becomes the $(k+1)$-st mover. The integrals are taken over the space of $(\pi_1, s_1, \ldots, \pi_n, s_n)$; recall that $\beta_k(h_k)$ is a probability distribution over this space. The expression (33) provides a more explicit representation of the value function: $V_k(\pi_j, s_j|h_k) = \max_{\tau \geq 0} U_k(\tau|\pi_j, s_j, h_k)$ for $k < n-1$.

We observe that $V_k(\pi_j, s_j|h_k)$ depends only on $\pi^{(1,\ldots,k)}$ and π_j, where $\pi^{(\ell)}$ is the precision of the ℓ-th mover. When $k = n-1$, the value function $V_{n-1}(\pi_j, s_j|h_{n-1}) = u(\Pi_{n-1} + \pi_j)$ depends only on these variables. We can then inductively show that U_k is a function of $\pi^{(1,\ldots,k)}$, π_j, and τ, but not of any other variables, if V_{k+1} is determined only by $\pi^{(1,\ldots,k+1)}$ and π_j. Consequently, V_k, $\bar{V}_k$, and t_{k+1} are fully determined by $\pi^{(1,\ldots,k)}$ and π_j. In particular, we no longer need to distinguish $\bar{V}_k$ from V_k because neither depends on identities.

We further simplify the expression (33) by observing that its last integral equals 0. Suppose not. Find a decreasing sequence $\{\tau_n\}_{n=1}^{\infty}$ convergent to τ such that the event $\{t^{(k+1)} = \tau_n\}$ has probability 0 in $\beta_k(h_k)$. Such a sequence exists because a distribution (in this case, that of $\tau^{(k+1)}$) may have at most countably many atoms. Since the event $\{t^{(k+1)} = \tau\}$ is a subset of the event $\{t^{(k+1)} < \tau_n\}$ corresponding to the first integral, the limit superior of the difference $U_k(\tau_n|\pi_j, h_k) - U_k(\tau|\pi_j, h_k)$ converges to a positive value. Hence, τ is suboptimal even though it is chosen with a positive probability in equilibrium; this is a contradiction.

With the simplified objective function (33), we show that $t_k(\pi_j|h_{k-1})$ is non-increasing. We prove the monotonicity from the *decreasing* difference condition: $U_k(\tau|\pi_j, h_k) - U_k(\tau'|\pi_j, h_k)$

is decreasing in π_j when $\tau > \tau' \geq 0$. The difference is equal to

$$\int_{\{\tau > t^{(k+1)} > \tau'\}} e^{-r\tau} \left[e^{-r(t^{(k+1)} - \tau)} \bar{V}_{k+1} - u(\Pi_k + \pi_j) \right] \mathrm{d}\beta_k(h_k) \\ - \int_{\{t^{(k+1)} > \tau\}} e^{-r\tau'} \left(1 - e^{-r(\tau - \tau')} \right) u(\Pi_k + \pi_j) \; \mathrm{d}\beta_k(h_k)$$

Thus, to prove the difference is decreasing in π_j, it suffices to show $\partial V_{k+1}(\pi_j | h_{k+1}) / \partial \pi_j \leq u'(\Pi_k + \pi_j)$ almost everywhere in π_j. Note that V_{k+1} can be written as $\mathbb{E}[e^{-r\widetilde{t}} u'(\widetilde{\Pi} + \pi_j) | \pi_j, h_{k+1}]$ with random variables $\widetilde{t}$ and $\widetilde{\Pi}$ that signify the additional waiting time toward the ultimate action time and the corresponding public precision at the time of the action, respectively. Since these random variables are optimally chosen, we can apply the envelope theorem and obtain

$$\frac{\partial V_{k+1}}{\partial \pi_j} = \mathbb{E} \left[e^{-r\widetilde{t}} u' \left(\widetilde{\Pi} + \pi_j \right) \middle| \pi_j, h_{k+1} \right] \leq u'(\Pi_k + \pi_j).$$

We prove that $t_{k+1}(\pi_j | h_k)$ is differentiable almost everywhere. Due to the monotonicity, we can simply write

$$V_k(\pi_j | \pi^{(1,\dots,k)}) = \int_{\pi_j}^{\pi^{(k)}} e^{-rt_{k+1}(\pi^{(k+1)} | \pi^{(1,\dots,k)})} V_{k+1}(\pi_j | \pi^{(1,\dots,k+1)}) \, \mathrm{d}F_k(\pi^{(k+1)} | \pi^{(k)}) \\ + F_{k+1}(\pi_j | \pi^{(k)}) e^{-rt_{k+1}(\pi_j | \pi^{(1,\dots,k)})} u(\Pi_k + \pi_j).$$

By rearranging the above expression, we can express

$$t_k(\pi_j | h_k) = -\frac{1}{r} \cdot \log \left[\frac{V_k(\pi_j | h_k) - \int_{\pi_j}^{\overline{\pi}} e^{-rt_{k+1}(\pi^{(k+1)} | h_k)} \bar{V}_{k+1} \, \mathrm{d}F_k(\pi^{(k+1)})}{F_k(\pi_j) u(\Pi_k + \pi_j)} \right]$$

as a combination of functions that are differentiable almost everywhere.

We uniquely find the optimal value of $t_{k+1}(\pi_j | h_k)$ by the first-order condition (11) as in the text. The optimality is guaranteed by the decreasing difference condition.

Proof of Theorem 5

By repeating the proof of Theorem 2, we can reduce the decision problem to the maximization of the value function

$$V_k(q_j | q^{(1,\dots,k)}) = \max_{\hat{q} \in [0, q^{(k)}]} \Bigg[\int_{\hat{q}}^{q^{(k)}} e^{-rt_{k+1}(q^{(k+1)} | q^{(1,\dots,k)})} V_{k+1}(q_j | q^{(1,\dots,k+1)}) \, \mathrm{d}F_{k+1}(q^{(k+1)} | q^{(k)}) \\ + F_{k+1}(\hat{q} | q^{(k)}) e^{-rt_{k+1}(\hat{q} | q^{(1,\dots,k)})} u(\Pi_k + \pi(q_j)) \Bigg].$$

The equilibrium waiting time is uniquely characterized by the first-order condition

$$-rt'_{k+1}(q_j|q^{(1,\dots,k)}) = (n-k-1)h(q_j)\Gamma(\pi(q_j),\Pi_k)$$

with the initial condition $t_{k+1}(q^{(k)}|q^{(1,\dots,k)}) = 0$.

Note that even when agents observe precision π_i instead of quantile q_i, any symmetric, separating equilibrium must be observationally equivalent to the above equilibrium with observable quantiles. The unobservability matters only when the distribution F has an atom. Let $\hat{\pi}$ be an atom. When agents with precision $\hat{\pi}$ is indifferent between waiting and moving, the type $\hat{\pi}$ needs to provide the same speed of information flow as in the equilibrium with observable quantiles to keep the agent indifferent. Therefore, the two equilibria need to be observationally equivalent.

C.2 Proof of Theorem 3

We prove Theorem 3 in the following steps. We first show that the equilibrium precision is distributed within $[0,\pi^*]$ in Appendix C.2.1. We then introduce notations for the main parts of the proof in Appendix C.2.2 and explicitly calculate the equilibrium condition in Appendix C.2.3. The main part of the proof involves careful evaluation of functions by providing upper and lower bounds; these bounds are provided in Appendix C.2.4. In Appendix C.2.5, we transform the equilibrium condition into a fixed-point problem show that the corresponding operator can be seen as a contraction mapping. Using this technique, we complete the proof of Theorem 3 in Appendix C.2.6.

In the proof, we frequently use the following elementary facts without mentioning them: $ax - by = a(x-y) + (a-b)y$ for all $a,b,x,y \in \mathbb{R}$; and $|e^{-x} - e^{-y}| = e^{-\min\{x,y\}}|x-y|$ for all $x,y \in [0,\infty)$.

C.2.1 Properties of the equilibrium distribution

We first show that the largest precision is given as the individually optimal precision π^*, defined by equation (14).

Lemma 2. *The largest precision in the support of the is π^* in any symmetric equilibrium.*

Proof. Let $\overline{\pi}$ denote the maximum precision in the support. By choosing this precision, agents receives the payoff of $U(\overline{\pi}) = u(\pi_0 + \overline{\pi}) - C(\overline{\pi})$. Instead, by choosing π^* and acting at absolute time $T = 0$, agents obtain the payoff of $U(\pi^*) = u(\pi_0 + \pi^*) - C(\pi^*)$, which should be lower than or equal to the equilibrium payoff $U(\overline{\pi})$. Since π^* is the unique maximizer of $U(\pi)$, the maximum precision $\overline{\pi}$ must coincide with π^*. □

We then eliminate the possibility of equilibrium in pure strategies. Note that the statement below does not elimitate the possibility of an atom at $\pi = 0$.[52]

Lemma 3. *An equilibrium mixed strategy has a support $[\underline{\pi},\pi^*]$ with some $\underline{\pi} \in [0,\pi^*)$. The mixed-strategy distribution F is absolutely continuous on $[0,\pi^*]$.*

[52] A function F is absolutely continuous on $[a,b]$ if and only if $F(x) - F(y) = \int_y^x F'(\pi)\,\mathrm{d}\pi$ for all $x,y \in [a,b]$. It is possible for $F(a)$ to have a positive value.

Proof. Suppose $F(\pi)$ is an equilibrium mixed-strategy distribution and let S be its support. We know the largest element of S is π^* due to Lemma 2. Let $\tilde{T}(q)$ denote the random variables that represent the equilibrium action time (in absolute time) for quantile q. Let $\tilde{\Pi}(q)$ be the corresponding accumulated public precision at the time of the action. Each π corresponds to the set $Q(\pi) = [\lim_{\pi' \nearrow \pi} F(\pi'), F(\pi)]$ of quantiles: $\tilde{T}(q)$ is an optimal action time for precision π as long as $q \in Q(\pi)$. Let $U(q;\pi) = \mathbb{E}[e^{-r\tilde{T}(q)} u(\tilde{\Pi}(q) + \pi)]$ be the payoff that agents with precision π receives when they employ the optimal waiting time for quantile q. This payoff is related to the equilibrium payoff: $V_0(\pi) = U(q;\pi)$ for all $q \in Q(\pi)$.

To show the first part of this lemma, we begin with the case that π^* is an isolated point in S. Let π^{**} denote the second largest element. If S has no other element than π^*, we set $\pi^{**} = 0$. For all $\pi \in (\pi^{**}, \pi^*)$, the difference $V_0'(\pi) - C'(\pi)$ is bounded above by

$$
\begin{aligned}
&\mathbb{E}\left[e^{-r\tilde{T}(F(\pi^{**}))} u'\left(\tilde{\Pi}(F(\pi^{**})) + \pi\right)\right] - c\pi \\
&\leq F(\pi^{**})^{n-1} u'(\pi_0 + \pi) + \left(1 - F(\pi^{**})^{n-1}\right) u'(\pi_0 + \pi^* + \pi) - c\pi \\
&\leq F(\pi^{**})^{n-1} \left\{u'(\pi_0 + \pi) - u'(\pi_0 + \pi^*)\right\} \\
&\quad - \left(1 - F(\pi^{**})^{n-1}\right) \left\{u'(\pi_0 + \pi^*) - u'(\pi_0 + \pi^* + \pi)\right\} + c(\pi^* - \pi).
\end{aligned} \tag{34}
$$

Note that $F(\pi^{**})^{n-1}$ is the probability that none of the other agents has precision π^*. Also, the last expression (34) follows from the fact $u'(\pi_0 + \pi^*) = c\pi^*$. Since (34) converges to a negative value as π goes to π^*, the value function V_0 decreases on $(\pi^* - \Delta, \pi^*]$ with some small $\Delta > 0$. That is, π^* is not an optimal choice; a contradiction.

We then show that the support S consists only of a single connected component. Let $[\pi_*, \pi^*]$ be the connected component that contains π^*. Let π^{**} be the largest element of $S \setminus [\pi_*, \pi^*]$. If $S = [\pi_*, \pi^*]$, we set $\pi^{**} = 0$. On the one hand, the value function V_0 on $[\pi^{**}, \pi_*]$ is expressed as

$$
V_0(\pi) = \mathbb{E}\left[e^{-r\tilde{T}(F(\pi^{**}))} u\left(\tilde{\Pi}(F(\pi^{**})) + \pi\right)\right], \tag{35}
$$

which is left differentiable at π_*. On the other hand, $V_0(\pi) - C(\pi)$ is constant on $[\pi_*, \pi^*]$ and thus V_0 is right differentiable at π_*. By Theorem 1 of Milgrom and Segal (2002) the value function V_0 is differentiable at π_* and the derivative is given by

$$
V_0'(\pi_*) = \mathbb{E}\left[e^{-r\tilde{T}(F(\pi^{**}))} u'\left(\tilde{\Pi}(F(\pi^{**})) + \pi_*\right)\right]. \tag{36}
$$

Again by the constancy of $V_0(\pi) - C(\pi)$, this derivative $V_0'(\pi_*)$ must coincide with $C'(\pi_*) = c\pi_*$. By the above arguments, for all $\pi \in (\pi^{**}, \pi_*)$,

$$
V_0'(\pi) = \mathbb{E}\left[e^{-r\tilde{T}(F(\pi^{**}))} u'\left(\tilde{\Pi}(F(\pi^{**}) + \pi\right)\right] > V_0'(\pi_*) = c\pi_* > c\pi. \tag{37}
$$

That is, $V_0(\pi) - C(\pi)$ is increasing on the interval (π^{**}, π_*). Consequently,$V_0(\pi^{**}) - C(\pi^{**}) < V_0(\pi_*) - C(\pi_*)$ and thus π^{**} is suboptimal. Therefore, $S = [\pi_*, \pi^*]$ is only the possibility.

We switch to the second part of this lemma. Let $S = [\underline{\pi}, \pi^*]$ be the support of the equilibrium distribution F. For each $\pi \in S$, let $q(\pi)$ denote the midpoint of the interval $Q(\pi)$. We first verify

$$V_0'(\pi) = \mathbb{E}\left[e^{-r\tilde{T}(q(\pi))} u\left(\tilde{\Pi}(q(\pi)) + \pi\right)\right]$$

not only on $(\underline{\pi}, \pi^*)$ but also at $\underline{\pi}$ and π^*. We have already discussed the differentiability at $\underline{\pi}$ in proving (36). The differentiability at π^* is similarly shown. The value function V_0 is right differentiable at π^* because $V_0(\pi) - C(\pi)$ is constant on S. The left differentiability follows from $V_0(\pi) = u(\pi_0 + \pi)$ for $\pi \geq \pi^*$. Both semiderivatives coincide, again, due to Theorem 1 of Milgrom and Segal (2002).

Consider $x, y \in S$ such that $x > y > 0$. Then,

$$\begin{aligned}
c(x-y) &= V_0'(x) - V_0'(y) \\
&= \mathbb{E}\left[e^{-r\tilde{T}(q(x))}\left\{u'\left(\tilde{\Pi}(q(x)) + x\right) - e^{-r\{\tilde{T}(q(y)) - \tilde{T}(q(x))\}} u'\left(\tilde{\Pi}(q(y)) + y\right)\right\}\right] \\
&\geq \mathbb{E}\left[u'\left(\tilde{\Pi}(q(x)) + x\right) - u'\left(\tilde{\Pi}(q(y)) + y\right)\right] \\
&\geq \alpha\,\mathbb{E}\left[\tilde{\Pi}(q(y)) - \tilde{\Pi}(q(x))\right] - \beta(x-y),
\end{aligned}$$

where $\alpha = \pi_0 \bar{\Pi}^{-4}$ and $\beta = 2\bar{\Pi}\pi_0^{-4}$ with $\bar{\Pi} = \pi_0 + n\pi^*$. In evaluating the expectation, observe that $\tilde{\Pi}(q(x)) - \tilde{\Pi}(q(y)) \geq ky$ when k agents have a quantile within the interval $(q(y), q(x))$. Hence, $\mathbb{E}[\tilde{\Pi}(q(y)) - \tilde{\Pi}(q(x))]$ is at least as high as $(n-1)(q(x) - q(y))y$. Here, $(n-1)(q(x) - q(y))$ is the mean of the binomial distribution with $n-1$ trials and success probability $q(x) - q(y)$. Therefore,

$$q(x) - q(y) \leq \frac{\beta + c}{(n-1)\alpha} \cdot \frac{x-y}{y}. \tag{38}$$

The inequality (38) eliminate most of the singularity of F. First, F cannot have an atom except at $\pi = 0$. If $\pi > 0$ is an atom, then set either x or y to be π and let the other variable very close to π. Then, the left-hand side of (38) remains positive even when the right-hand side shrinks to 0; a contradiction. This implies $q(\pi) = F(\pi)$ except at $\pi = 0$. Second, since q and thus F are Lipschitz continuous on $[\hat{\pi}, \pi^*]$ for any $\hat{\pi} \in (0, \pi^*)$, F is absolutely continuous on $[\hat{\pi}, \pi^*]$. Finally, this second argument implies the absolute continuity of F on $[0, \pi^*]$ because $\hat{\pi}$ can be arbitrarily close to 0. □

C.2.2 Notations

We now introduce notations. In this proof, the hazard rate $h_1(\pi^{(1)})$ for the first order statistic plays a central role. To make it look distinct, we denote $h_* = h_1$. From a hazard rate h_*, we induce $F_*(\pi) = e^{-H_*(\pi)}$ and $f_*(\pi) = h_*(\pi)F_*(\pi)$, where $H_*(\pi) = \int_\pi^{\pi^*} h_*(\pi)\,\mathrm{d}\pi$. When we consider an alternative hazard rate, e.g., h_{**}, we analogously define F_{**}, f_{**}, and H_{**}.

We outline the proof. In Appendix C.2.3, we derive V_0'' to explicitly calculate the second-order indifference condition $V_0''(\pi) = c$. We express the key functions mentioned in Section 4, namely N_k^n and D_k^n, as multiple integrals in the following form:

$$\begin{aligned} I_k^n(\pi; h_*, \omega) = &\int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} e^{-rt_1^n(\pi^{(1)}; h_*)} h_*(\pi^{(1)}) \cdots \int_\pi^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} e^{-rt_k^n(\pi^{(k)}; \pi^{(1,\dots,k-1)}, h_*)} h_*(\pi^{(k)}) \\ &\cdot e^{-rt_{k+1}^n(\pi; \pi^{(1,\dots,k)}, h_*)} \omega(\pi^{(1,\dots,k)}, \pi). \end{aligned}$$

When $k = 0$, I_0^n simply equals the second line of the definition. Here, the equilibrium waiting time

$$t_k^n(\pi^{(k)}; \pi^{(1,\dots,k-1)}, h_*) = \frac{n-k}{n-1} \cdot \frac{1}{r} \int_{\pi^{(k)}}^{\pi^*} \Gamma(\pi, \Pi_{k-1}) h_*(\pi) \, \mathrm{d}\pi$$

is regarded as a function of h_*. We allow $n = \infty$; in this case, the fraction $\frac{n-k}{n-1}$ is set to be 1.

C.2.3 Equilibrium condition

The first order derivative of $V_k(\pi|\pi^{(1,\dots,k)})$ is, due to the envelope theorem,

$$\begin{aligned} \frac{\partial V_k}{\partial \pi} = &\int_\pi^{\pi^{(k)}} e^{-rt_{k+1}(\pi^{(k+1)}|\pi^{(1,\dots,k)})} \frac{\partial V_{k+1}}{\partial \pi}(\pi|\pi^{(1,\dots,k+1)}) f_{k+1}(\pi^{(k+1)}|\pi^{(k)}) \, \mathrm{d}\pi^{(k+1)} \\ &+ F_{k+1}(\pi|\pi^{(k)}) \cdot e^{-rt_{k+1}(\pi|\pi^{(1,\dots,k)})} u'(\pi + \Pi_k) \end{aligned}$$

almost everywhere. (Hereafter, we do not mention the derivatives are not defined on the full domain.) Consequently, the second-order derivative is

$$\begin{aligned} \frac{\partial^2 V_k}{\partial \pi^2} = &\int_\pi^{\pi^{(k)}} e^{-rt_{k+1}(\pi^{(k+1)}|\pi^{(1,\dots,k)})} \frac{\partial^2 V_{k+1}}{\partial \pi^2}(\pi|\pi^{(1,\dots,k+1)}) f_{k+1}(\pi^{(k+1)}|\pi^{(k)}) \, \mathrm{d}\pi^{(k+1)} \\ &+ F_{k+1}(\pi|\pi^{(k)}) \cdot e^{-rt_{k+1}(\pi|\pi^{(1,\dots,k)})} u''(\pi + \Pi_k) \\ &+ f_{k+1}(\pi|\pi^{(k)}) \cdot e^{-rt_{k+1}(\pi|\pi^{(1,\dots,k)})} D(\pi, \Pi_k), \end{aligned}$$

where $D(\pi, \Pi_k)$ is as defined in (17). We can recursively construct V_k from the terminal condition $\partial^2 V_{n-1}/\partial \pi^2(\pi|\pi^{(1,\dots,n-1)}) = u''(\pi + \Pi_{n-1})$:

$$\frac{\partial^2 V_0}{\partial \pi^2}(\pi) = \sum_{k=0}^{n-2} D_k^n(\pi; h_*) f_*(\pi) - \sum_{k=0}^{n-1} N_k^n(\pi; h_*) F_*(\pi),$$

where

$$N_k^n(\pi;h_*) = \int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} e^{-rt_1(\pi^{(1)})} f_1(\pi^{(1)}|\pi^{(0)}) \cdots \int_\pi^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} e^{-rt_k(\pi^{(k)}|\pi^{(1,\dots,k-1)})} f_k(\pi^{(k)}|\pi^{(k-1)})$$
$$\cdot\, e^{-rt_{k+1}(\pi|\pi^{(1,\dots,k)})} \left\{-u''(\pi+\Pi_k)\right\} F_{k+1}(\pi|\pi^{(k)})/F_*(\pi) \tag{39}$$
$$D_k^n(\pi;h_*) = \int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} e^{-rt_1(\pi^{(1)})} f_1(\pi^{(1)}|\pi^{(0)}) \cdots \int_\pi^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} e^{-rt_k(\pi^{(k)}|\pi^{(1,\dots,k-1)})} f_k(\pi^{(k)}|\pi^{(k-1)})$$
$$\cdot\, e^{-rt_{k+1}(\pi|\pi^{(1,\dots,k)})} D(\pi|\Pi_k) f_{k+1}(\pi|\pi^{(k)})/f_*(\pi).$$

When $k=0$, N_k^n and D_k^n are identical to the second lines of the above expressions. Also, when $k=n-1$, we set $F_n(\pi)=1$ and $f_n(\pi)=0$. Since $D_{n-1}^n=0$, we can align the two summations as below:

$$\frac{\partial^2 V_0}{\partial \pi^2}(\pi) = \sum_{k=0}^{n-1} D_k^n(\pi;h_*) f_*(\pi) - \sum_{k=0}^{n-1} N_k^n(\pi;h_*) F_*(\pi).$$

The functions $N_k^n(\pi;h_*)$ and $D_k^n(\pi;h_*)$ have a common structure. Their almost identical multiple integrals commonly contain

$$f_1(\pi^{(1)}|\pi^{(0)}) \cdots f_k(\pi^{(k)}|\pi^{(k-1)}) = \prod_{i=1}^k \frac{f_i(\pi^{(i)}|\pi^{(i-1)})}{F_i(\pi^{(i)}|\pi^{(i-1)})} \cdot \left(\frac{F(\pi^{(i)})}{F(\pi^{(i-1)})}\right)^{n-i}$$
$$= \alpha_{n,k} \Phi_{n,k}(\pi^{(1,\dots,k)};h_*) F_*(\pi^{(k)}) \prod_{i=1}^k h_*(\pi^{(i)}), \tag{40}$$

where $\alpha_{n,k} = (n-1)^{-k}(n-1)\cdots(n-k)$ and

$$\Phi_{n,k}(\pi^{(1,\dots,k)};h_*) = \left(\frac{F_*(\pi^{(1)})\cdots F_*(\pi^{(k)})}{F_*(\pi^{(k)})^k}\right)^{\frac{1}{n-1}} = e^{(n-1)^{-1}\sum_{i=1}^k [H_*(\pi^{(k)})-H_*(\pi^{(i)})]}.$$

Since $\Phi_{n,k}(\pi^{(1,\dots,k)};h_*) F_{k+1}(\pi|\pi^{(k)}) = \Phi_{n,k+1}(\pi^{(1,\dots,k)},\pi;h_*)$, equation (40) implies

$$N_k^n(\pi;h_*) = I_k^n(\pi;h_*,\omega_{n,k}^N(\cdot;h_*))$$
$$D_k^n(\pi;h_*) = I_k^n(\pi;h_*,\omega_{n,k}^D(\cdot;h_*)),$$

where $\omega_{n,k}^N(\pi^{(1,\dots,k)},\pi;h_*) = \alpha_{n,k}\Phi_{n,k+1}(\pi^{(1,\dots,k)},\pi;h_*)\cdot\{-u''(\pi+\Pi_k)\}$ and $\omega_{n,k}^D(\pi^{(1,\dots,k)},\pi;h_*) = \alpha_{n,k+1}\Phi_{n,k+1}(\pi^{(1,\dots,k)},\pi;h_*)\cdot D(\pi,\Pi_k)$.

For the case of $n=\infty$, we analogously define $N_k^\infty(\pi;h_*) = I_k^\infty(\pi;h_*,\omega_{\infty,k}^N)$ and $D_k^\infty(\pi;h_*) = I_k^\infty(\pi;h_*,\omega_{\infty,k}^D)$. Since both $\alpha_{n,k}$ and $\Phi_{n,k}$ converge (pointwise) to 1 as $n\to\infty$, we naturally define $\omega_{\infty,k}^N(\pi^{(1,\dots,k)},\pi) = -u''(\Pi_k+\pi)$ and $\omega_{n,k}^D(\pi^{(1,\dots,k)},\pi) = D(\pi,\Pi_k)$. We later observe that N_k^n and D_k^n appropriately converge to N_k^∞ and D_k^∞, respectively, in Lemma 13.

C.2.4 Preliminary results on bounds

We first provide bounds on several important functions.

Lemma 4. *There exist positive constants $\bar{\Gamma}$, d^*, and d^{**} and a function $d_* : (0, \pi^*] \to (0, \infty)$ such that, for all $n \in \{2, 3, \ldots\}$, $\pi \in [0, \pi^*]$, $\Pi \in [\pi_0, \infty)$, and a Lebesgue-integrable function $h_* : [\pi, \pi^*] \to [0, \infty)$, we have $\Gamma(\pi, \Pi) \le \bar{\Gamma}$, $D(\pi, \Pi) \le d^*\pi$, $D(\pi, \Pi) \le d^{**}(\Pi + \pi)^{-3}$,*

$$\sum_{k=0}^{n-1} N_k^n(\pi; h_*) \le -u''(\pi_0)/F_*(\pi), \text{ and}$$

$$e^{-\bar{\Gamma} H_*(\pi)} d_*(\pi) \le \sum_{k=0}^{n-1} D_k^n(\pi; h_*) \le d^*\pi \cdot F(\pi)/F_*(\pi).$$

Proof. Proposition 8 proves $\Gamma(\pi, \Pi) \le \Gamma(\pi_0 \cdot (u(\pi_0)/2u_0)^{1/2}, \pi_0) \equiv \bar{\Gamma}$. This proposition also implies $\Gamma(\pi, \Pi) \le \gamma^*\pi$ and $\Gamma(\pi, \Pi) \le \gamma^{**}(\Pi + \pi)^{-1}$ with $\gamma^* = \pi_0^{-2} u(\pi_0)^{-1}$ and $\gamma^{**} = \pi^* \pi_0^{-1} u(\pi_0)^{-1}$. Since

$$u'(\Pi + \pi) - u'(\Pi + 2\pi) = \frac{2\pi\Pi + 3\pi^2}{(\Pi + \pi)^2(\Pi + 2\pi)^2} \le \frac{2\pi(\Pi + 2\pi)}{(\Pi + \pi)^2(\Pi + 2\pi)^2} \le \frac{2\pi}{(\Pi + \pi)^3},$$

we can use $d^* = \pi_0^{-4}\{u(\pi_0)^{-1} + 2\pi_0\}$ and $d^{**} = \pi^*\{\pi_0^{-1} u(\pi_0)^{-1} + 2\}$ to obtain the two desired upper bounds for $D(\pi, \Pi)$.

We evaluate $\sum_{k=0}^{n-1} N_k^n(\pi; h_*)$. Let $p(k)$ be the probability that the number of agents with precision more than π is equal to k out of $n-1$ agents. By (39), we have $N_k^n(\pi; h_*)F_*(\pi) \le -u''(\pi_0) \cdot p(k)$. Therefore, $\sum_{k=0}^{n-1} N_k^n(\pi; h_*)F_*(\pi) \le -u''(\pi_0)$.

We need more delicate arguments for the other summation $\sum_{k=0}^{n-1} D_k^n(\pi; h_*)$. Since

$$\frac{f_{k+1}(\pi|\pi^{(k)})}{f_*(\pi)} \cdot F_*(\pi) = F_{k+1}(\pi|\pi^{(k)}) \cdot \frac{f_{k+1}(\pi|\pi^{(k)})}{F_{k+1}(\pi|\pi^{(k)})} \cdot \frac{F_*(\pi)}{f_*(\pi)} = \frac{n-k-1}{n-1} \cdot F_{k+1}(\pi|\pi^{(k)}),$$

we obtain

$$D_k^n(\pi; h_*)F_*(\pi) \le \left(1 - \frac{k}{n-1}\right) \cdot d^*\pi \cdot p(k).$$

Since $p(k)$ is the probability mass function of a binomial distribution with $n-1$ trials and success probability $1 - F(\pi)$, we obtain

$$\sum_{k=0}^{n-1} D_k^n(\pi; h_*)F_*(\pi) \le d^*\pi \cdot \left(1 - \frac{(n-1)(1 - F(\pi))}{n-1}\right) \le d^*\pi \cdot F(\pi).$$

The lower bound of $\sum_{k=0}^{n-1} D_k^n(\pi; h_*)$ is given by

$$\sum_{k=0}^{n-1} D_k^n(\pi; h_*) \ge D_0^n(\pi; h_*) = e^{-rt_1(\pi; h_*)} D(\pi, \pi_0) \ge e^{-\bar{\Gamma} H_*(\pi)} d_*(\pi)$$

with $d_*(\pi) = \min_{\pi' \in [\pi, \pi^*]} D(\pi', \pi_0)$. □

We repeatedly use the following lemma to evaluate differences of multiple integrals. We allow for the case of $n = \infty$.

Lemma 5. *Let $n, m \in \{2, 3, \ldots, \infty\}$. Let $h_*, h_{**} : [\breve{\pi}, \pi^*] \to [0, \infty)$ and $\omega_*, \omega_{**} : [\breve{\pi}, \pi^*]^{k+1} \to [0, \bar{\omega}]$ be Lebesgue-measurable functions. The difference $|I_k^n(\pi; h_*, \omega_*) - I_k^m(\pi; h_{**}, \omega_* *)|$ is bounded above by*

$$\sum_{j=0}^{k-1} \left\{ \alpha_j \cdot \frac{H_{**}(\breve{\pi})^j}{j!} \cdot \frac{H_*(\breve{\pi})^{k-j-1}}{(k-j-1)!} \right\} + (\alpha_k + \varepsilon_\omega) \cdot \frac{H_{**}(\breve{\pi})^k}{k!},$$

where $\alpha_j = \bar{\omega}[(1 + \bar{\Gamma} H_*(\breve{\pi}))\varepsilon_h + \eta_{j+1}^{n,m} \bar{\Gamma} H_*(\breve{\pi})^2]$, $\varepsilon_h = \sup_{\pi \in [\breve{\pi}, \pi^*]} \int_\pi^{\pi^*} \big(h_*(\pi') - h_{**}(\pi')\big) \, \mathrm{d}\pi'$, $\varepsilon_\omega = \sup_{x \in [\breve{\pi}, \pi^*]^{k+1}} |\omega_*(x) - \omega_{**}(x)|$, and $\eta_i^{n,m} = |\frac{n-i}{n-1} - \frac{m-i}{m-1}|$.

Proof. If either $H_*(\breve{\pi})$ or $H_{**}(\breve{\pi})$ is infinity, the statement is obvious. We assume they are finite.

To simplify notations, we write t_i^* and t_i^{**} to denote $t_i^n(\pi^{(i)}; \pi^{(1,\ldots,i-1)}, h_*)$ and $t_i^m(\pi^{(i)}; \pi^{(1,\ldots,i-1)}, h_{**})$, respectively. We decompose the difference between $I_{k,0} = I_k^n(\pi; h_*, \omega)$ and $I_{k,k+1} = I_k^m(\pi; h_{**}, \omega)$ as follows. For each $j = 1, \ldots, k$, let

$$\begin{aligned} I_{k,j} = &\int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} e^{-rt_{1**}} h_{**}(\pi^{(1)}) \cdots \int_\pi^{\pi^{(j-1)}} \mathrm{d}\pi^{(j)} e^{-rt_j^{**}} h_{**}(\pi^{(j)}) \\ &\int_\pi^{\pi^{(j)}} \mathrm{d}\pi^{(j+1)} e^{-rt_{j+1}^*} h_*(\pi^{(j-1)}) \cdots \int_\pi^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} e^{-rt_k^*} h_*(\pi^{(k)}) \\ &\cdot e^{-rt_{k+1}^n(\pi; \pi^{(1,\ldots,k)}, h_*)} \omega_*(\pi^{(1,\ldots,k)}, \pi). \end{aligned}$$

We evaluate the difference $|I_{k,0} - I_{k,k+1}|$ by providing an upper bounds to $|I_{k,j} - I_{k,j+1}|$ for each j.

When $j < k$,

$$\begin{aligned} |I_{k,j} - I_{k,j+1}| \leq \bar{\omega} \cdot &\int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} h_{**}(\pi^{(1)}) \cdots \int_\pi^{\pi^{(j-1)}} \mathrm{d}\pi^{(j)} h_{**}(\pi^{(j)}) \\ &\left| \int_\pi^{\pi^{(j)}} \mathrm{d}\pi^{(j+1)} \left\{ e^{-rt_{j+1}^{**}} h_{**}(\pi^{(j+1)}) - e^{-rt_{j+1}^*} h_*(\pi^{(j+1)}) \right\} \right. \\ &\left. \int_\pi^{\pi^{(j+1)}} \mathrm{d}\pi^{(j+2)} h_*(\pi^{(j+2)}) \cdots \int_\pi^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} h_*(\pi^{(k)}) \right|, \\ \leq \bar{\omega} \cdot \frac{H_*(\hat{\pi})^{k-j-1}}{(k-j-1)!} \cdot &\int_\pi^{\pi^{(0)}} \mathrm{d}\pi^{(1)} h_{**}(\pi^{(1)}) \cdots \int_\pi^{\pi^{(j-1)}} \mathrm{d}\pi^{(j)} h_{**}(\pi^{(j)}) \\ &\left| \int_\pi^{\pi^{(j)}} \mathrm{d}\pi^{(j+1)} \left\{ e^{-rt_{j+1}^{**}} h_{**}(\pi^{(j+1)}) - e^{-rt_{j+1}^*} h_*(\pi^{(j+1)}) \right\} \right|. \end{aligned} \tag{41}$$

We used the following inequality to obtain (41):

$$\int_{\pi}^{\pi^{(j+1)}} \mathrm{d}\pi^{(j+2)} h_*(\pi^{(j+2)}) \cdots \int_{\pi}^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} h_*(\pi^{(k)}) = \frac{\{H_*(\pi) - H_*(\pi^{(j+1)})\}^{k-j-1}}{(k-j-1)!} \leq \frac{H_*(\pi)^{k-j-1}}{(k-j-1)!}.$$

The j-th integral (41) is at most

$$\left| \int_{\hat{\pi}}^{\pi^*} \left\{ h_*(\pi^{(j+1)}) \left(e^{-rt^*_{j+1}} - e^{-rt^{**}_{j+1}} \right) + e^{-rt^{**}_{j+1}} \left(h_*(\pi^{(j+1)}) - h_{**}(\pi^{(j+1)}) \right) \right\} \mathrm{d}\pi^{(j+1)} \right| \leq \int_{\hat{\pi}}^{\pi^*} h_*(\pi^{(j+1)}) \left| rt^*_{j+1} - rt^{**}_{j+1} \right| \mathrm{d}\pi^{(j+1)} + \varepsilon_h$$

Since $t^n_i = \frac{n-i}{n-1} t^\infty_i$, the difference $|rt^*_{j+1} - rt^{**}_{j+1}|$ is bounded above by

$$\eta^{n,m}_{j+1} \left| rt^{\infty *}_{j+1} \right| + \left| rt^{\infty *}_{j+1} - rt^{\infty **}_{j+1} \right| \leq \eta^{n,m}_{j+1} \bar{\Gamma} H_*(\breve{\pi}) + \bar{\Gamma} \varepsilon_h,$$

where $t^{\infty *}_{j+1} = t^\infty_{j+1}(\pi^{(j+1)}; \pi^{(1,\ldots,j)}; h_*)$, and $t^{\infty **}_{j+1} = t^\infty_{j+1}(\pi^{(j+1)}; \pi^{(1,\ldots,j)}; h_{**})$. Therefore, the j-th integral (41) is at most $(1 + \bar{\Gamma} H_*(\breve{\pi}))\varepsilon_h + \eta^{n,m}_{j+1} \bar{\Gamma} H_*(\breve{\pi})^2$. Finally, by calculating the remaining integrals, we obtain

$$|I_{k,j} - I_{k,j+1}| \leq \bar{\omega} \left[\left(1 + \bar{\Gamma} H_*(\breve{\pi})\right) \varepsilon_h + \eta^{n,m}_{j+1} \bar{\Gamma} H_*(\breve{\pi})^2 \right] \cdot \frac{H_{**}(\pi)^j}{j!} \cdot \frac{H_*(\pi)^{k-j-1}}{(k-j-1)!}.$$

Lastly, we evaluate $|I_{k,k} - I_{k,k+1}|$. The difference corresponding to (41) in this case is at most

$$\left| \omega_*(\pi^{(1,\ldots,k)}, \pi) - \omega_{**}(\pi^{(1,\ldots,k)}, \pi) \right| + \bar{\omega} \left| e^{-rt^n_{k+1}(\pi; \pi^{(1,\ldots,k)}, h_*)} - e^{-rt^m_{k+1}(\pi; \pi^{(1,\ldots,k)}, h_{**})} \right| \leq \varepsilon_\omega + \bar{\omega} \left[\left(1 + \bar{\Gamma} H_*(\breve{\pi})\right) \varepsilon_h + \eta^{n,m}_{k+1} \bar{\Gamma} H_*(\breve{\pi})^2 \right].$$

We therefore obtain the desired expression. □

C.2.5 Contraction mapping

We aim to find a unique fixed point of the following operator

$$\mathcal{P}^n h_*(\pi) = \frac{c/F_*(\pi; h_*) + \sum_{k=0}^{n-1} N^n_k(\pi; h_*)}{\sum_{k=0}^{n-1} D^n_k(\pi; h_*)}.$$

We define a *fixed point* of the operator $\mathcal{P}^n$ on $(\hat{\pi}, \pi^*]$ as a Lebesgue-measurable function $h_* : (\hat{\pi}, \pi^*] \to [0, \infty)$ such that $h_* = \mathcal{P}^n h_*$ on $(\hat{\pi}, \pi^*]$. The fixed point is *maximal* if $H_*(\hat{\pi}) = \infty$.

We allow both $n < \infty$ and $n = \infty$. When $n = \infty$, we need the following additional step to guarantee that the series $\sum_{k=0}^\infty N^\infty_k$ and $\sum_{k=0}^\infty D^\infty_k$ properly converge.

Lemma 6. *Let $h_* : [\hat{\pi}, \pi^*] \to [0, \infty)$ be a Lebesgue-integrable function. Both $\sum_{k=0}^{\infty} N_k^{\infty}(\pi; h_*)$ and $\sum_{k=0}^{\infty} D_k^{\infty}(\pi; h_*)$ uniformly converge on $[\hat{\pi}, \pi^*]$ as functions of π. Consequently, $\mathcal{P}^{\infty} h_*$ is continuous on $[\hat{\pi}, \pi^*]$.*

Proof. Let ω_k be either $\omega_{\infty,k}^N$ or $\omega_{\infty,k}^D$. In either case ω_k is bounded above by $\bar{\omega} = \pi_0^{-3} + d^* \pi^*$, where d^* is as in Lemma 4. Since

$$|I_k^{\infty}(\pi; h_*, \omega_k)| \le \bar{\omega} \int_{\pi}^{\pi^{(0)}} \mathrm{d}\pi^{(1)} h_*(\pi^{(1)}) \cdots \int_{\pi}^{\pi^{(k-1)}} \mathrm{d}\pi^{(k)} h_*(\pi^{(k)}) = \bar{\omega} \cdot \frac{H_*(\hat{\pi})^k}{k!}, \tag{42}$$

in either case, both series uniformly converge by the Weierstrass M-test. □

We show the operator $\mathcal{P}^n$ is a contraction mapping on an appropriate domain. Assume that the operator $\mathcal{P}^n$ has a unique fixed point $\hat{h}_*$ on $(\hat{\pi}, \pi^*]$ with some $\hat{\pi} \in (0, \pi^*]$. (When $\hat{\pi} = \pi^*$, the assumption is vacuous.) If $\hat{h}_*$ is not maximal, we slightly extend its domain to $(\hat{\pi} - \Delta, \pi^*]$ with some small $\Delta \in (0, \hat{\pi})$. Let $\mathcal{H}_{M,\Delta}(\hat{h}_*)$ be the space of Lebesgue-measurable functions $h_* : (\hat{\pi} - \Delta, \pi^*] \to [0, M]$ that are identical to $\hat{h}_*$ on $(\hat{\pi}, \pi^*]$. We endow this space with the uniform norm

$$\|h_* - h_{**}\|_{\infty, \hat{\pi}-\Delta} = \sup_{\pi \in [\hat{\pi}-\Delta, \pi^*]} |h_*(\pi) - h_{**}(\pi)|.$$

Lemma 7. *Let $\varepsilon \in (0, \pi^*/2)$. There exist $M_\varepsilon \ge \varepsilon^{-1}$ and $\Delta_\varepsilon > 0$ such that, for all $n \in \{2, 3, \ldots, \infty\}$, $\hat{\pi} \in [2\varepsilon, \pi^*]$, $M \ge M_\varepsilon$, and $\Delta \in (0, \min\{\Delta_\varepsilon, M^{-1}\}]$, if either*
(i) *the operator $\mathcal{P}^n$ has a fixed point $\hat{h}_*$ on $(\hat{\pi}, \pi^*]$ such that $\hat{H}_*(\hat{\pi}) \le \varepsilon^{-1}$ or*
(ii) *$\hat{\pi} = \pi^*$,*
then $\mathcal{P}^n$ is a contraction mapping in the non-empty space $\mathcal{H}_{M,\Delta}(\hat{h})$, endowed with the uniform norm.

Proof. Let $\Omega = 1 + (1 + \bar{\Gamma})(1 + \varepsilon^{-1})$ and $M_\varepsilon = \max\{e^{\Omega}(c - u''(\pi_0)) d_*(\varepsilon)^{-1}, \varepsilon^{-1}\}$ be sufficiently large constants. We first show that the operator $\mathcal{P}^n$ maps $\mathcal{H}_{M,\Delta}(\hat{h}_*)$ to itself for all $M \ge M_\varepsilon$ and $\Delta \in (0, M^{-1}]$. By Lemma 4,

$$\mathcal{P}^n h_*(\pi) \le \frac{c e^{H_*(\pi)} - u''(\pi_0) e^{H_*(\pi)}}{e^{-\bar{\Gamma} H_*(\pi)} d_*(\pi)} \le M_\varepsilon \le M \tag{43}$$

for all $h_* \in \mathcal{H}_{M,\Delta}(\hat{h}_*)$ and $\pi \in (\hat{\pi} - \Delta, \pi^*]$. Here, we used $F_*(\pi) = e^{-H_*(\pi)}$, $H_*(\pi) \le \hat{H}_*(\hat{\pi}) + M\Delta \le \varepsilon^{-1} + 1$, and $d_*(\pi) \ge d_*(\hat{\pi} - \Delta) \ge d_*(\varepsilon)$. Also, $\mathcal{P}^n h_*$ is continuous because all its components are continuous in π. Therefore, $\mathcal{P}^n h_* \in \mathcal{H}_{M,\Delta}(\hat{h}_*)$. The set $\mathcal{H}_{M,\Delta}(\hat{h}_*)$ is nonempty because (43) also implies $\hat{h}_* \le M$.

We then show that, with sufficiently small $\Delta \le M^{-1}$, $|\mathcal{P}_n h_*(\pi) - \mathcal{P}_n h_{**}(\pi)| \le 2^{-1} \|h_* - h_{**}\|_{\infty, \hat{\pi}-\Delta}$ holds for all $h_*, h_{**} \in \mathcal{H}_{M,\Delta}(\hat{h})$. To simplify notations, let $\alpha_* = c e^{H_*(\pi)} + \sum_{k=0}^{n-1} N_k^n(\pi; h_*)$ and $\beta_* = \sum_{k=0}^{n-1} D_k^n(\pi; h_*)$. Define α_{**} and β_{**} analogously. Then,

$$|\mathcal{P}^n h_*(\pi) - \mathcal{P}^n h_{**}(\pi)| \le \left| \frac{\alpha_*(\beta_{**} - \beta_*) + \beta_*(\alpha_* - \alpha_{**})}{\beta_* \beta_{**}} \right| \le \frac{\alpha_* |\beta_* - \beta_{**}|}{\beta_* \beta_{**}} + \frac{|\alpha_* - \alpha_{**}|}{\beta_*}.$$

By Lemma 4, α_* is bounded above by $-u''(u_0)$ and both β_*^{-1} and β_{**}^{-1} are by $e^{\Omega}d_*(\varepsilon)^{-1}$. Therefore,

$$\begin{aligned}
&\kappa^{-1}|\mathcal{P}^n h_*(\pi) - \mathcal{P}^n h_{**}(\pi)| \\
&\le |\alpha_* - \alpha_{**}| + |\beta_* - \beta_{**}| \\
&\le c|e^{H_*(\pi)} - e^{H_{**}(\pi)}| + \sum_{k=0}^{n-1} |N_k^n(\pi; h_*) - N_k^n(\pi; h_{**})| + \sum_{k=0}^{n-1} |D_k^n(\pi; h_*) - D_k^n(\pi; h_{**})| \qquad (44)
\end{aligned}$$

with a finite constant $\kappa = -u''(\pi_0)e^{3\Omega}d_*(\varepsilon)^{-2} + e^{\Omega}d_*(\varepsilon)^{-1}$.

We evaluate each component of (44) from above. First observe

$$|e^{H_*(\pi)} - e^{H_{**}(\pi)}| \le e^{\bar{\Gamma}(\hat{H}_*(\hat{\pi})+M\Delta)}\Delta\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta} \le e^{\Omega}\Delta\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta}$$

due to the fact $|e^x - e^y| \le e^{\max\{x,y\}}|x-y|$. We apply Lemma 5 to evaluate $|N_k^n(\pi; h_*) - N_k^n(\pi; h_{**})|$ and $|D_k^n(\pi; h_*) - D_k^n(\pi; h_{**})|$. Note $\Phi_{n,k+1}(\pi^{(1,\dots,k)}, \pi; h_*) \le e^{H_*(\pi)} \le e^{\Omega}$. Also, again by $|e^x - e^y| \le e^{\max\{x,y\}}|x-y|$,

$$\begin{aligned}
&|\Phi_{n,k+1}(\pi^{(1,\dots,k)}, \pi; h_*) - \Phi_{n,k+1}(\pi^{(1,\dots,k)}, \pi; h_{**})| \\
&\le e^{\Omega} \cdot \frac{1}{n}\sum_{i=1}^{k-1} \int_{\pi^{(k)}}^{\pi^{(i)}} |h_*(\pi) - h_{**}(\pi)|\,\mathrm{d}\pi. \\
&\le \frac{k}{n} \cdot e^{\Omega}\Delta\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta}.
\end{aligned}$$

By Lemma 5, both $|N_k^n(\pi; h_*) - N_k^n(\pi; h_{**})|$ and $|D_k^n(\pi; h_*) - D_k^n(\pi; h_{**})|$ are bounded above by

$$\lambda \left(\sum_{j=0}^{k-1} \frac{H_{**}(\hat{\pi})^j}{j!} \cdot \frac{H_*(\hat{\pi})^{k-j-1}}{(k-j-1)!} + \frac{H_{**}(\hat{\pi})^k}{k!} \right) \Delta\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta}$$

with a constant $\lambda = 2e^{\Omega}\Omega\{d^*\pi^* - u''(\pi_0)\}$. Here, d^* is the constant given in Lemma 4.

We now know the difference $|\mathcal{P}_n h_*(\pi) - \mathcal{P}_n h_{**}(\pi)|$ is bounded above by

$$\kappa \left\{ ce^{\Omega} + \lambda \left(\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} \frac{H_{**}(\hat{\pi})^j}{j!} \cdot \frac{H_*(\hat{\pi})^{k-j-1}}{(k-j-1)!} + \sum_{k=0}^{n-1} \frac{H_{**}(\hat{\pi})^k}{k!} \right) \right\} \Delta\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta}. \qquad (45)$$

The first (double) summation in (45) is at most

$$\sum_{j=0}^{\infty} \frac{H_{**}(\hat{\pi})^j}{j!} \sum_{k=j+1}^{\infty} \frac{H_*(\hat{\pi})^{k-j-1}}{(k-j-1)!} = \sum_{j=0}^{\infty} \frac{H_{**}(\hat{\pi})^j}{j!} e^{H_*(\hat{\pi})} = e^{H_*(\hat{\pi})+H_{**}(\hat{\pi})} \le e^{2\Omega}.$$

Similarly, the second summation is bounded above by $e^{H_{**}(\hat{\pi})} \le e^{\Omega}$. Therefore, due to (45), the desired inequality $|\mathcal{P}_n h_*(\pi) - \mathcal{P}_n h_{**}(\pi)| \le 2^{-1}\|h_* - h_{**}\|_{\infty,\hat{\pi}-\Delta}$ holds when $\Delta \le \Delta_\varepsilon \equiv$

$2^{-1}\kappa^{-1}\{ce^{\Omega}+\lambda(e^{\Omega}+e^{2\Omega})\}^{-1}$. Lastly, observe that Δ_ε and M_ε depend only on ε. □

Now we can uniquely construct a fixed point by the contraction mapping theorem.

Lemma 8. *Let $n \in \{2, 3, \ldots, \infty\}$. For all $\hat{\pi} \in (0, \pi^*)$, a fixed point of $\mathcal{P}^n$ is unique, if exists, on $(\hat{\pi}, \pi^*]$. Consequently, a maximal fixed point of $\mathcal{P}^n$ uniquely exists. The fixed point $h_*(\pi)$ is positive and continuous.*

Proof. To show the uniqueness, consider two fixed points h_* and h_{**} on $(\hat{\pi}, \pi^*]$. Suppose $h_*(\pi) \neq h_{**}(\pi)$ with some $\pi \in (\hat{\pi}, \pi^*]$, and let π_* be the supremum of such π. Define $\hat{h}(\pi) = h_*(\pi)$ on $(\pi_*, \pi^*]$. By Lemma 7, there exist $M > 0$ and $\Delta > 0$ such that both h_* and h_{**} are elements of the space $\mathcal{H}_{M,\Delta}(\hat{h}_*)$ and $\mathcal{P}^n$ is a contraction mapping on that space. (The domains of the functions h_* and h_{**} are restricted to $(\pi_* - \Delta, \pi^*]$.) Since $\mathcal{P}^n$ is a contraction mapping, $\|h_* - h_{**}\|_{\infty, \hat{\pi}-\Delta} = \|\mathcal{P}^n h_*(\pi) - \mathcal{P}^n h_{**}(\pi)\|_{\infty, \hat{\pi}-\Delta} < \|h_* - h_{**}\|_{\infty, \hat{\pi}-\Delta}$; a contradiction.

We turn to the existence of a maximal fixed point. Let S be the set of $\hat{\pi}$ such that a fixed point of $\mathcal{P}^n$ exists on $(\hat{\pi}, \pi^*]$. Let $\hat{\pi}_*$ be the infimum of S. We observe that $\hat{\pi}_*$ is indeed the mimimum of S. Let h_s denote the unique fixed point on $(s, \pi^*]$ and, for each $\pi \in (\hat{\pi}_*, \pi^*]$, set the value of $h_{\hat{\pi}_*}(\pi)$ to be $h_s(\pi)$ with some $s \in S \cap [0, \pi)$. The function $h_{\hat{\pi}_*}$ is well-defined due to the uniqueness part of this proposition. More important, function $h_{\hat{\pi}_*}$ is a fixed point on $(\hat{\pi}_*, \pi^*]$ because this function is a fixed point on $(\hat{\pi}, \pi^*]$ for all $\hat{\pi} > \hat{\pi}_*$.

The thus constructed function $h_{\hat{\pi}_*}$ needs to be maximal. If not, Lemma 7 extends the fixed point to a larger domain $(\hat{\pi}_* - \Delta, \pi^*]$ by the contraction mapping theorem. Thus $\hat{\pi}_* - \Delta \in S$, which contradicts the definition of $\hat{\pi}_*$. □

We now know a fixed-point hazard rate is unique, but the corresponding (equilibrium) distribution has other hazard rates that differ from the fixed point in sets of measure 0. The next lemma provides a remedy for this problem.

Lemma 9. *Let $n \in \{2, 3, \ldots, \infty\}$. Suppose that a distribution function F_* has a hazard rate $h_* : (\underline{\pi}, \pi^*] \to [0, \infty)$ such that $H_*(\underline{\pi}) = \infty$ and $h_* = \mathcal{P}^n h_*$ almost everywhere on $(\underline{\pi}, \pi^*]$. Define $h_{**} = \mathcal{P}^n h_*$ on $(\underline{\pi}, \pi^*]$. This hazard rate h_{**} is another hazard rate of the distribution F_* that satisfies $h_{**}(\pi) = \mathcal{P}^n h_{**}(\pi)$ for all $\pi \in (\underline{\pi}, \pi^*]$. Moreover, both h_{**} and $f_{**} = h_{**}F_*$ are continuous on $(\underline{\pi}, \pi^*]$.*

Proof. Since $h_* = h_{**}$ almost everywhere, we obtain $H_*(\pi) = H_{**}(\pi)$, $F_*(\pi) = e^{-H_*(\pi)} = e^{-H_{**}(\pi)} = F_{**}(\pi)$, and $\mathcal{P}^n h_{**}(\pi) = \mathcal{P}^n h_*(\pi) = h_{**}(\pi)$ for all $\pi \in (\underline{\pi}, \pi^*]$. The functions h_{**} and f_{**} are continuous because $\mathcal{P}^n h$ is continuous for any hazard rate h. □

C.2.6 Proof of Theorem 3

Let F be an equilibrium mixed-strategy distribution. By Lemma 3, the support of F is a non-degenerate interval $[\underline{\pi}, \pi^*]$ and there exists a Lebesgue-measurable function $f : (\underline{\pi}, \pi^*] \to (0, \infty)$ such that $F(\pi) = F(0) + \int_{\underline{\pi}}^{\pi} f(\pi')\,\mathrm{d}\pi'$. Define the corresponding hazard rate for the first order statistic by $h_*(\pi) = (n-1)f(\pi)/F(\pi)$ on $(\underline{\pi}, \pi^*]$. This hazard rate satisfies $h_* = \mathcal{P}^n h_*$ almost everywhere as a necessary condition.

We eliminate the possibility of $\underline{\pi} = 0$. Let $\alpha = (n-1)d^*/c$. By Lemma 4,

$$f(\pi) = \frac{h_*(\pi)}{n-1} \cdot F(\pi) = \mathcal{P}^n h_*(\pi) \cdot \frac{F(\pi)}{n-1} \geq \frac{c/F_*(\pi)}{d^*\pi F(\pi)/F_*(\pi)} \cdot \frac{F(\pi)}{n-1} = \frac{1}{\alpha\pi}$$

for all $\pi \in (0, \pi^*]$. Since

$$1 = F(\pi^*) \geq \int_{\underline{\pi}}^{\pi^*} f(\pi)\,\mathrm{d}\pi = \alpha^{-1}(\log \pi^* - \log \underline{\pi}),$$

we obtain the desired inequality $\underline{\pi} \geq e^{-\alpha}\pi^*$ (> 0). Consequently, $F(0) = 0$ and the function f is a density of F.

The hazard rate h_* is now defined on the entire domain $(\underline{\pi}, \pi^*]$ and satisfies $h_* = \mathcal{P}h_*$ almost everywhere. By Lemma 9, we can alternatively find a continuous hazard rate h_* such that $h_* = \mathcal{P}h_*$ holds on the whole domain. The corresponding density $f(\pi) = (n-1)^{-1}h_*(\pi)F(\pi)$ is also continuous.

It remains to show that the unique maximal fixed point h_* constitutes an equilibrium. We have already observed that $\pi > \pi^*$ is suboptimal in the proof of Lemma 2. We have also seen that $\pi < \underline{\pi}$ is suboptimal in the proof of Lemma 3 (due to (37)). Thus, it suffices to show $V_0(\pi) - C(\pi)$ is constant on $[\underline{\pi}, \pi^*]$. First observe that $V_0'(\pi) - C'(\pi)$ is constant by the definition of $\mathcal{P}^n$. Second, $V_0'(\pi^*) - C'(\pi^*) = u'(\pi_0 + \pi^*) - c\pi^* = 0$. The two observations imply $V_0(\pi) - C(\pi)$ is constant on $[\underline{\pi}, \pi^*]$.

C.3 Proof of Theorem 4

For each $n \in \{2, 3, \ldots\}$, let F^n be the equilibrium mixed-strategy distribution in the game with n agents. Let $F_*^n(\pi) = \{F^n(\pi)\}^{n-1}$ be the distribution for the first order statistic and h_*^n the corresponding hazard rate. We show that h_*^n (weakly-*) converges to the unique fixed point of the limit operator $\mathcal{P}^\infty$.

We always regard F_*^n as an element of the space of probability distributions on the compact set $[0, \pi^*]$. We endow this space with the topology of weak convergence. With this topology, the space is compact and metrizable. In particular, any subsequence of $\{F_*^n\}$ has a weakly convergent subsequence.

When $\{F_*^{n(m)}\}$ is weakly convergent, the corresponding subsequence $\{h_*^{n(m)}\}$ of hazard rates is also convergent (with proper restrictions on the domains), as we see in the next lemma.

Lemma 10. *Suppose a subsequence $\{F_*^{n(m)}\}$ weakly converges to another probability distribution F_*^∞ on $[0, \pi^*]$. Let $\underline{\pi}^\infty$ be the smallest element in the support of F_*^∞.*

(i) *The distribution F_*^∞ is absolutely continuous on $[0, \pi^*]$.*[53]

(ii) *Let $\hat{\pi} \in (\underline{\pi}^\infty, \pi^*)$. There exists $L > 0$ such that $h_*^{n(m)}$ is integrable on $[\hat{\pi}, \pi^*]$ for all $m \geq L$. The sequence $\{h_*^{n(m)}\}_{m=L}^\infty$ weakly-* converges to $h_*^\infty(\pi) = f_*^\infty(\pi)/F_*^\infty(\pi)$ in L^1 on $[\hat{\pi}, \pi^*]$.*[54]

[53] The distribution F_*^∞ may have an atom at $\pi = 0$. See footnote 52.

[54] Here, an element f in the L^1-space is canonically identified with the linear operator $f^*(g) =$

Proof. We first show (i). Take arbitrary $\breve{\pi} \in (0, \pi^*)$ such that $F_*^\infty(\breve{\pi}) > 0$. Let $p = F_*^\infty(\breve{\pi})/2$ and $\varepsilon = \min\{\breve{\pi}, -(\log p)^{-1}\}$. Take $M_\varepsilon \geq \varepsilon^{-1}$ and $\Delta_\varepsilon > 0$ as in Lemma 7. Let $\Delta = \min\{\Delta_\varepsilon, M_\varepsilon^{-1}\}$ and $\hat{\pi} = \min\{\pi^*, \breve{\pi} + \Delta/2\}$.

We verify $H^{n(m)}(\hat{\pi}) \leq \varepsilon^{-1}$ for sufficiently large m. Since $F_*^{n(m)}$ weakly converges to F_*^∞, there exists L such that $m \geq L$ implies that $F_*^{n(m)}(\hat{\pi}) \geq F_*^\infty(\breve{\pi})/2 = p$. When $m \geq L$, we obtain $H_*^{n(m)}(\hat{\pi}) = -\log F_*^{n(m)}(\hat{\pi}) \leq -\log p \leq \varepsilon^{-1}$.

By Lemma 7 and the contraction mapping theorem, the unique fixed point $h_*^{n(m)}$ of $\mathcal{P}^n$ is defined at least on $[\breve{\pi} - \Delta/2, \pi^*]$ and thus $f_*^{n(m)} \leq h_*^{n(m)}(\pi) \leq M_\varepsilon$ on that domain. As a result,

$$|F_*^\infty(x) - F_*^\infty(y)| \leq \limsup_{m\to\infty} |F_*^{n(m)}(x) - F_*^{n(m)}(y)| \leq M|x - y|$$

for all $x, y \in [\breve{\pi} - \Delta/2, \pi^*]$. Therefore, the limit distribution F_*^∞ is (Lipschitz and thus) absolutely continuous on $[\breve{\pi} - \Delta/2, \pi^*]$.

We see that the above result implies the absolute continuity on $[0, \pi^*]$. First observe that F_*^∞ is absolutely continuous on $[\underline{\pi}^\infty, \pi^*]$ because $\breve{\pi}$ can be arbitrarily close to $\underline{\pi}^\infty$. The proof is completed if $\underline{\pi}^\infty = 0$ or $F(\underline{\pi}^\infty) = 0$. When $\underline{\pi}^\infty > 0$ and $F(\underline{\pi}^\infty) > 0$, we set $\breve{\pi} = \underline{\pi}^\infty$ and find a density on the whole domain. This completes the proof of (i).

We turn to (ii). For all bounded measurable function g on $[\hat{\pi}, \pi^*]$,

$$\begin{aligned}
&\left|\int_{\hat{\pi}}^{\pi^*} g(\pi)(h_*^{n(m)}(\pi) - h_*^\infty(\pi))\,\mathrm{d}\pi\right| \\
&= \left|\int_{\hat{\pi}}^{\pi^*} \left(\frac{g(\pi)}{F_*^{n(m)}(\pi)} f_*^{n(m)}(\pi) - \frac{g(\pi)}{F_*^\infty(\pi)} f_*^\infty(\pi)\right)\mathrm{d}\pi\right| \\
&\leq \left|\int_{\hat{\pi}}^{\pi^*} \left(\frac{g(\pi)}{F_*^{n(m)}(\pi)} - \frac{g(\pi)}{F_*^\infty(\pi)}\right) f_*^\infty(\pi)\,\mathrm{d}\pi\right| \\
&\quad + \left|\int_{\hat{\pi}}^{\pi^*} \frac{g(\pi)}{F_*^\infty(\pi)} \left(f_*^{n(m)}(\pi) - f_*^\infty(\pi)\right)\mathrm{d}\pi\right|.
\end{aligned} \tag{46}$$

Observe that $F_*^{n(m)}$ converges pointwise to F_*^∞ on $[\hat{\pi}, \pi^*]$ because F_*^∞ is (absolutely) continuous on $[0, \pi^*]$. Thus, the first integral of the upper bound (46) converges to 0. The second integral also converges to 0 by the definition of weak convergence.

Lastly, we verify the integrability of h_*^∞ and $h_*^{n(m)}$ for large m. First,

$$\int_{\hat{\pi}}^{\pi^*} h^\infty(\pi)\,\mathrm{d}\pi = \int_{\hat{\pi}}^{\pi^*} \frac{f_*^\infty(\pi)}{F_*^\infty(\pi)}\,\mathrm{d}\pi \leq \frac{1 - F_*^\infty(\hat{\pi})}{F_*^\infty(\hat{\pi})} < \infty.$$

Second, the integrability of $h_*^{n(m)}$ follows from (46) with $g = 1$. □

We prove three lemmas to guarantee that the series $\sum_{k=0}^{n(m)-1} N_k^{n(m)}(\pi; h^{n(m)})$ and the series $\sum_{k=0}^{n(m)-1} D_k^{n(m)}(\pi; h^{n(m)})$ properly converge to their limit counterparts. Lemmas 12 and 13 are directly related to the convergence, whereas we use Lemma 11 to prove Lemma 12. Let $\mathcal{L}_{M,\hat{\pi}}$ be the set of Lebesgue-measurable functions $g : (\hat{\pi}, \pi^*] \to [0, M]$. Note that,

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$\int_{\hat{\pi}}^{\pi^*} g(\pi)f(\pi)\,\mathrm{d}\pi$ on the L^∞-space.

when M is large enough, we have $h_*^{n(m)} \le M$ and thus $h_*^{n(m)} \in \mathcal{L}_{M,\hat{\pi}}$ for sufficiently large m (due to (43)).

Lemma 11. *Suppose that a sequence $\{g_k\}_{k=0}^\infty$ in $\mathcal{L}_{M,\hat{\pi}}$ weakly-* converges to $g \in \mathcal{L}_{M,\hat{\pi}}$ in L^1. Then, $\sup_{\pi' \in (\hat{\pi},\pi^*]} |\int_{\pi'}^{\pi^*}(g_k(\pi) - g(\pi))\,\mathrm{d}\pi|$ converges to 0 as $k \to \infty$.*

Proof. We discretize the interval $(\hat{\pi}, \pi^*]$ into ℓ pieces as follows. Let $I_k(x,y) = |\int_x^y (g_k(\pi) - g(\pi))\,\mathrm{d}\pi|$. For all $\pi \in (\hat{\pi}, \pi^*]$,

$$I_k(\pi,\pi^*) \le I_k(\pi, \pi^* i/\ell) + I_k(\pi^* i/\ell, \pi^*) \le \frac{2M\pi^*}{\ell} + I_k(\pi^* i/\ell, \pi^*)$$

where i is the smallest integer j such that $\pi^* j/\ell \ge \pi$.

Set $K_1 = 2$ and define K_ℓ as the smallest $K \ge 2$ such that $k \ge K$ implies $I_k(\pi^* i/\ell, \pi^*) \le 2M\pi^*/\ell$ for all $i \in \{1,\dots,\ell\}$. Such K exists because g_k weakly-* converges to g. Define $\ell(k)$ as the largest ℓ such that $k \ge K_\ell$. Then,

$$I_k(\pi,\pi^*) \le \frac{2M\pi^*}{\ell(k)} + I_k(\pi^* i/\ell(k), \pi^*) \le \frac{4M\pi^*}{\ell(k)} \to 0$$

as $k \to \infty$. □

Lemma 12. *Suppose that a sequence $\{h_m\}_{m=0}^\infty$ in $\mathcal{L}_{M,\hat{\pi}}$ weakly-* converges to $h_\infty \in \mathcal{L}_{M,\hat{\pi}}$ in L^1. For each k, let $\omega_k : [0,\pi^*]^{k+1} \to [0,\bar{\omega}]$ be a Lebesgue-measurable function. Then,*

$$\lim_{m\to\infty} \sum_{k=0}^\infty I_k^\infty(\hat{\pi}; \omega_k, h_m) = \sum_{k=0}^\infty I_k^\infty(\hat{\pi}; \omega, h_\infty).$$

Proof. By Lemma 5, the difference $|I_k^\infty(\hat{\pi};\omega_k,h_m) - I_k^\infty(\hat{\pi};\omega_k,h_m)|$ is bounded above by

$$\delta_k^m = \alpha\gamma_m \cdot \left(\sum_{j=0}^{k-1} \frac{H_m(\hat{\pi})^j}{j!} \cdot \frac{H_\infty(\hat{\pi})^{k-j-1}}{(k-j-1)!} + \frac{H_m(\hat{\pi})^k}{k!} \right),$$

where $\alpha_m = \bar{\omega}(1 + \bar{\Gamma} H_\infty(\hat{\pi}))$ and $\gamma_m = \sup_{\pi' \in (\hat{\pi},\pi^*]} |\int_{\pi'}^{\pi^*}(h_m(\pi) - h_\infty(\pi))\,\mathrm{d}\pi|$. The series $\sum_{k=0}^\infty \delta_k^m$ is at most $\alpha\gamma_m e^{H_m(\hat{\pi})} \cdot (e^{H_\infty(\hat{\pi})} + e^{H_m(\hat{\pi})})$.[55] Therefore, $|\sum_{k=0}^\infty I_k^\infty(\hat{\pi};\omega_k,h_\circ^n) - \sum_{k=0}^\infty I_k^\infty(\hat{\pi};\omega_k,h_\infty)|$ is bounded above by

$$\alpha\gamma_m e^{H_m(\hat{\pi})} \cdot (e^{H_\infty(\hat{\pi})} + e^{H_m(\hat{\pi})}) \le \alpha\gamma_m e^{H_\infty(\hat{\pi})+\gamma_m} \cdot (e^{H_\infty(\hat{\pi})} + e^{H_\infty(\hat{\pi})+\gamma_m}) \to 0.$$

Here, the inequality follows from $|H_m(\hat{\pi}) - H_\infty(\hat{\pi})| \le \gamma_m$. Also, γ_m converges to 0 due to Lemma 11. □

Lemma 13. *Let $h_* \in \mathcal{L}_{M,\hat{\pi}}$. For each k, let $\omega_k : [0,\pi^*]^{k+1} \to [0,\bar{\omega}]$ be a Lebesgue-measurable function such that $\omega_k(\pi^{(1,\dots,k)},\pi) \le \gamma(\Pi_k + \pi)^{-3}$ with some constant $\gamma > 0$. Let $\beta_{n,k}$ be either*

[55] See the last paragraph of the proof of Lemma 7 for more detailed explanations on very similar calculations.

$\alpha_{n,k}$ *or* $\alpha_{n,k+1}$. *Then,*

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} I_k^n(\hat{\pi};\beta_{n,k}\Phi_{n,k+1}\omega_k,h_*)=\sum_{k=0}^{n-1} I_k^\infty(\hat{\pi};\omega_k,h_*).$$

Proof. Since $I_0^n=I_0^\infty$, we focus on the case of $k\geq 1$ in evaluating $|I_k^n-I_k^\infty|$ from above.

We first evaluate $|\omega_k-\beta_{n,k}\Phi_{n,k+1}\omega_k|$. This difference is bounded above by $\gamma((k+1)\hat{\pi})^{-3}(|1-\beta_{n,k}|+\beta_{n,k}|\Phi_{n,k+1}-1|)$. Note $\beta_{n,k}|\Phi_{n,k+1}-1|\leq e^{\frac{k}{n-1}H_*(\hat{\pi})}-1$. Also, since $\alpha_{n,1}=1$,

$$|1-\beta_{n,k}|=\sum_{i=1}^{k}(\alpha_{n,i}-\alpha_{n,i+1})\leq\sum_{i=1}^{k}\frac{\alpha_{n,i}\cdot i}{n-1}\leq\frac{2k(k+1)}{n-1}.$$

Therefore, $|\omega_k-\beta_{n,k}\Phi_{n,k+1}\omega_k|$ is at most

$$\frac{\gamma}{\hat{\pi}^3}\left\{\frac{2k(k+1)}{(k+1)^3(n-1)}+\left(e^{\frac{k}{n-1}H_*(\hat{\pi})}-1\right)\right\}\leq\frac{\gamma}{\hat{\pi}^3}\left\{\frac{2}{n-1}+\left(e^{\frac{k}{n-1}H_*(\hat{\pi})}-1\right)\right\}.$$

Let λ_n denote the right-hand side.

By Lemma 5, the difference $|I_k^n-I_k^\infty|$ is bounded above by

$$\delta_k^n=\kappa_n\cdot\sum_{j=1}^{k-1}\frac{H_*(\hat{\pi})^j}{(j-1)!}\cdot\frac{H_*(\hat{\pi})^{k-j-1}}{(k-j-1)!}+(\kappa_n+\lambda_n)\cdot\frac{H_*(\hat{\pi})^k}{(k-1)!}$$

with $\kappa_n=\bar{\omega}\bar{\Gamma}H_*(\hat{\pi})/(n-1)$. Therefore, $\sum_{k=0}^{n-1}|I_k^n-I_k^\infty|$ is bounded above by[56]

$$\sum_{k=1}^{n-1}\delta_k^n\leq\kappa_n H_*(\hat{\pi})e^{2H_*(\hat{\pi})}+(\kappa_n+\lambda_n)H_*(\hat{\pi})e^{H_*(\hat{\pi})}.$$

This bound converges to 0 as $n\to\infty$. □

Proof of Theorem 4. We keep the supposition that $\{F_*^{n(m)}\}$ weakly converges to F_*^∞. Let $\underline{\pi}^\infty$ denote the minimum of the support of F_*^∞. By Lemma 10, there exists a Lebesgue-measurable function $f_*^\infty:[0,\pi^*]\to[0,\infty)$ such that $F^\infty(\pi)=F^\infty(0)+\int_0^\pi f_*^\infty(\pi')\,\mathrm{d}\pi'$ for all $\pi\in[0,\pi^*]$. Thus, the hazard rate $h_*^\infty=f_*^\infty/F_*^\infty$ is defined on $(\underline{\pi}^\infty,\pi^*]$.

We show that $\mathcal{P}^{n(m)}h_*^{n(m)}$ converges pointwise to $\mathcal{P}^\infty h_*^\infty$. We first observe $\sum_{k=0}^{n(m)-1}N_k^{n(m)}(\pi;h_*^{n(m)})$ converges pointwise to $\sum_{k=0}^{\infty}N_k^\infty(\pi;h_*^\infty)$ on $(\underline{\pi}^\infty,\pi^*]$. The absolute value of their difference

[56] We have already seen similar calculations in Lemmas 7 and 12. See the last paragraph of the proof of Lemma 7.

is bounded above by

$$\sum_{k=0}^{n(m)-1} \left| N_k^{\infty}(\pi; h_*^{n(m)}) - N_k^{\infty}(\pi; h_*^{\infty}) \right|$$
$$+ \sum_{k=0}^{n(m)-1} \left| N_k^{n(m)}(\pi; h_*^{n(m)}) - N_k^{\infty}(\pi; h_*^{n(m)}) \right| + \sum_{k=n}^{\infty} \left| N_{\infty,k}(\pi; h_*^{n(m)}) \right|.$$

The first summation converges to 0 by Lemma 12. The second also converges to 0 by Lemma 13 because $-u''(\Pi_k + \pi) = (\Pi_k + \pi)^{-3}$ satisfies the assumption $\omega_k(\pi^{(1,\ldots,k)}, \pi) \leq \gamma(\Pi_k + \pi)^{-3}$. The convergence of the third to 0 follows from (42). Similarly, $\sum_{k=0}^{n(m)-1} D_k^{n(m)}(\pi; h_*^{n(m)})$ converges pointwise to $\sum_{k=0}^{\infty} D_k^{\infty}(\pi; h_*^{\infty})$ because $D(\pi, \Pi) \leq d^{**}(\Pi + \pi)^{-3}$ by Lemma 4. Also, $F_*^{n(m)}(\pi)$ converges pointwise to $F_*^{\infty}(\pi)$ by the definition of weak convergence. These convergence results imply that $\mathcal{P}^{n(m)} h_*^{n(m)}$ converges pointwise to $\mathcal{P}^{\infty} h_*^{\infty}$.

By the above convergence result, we obtain

$$\int_{\hat{\pi}}^{\pi^*} \left(\mathcal{P}^{\infty} h_*^{\infty}(\pi) - h_*^{\infty}(\pi) \right) \mathrm{d}\pi = \lim_{m\to\infty} \int_{\hat{\pi}}^{\pi^*} \mathcal{P}^{\infty} h_*^{n(m)}(\pi)\, \mathrm{d}\pi - \lim_{m\to\infty} \int_{\hat{\pi}}^{\pi^*} h_*^{n(m)}(\pi)\, \mathrm{d}\pi$$
$$= \lim_{m\to\infty} \int_{\hat{\pi}}^{\pi^*} \left(\mathcal{P}^{n(m)} h_*^{n(m)}(\pi) - h_*^{n(m)}(\pi) \right) \mathrm{d}\pi = 0$$

for all $\hat{\pi} \in (\underline{\pi}^{\infty}, \pi^*]$. By differentiate the above expression with respect to $\hat{\pi}$, we obtain $h_*^{\infty} = \mathcal{P}^{\infty} h_*^{\infty}$ almost everywhere on $(\underline{\pi}^{\infty}, \pi^*]$.

We show that $(\underline{\pi}^{\infty}, \pi^*]$ is the maximal domain; i.e., $\underline{\pi}^{\infty}$ is not an atom. Note that $\pi = 0$ is the only candidate of an atom (Lemma 10). Suppose $\underline{\pi}^{\infty} = 0$. By Lemma 4,

$$h_*^{\infty}(\pi) = \mathcal{P}^{\infty} h_*^{\infty}(\pi) = \lim_{m\to\infty} \mathcal{P}^{n(m)} h_*^{n(m)}(\pi) \geq \lim_{m\to\infty} \frac{c/F_*(\pi)}{d^*\pi/F_*(\pi)} = \frac{c}{d^*\pi}$$

almost everywhere on $(0, \pi^*]$. Therefore, $F_*^{\infty}(0) = e^{-H_*^{\infty}(0)} = 0$. That is, F_*^{∞} has no atom and $(\underline{\pi}^{\infty}, \pi^*]$ is the maximal domain of the fixed-point problem.

By Lemma 9, the limit distribution F_*^{∞} has a continuous density f_{**}^{∞} and the associated hazard rate $h_{**}^{\infty} = f_{**}^{\infty}/F_*^{\infty}$ is the unique maximal fixed point of $\mathcal{P}^{\infty}$. Thus, any convergent subsequence $\{F_{\infty}^{n(m)}\}$ needs to converge to this unique limit distribution F_*^{∞}. This further implies that the entire sequence $\{F_*^n(\pi)\}$ converges to this limit $F_*^{\infty}(\pi)$ because the space of probability distributions is compact and metrizable. □

C.4 Proof of Proposition 1

The first two statements are already shown in the text. We show (iii). Suppose that agent 1 chose the lowest precision $\underline{\pi}$ in the support of the equilibrium mixed-strategy distribution. Consider the timing game where the other $n-1$ agents follow the equilibrium mixed strategy. Let $\tilde{T}$ be the absolute time when all the other agents finish moving and $\tilde{\Pi}$ be the sum $\pi_0 + \sum_{i=2}^{n} \pi_i$. We treat these variables as random variables by regarding realizations of

$(\pi_2, \ldots, \pi_n)$ as samples.

Since $\underline{\pi}$ is an optimal choice, we obtain the following indifference condition and it first-order counterpart:[57]

$$\mathbb{E}\Big[e^{-r\tilde{T}} u\Big(\tilde{\Pi} + \underline{\pi}\Big)\Big] - C(\underline{\pi}) = u(\pi_0 + \pi^*) - C(\pi^*) \tag{47}$$

$$\mathbb{E}\Big[e^{-r\tilde{T}} u'\Big(\tilde{\Pi} + \underline{\pi}\Big)\Big] - c\underline{\pi} = 0. \tag{48}$$

The second equation (48) implies

$$\underline{\pi} \leq c^{-1}\, \mathbb{E}\big[e^{-r\tilde{T}}\big] u'(n\underline{\pi})$$

because $\tilde{\Pi} \geq (n-1)\underline{\pi}$. Since $u'(n\underline{\pi}) = n^{-2}\underline{\pi}^{-2}$, we have $n^2\underline{\pi}^3 \leq c^{-1}\, \mathbb{E}[e^{-r\tilde{T}}]$. Equation (47) provides an upper bound of the expected value of $e^{-r\tilde{T}}$:

$$\mathbb{E}\big[e^{-r\tilde{T}}\big] \leq \frac{u(\pi_0 + \pi^*) - \{C(\pi^*) - C(\underline{\pi})\}}{u(\pi_0)} \leq \frac{u(\pi_0 + \pi^*)}{u(\pi_0)}$$

By combining these inequalities, we obtain

$$\underline{\pi} \leq n^{-2/3} \left(\frac{u(\pi_0 + \pi^*)}{cu(\pi_0)} \right)^{1/3}.$$

This upper bound converges to 0 as n goes to infinity.

C.5 Proof of Proposition 2

Let $W(n)$ be the average surplus under the social planner with n agents. When $n = 1$, the social planner's problem coincides with the individual optimization problem that defines π^* (equation (14)). Hence, $W(1) = u(\pi_0 + \pi^*) - C(\pi^*)$, which is equal to the equilibrium payoff *for all* n due to Theorem 3.

Now it suffices to show that $W(n) \leq W(n+1)$ for all $n \in \{1, 2, \ldots\}$. Let $\hat{\pi}^{(1,\ldots,n)}$ be the solution of the social planner's problem for n agents. From this solution, define $\tilde{\pi}^{(1,\ldots,n+1)} = (\pi^{(1,\ldots,n)}, 0)$ for the problem with $n+1$ agents. Although $\tilde{\pi}^{(1,\ldots,n+1)}$ is not necessarily a solution for the planner's problem, we obtain that the average payoff of $n+1$ agents with $\tilde{\pi}^{(1,\ldots,n+1)}$ is greater than that of n agents with $\hat{\pi}^{(1,\ldots,n)}$ (i.e., $W(n)$) for the following reason: the first n agents obtain exactly the same payoffs in the two scenarios, but the last agent in the former receives the payoff of $u(\pi_0 + \hat{\pi}^{(1)} + \cdots + \hat{\pi}^{(n)})$, which is greater than, or equal to, the k-th agent's payoff $u(\pi_0 + \hat{\pi}^{(1)} + \cdots + \hat{\pi}^{(k)}) - C(\pi^{(k)}) \leq u(\pi_0 + \hat{\pi}^{(1)} + \cdots + \hat{\pi}^{(k)})$ for any $k \in \{1, \ldots, n\}$. Since this average surplus cannot exceed the maximized average surplus $W(n+1)$, we obtain the desired inequality: $W(n) \leq W(n+1)$.

[57] We obtain the differentiability and first-order derivative at $\pi = \underline{\pi}$ by the same argument as in Lemma 3.

C.6 Proof of Proposition 3

It suffices to show that, whenever a reversal $\pi^{(k)} < \pi^{(k+1)}$ occurs, the planner can increase the total surplus by switching the order of the k-th and $(k+1)$-st movers. While this switch does not affect the payoff of any other agent, the sum of the payoffs of these two agents changes from

$$u(\Pi_{k-1} + \pi^{(k)}) + u(\Pi_{k+1}) - C(\pi^{(k)}) - C(\pi^{(k+1)})$$

to

$$u(\Pi_{k-1} + \pi^{(k+1)}) + u(\Pi_{k+1}) - C(\pi^{(k)}) - C(\pi^{(k+1)}),$$

where $\Pi_\ell = \pi_0 + \sum_{i \leq \ell} \pi^{(i)}$. The latter payoff is higher by the assumption $\pi^{(k)} < \pi^{(k+1)}$. Therefore, $\pi^{(1)} \geq \cdots \geq \pi^{(n)}$ must hold at the optimum.

C.7 Proof of Proposition 4

The intensity of the counting process k_t is defined by

$$\lambda_t^* = \lim_{\varepsilon \searrow 0} \frac{\mathbb{E}_t[k_{t+\varepsilon}] - k_t}{\varepsilon},$$

where $\mathbb{E}_t$ is the expectation operator conditional on the information available at absolute time t. The probability that k_t increases by 1 within time interval $[t, t+\varepsilon]$ is given by $\varepsilon \cdot \alpha(\varepsilon)$, where

$$\alpha(\varepsilon) = (n - k_t) \cdot \frac{f(\pi_t^c) + o(\varepsilon)}{F(\pi_t^c)} \cdot \left(\frac{\mathrm{d}t_{k_t+1}}{\mathrm{d}\pi} \right)^{-1} = \lambda_t^* + o(\varepsilon)$$

and the terms represented by $o(\varepsilon)$, the little o, disappear as $\varepsilon \to 0$. The probability that k_t increases by more than 1 shrinks to 0 faster than ε. Therefore, the intensity is

$$\lim_{\varepsilon \searrow 0} \frac{\mathbb{E}_t[k_{t+\varepsilon}] - k_t}{\varepsilon} = \lim_{\varepsilon \searrow 0} \left[\frac{(k_t + 1 \cdot \varepsilon \lambda_t^*) - k_t}{\varepsilon} + o(\varepsilon) \right] = \lambda_t^*.$$

The properties of λ_t^* follow either from Proposition 8 or straightforwardly from the definition. The process $\hat{\pi}_t$ is non-decreasing because so is the public precision Π_t.

C.8 Proof of Corollary 1

The first part follows from $\hat{\pi}_t^c \leq \overline{\pi} \leq \hat{\pi}(\pi_0) \leq \hat{\pi}(\Pi_t)$. To show the second part, observe that the individually optimal precision π^* does not exceed $\hat{\pi}(\pi_0)$ if and only if $u'(\pi_0 + \hat{\pi}(\pi_0)) \geq c \cdot \hat{\pi}(\pi_0)$, or equivalently, $1 \geq c \cdot \hat{\pi}(\pi_0) \cdot (\pi_0 + \hat{\pi}(\pi_0))^2$.

C.9 Proof of Proposition 5

Consider history with k actions. We show $t_{k+1}(\underline{\pi}; \pi^{(1,\ldots,k)}) = \infty$ when $\pi^{(k+1)} = \cdots = \pi^{(n)} = \underline{\pi}$. Let $\Gamma_* = \inf_{\pi \in [\underline{\pi}, \pi^{(k)}]} \Gamma(\pi, \Pi_k)$. The minimum Γ_* is positive because $\Gamma(\pi, \Pi_k)$ is positive on the compact set $[\underline{\pi}, \pi^{(k)}]$ bounded away from 0. Therefore,

$$t_{k+1}(\underline{\pi}; \pi^{(1,\ldots,k)}) \geq \frac{\Gamma_*}{r} \int_{\underline{\pi}}^{\pi^{(k)}} h(\pi)\, \mathrm{d}\pi = \frac{\Gamma_*}{r} \Big(\log F(\pi^{(k)}) - \log F(\underline{\pi}) \Big) = \infty.$$

C.10 Proof of Propositions 6 and 7

Both propositions follow from the following observation. Let $\ell \in \{1, \ldots, k\}$ and suppose $\pi^{(\ell)}$ increases. When $\ell < k$, the increase reduces the absolute action time T_m for all $m > k$ and for each realization of $\pi^{(k+1)}, \ldots, \pi^{(n)}$ (due to Proposition 12, which does not rely on this proof) but does not change the conditional distribution $F(\pi)/F(\pi^{(k)})$. When $\ell = k$, the distribution $F(\pi)/F(\pi^{(k)})$ is improved in first-order stochastic dominance so that the absolute action time also improves in first-order stochastic dominance (once again, by applying Proposition 12).

C.11 Proof of Proposition 8

The expressions (26) follows from multiplying both denominator and numerator of

$$\Gamma(\pi, \Pi) = \frac{(\Pi + \pi)^{-1} - (\Pi + 2\pi)^{-1}}{u_0 - (\Pi + \pi)^{-1}}$$

by $(\Pi + \pi)(\Pi + 2\pi)$. The monotonicity of Γ in Π is obvious from the second expression of (26). The single-peakedness follows from

$$\frac{\mathrm{d}}{\mathrm{d}\pi} \left(2u_0\pi + (3u_0\Pi - 2) + \frac{\Pi^2 u(\Pi)}{\pi} \right) = 2u_0 - \frac{\Pi^2 u(\Pi)}{\pi^2}.$$

C.12 Proof of Corollary 2

The marginal waiting time is decreasing in Π_k by Proposition 8. The inequality follows from

$$\begin{aligned} &T_k(\pi^{(k+1)} | \pi^{(1,\ldots,k-1)}) - T_{k+1}(\pi^{(k+1)} | \pi^{(1,\ldots,k)}) \\ &= \int_{\pi^{(k+1)}}^{\pi^{(k)}} \big\{ (n-k-1)\Gamma(\pi, \Pi_{k-1}) - (n-k-2)\Gamma(\pi, \Pi_k) \big\} \frac{h(\pi)}{r}\, \mathrm{d}\pi \geq 0 \end{aligned}$$

The inequality is strict when the interval $[\pi^{(k+1)}, \pi^{(k)}]$ has a positive measure.

C.13 Proof of Corollary 3

The action time $T_k(\pi^{(k)} | \pi^{(1,\ldots,k-1)}) = r^{-1} \sum_{m=0}^{k-1} \int_{\pi^{(m+1)}}^{\pi^{(m)}} (n-m-1)\Gamma(\pi, \Pi_m) h(\pi)\, \mathrm{d}\pi$ is decreasing in $\pi^{(\ell)}$ ($\ell \in \{0, \ldots, k-1\}$) because so is $\Gamma(\pi, \Pi_{m-1})$ for $m \geq \ell$. In the case of $\ell = k$, an increase in $\pi^{(k)}$ shortens the interval for the last integral without changing anything else.

C.14 Proof of Proposition 9

Proposition 8 implies $\pi_t^c/\Gamma(\pi_t^c, \Pi_t) = (\Pi_t + 2\pi_t^c)(u_0\Pi_t + u_0\pi_t^c - 1)$. By combining the above expression and (21), we obtain the desired expression.

C.15 Proof of Proposition 10

First suppose $\pi^\alpha h(\pi)$ converges to a non-negative value with some $\alpha < 2$. By Proposition 8, we know $\Gamma(\pi, \Pi_k)/\pi$ converges to a positive constant $[\Pi_k(u_0\Pi_k - 1)]^{-1}$. Thus, $X_\alpha(\pi) = \pi^{-1}\Gamma(\pi, \Pi_k) \cdot \pi^\alpha h(\pi)$ is a non-negative, continuous function on $[0, \overline{\pi}]$. Hence,

$$\int_0^{\pi^{(k)}} \Gamma(\pi, \Pi_k)h(\pi)\,\mathrm{d}\pi = \int_0^{\pi^{(k)}} X_\alpha(\pi)\pi^{1-\alpha}\,\mathrm{d}\pi \leq \left(\max_{\pi\in[0,\overline{\pi}]} X_\alpha(\pi)\right)\int_0^{\varepsilon} \pi^{1-\alpha}\,\mathrm{d}\pi < \infty.$$

If $\pi^2 h(\pi)$ converges to a finite value or diverges to infinity, then $X_2(\pi)$ is a continuous function from $[0, \overline{\pi}]$ to $(0, \infty]$. Thus,

$$\int_0^{\pi^{(k)}} \Gamma(\pi, \Pi_k)h(\pi)\,\mathrm{d}\pi = \int_0^{\pi^{(k)}} X_2(\pi)\pi^{-1}\,\mathrm{d}\pi \geq \left(\min_{\pi\in[0,\overline{\pi}]} X_2(\pi)\right)\int_0^{\pi^{(k)}} \pi^{-1}\,\mathrm{d}\pi = \infty.$$

C.16 Proof of Proposition 11

Let $G_n(\tau)$ denote the probability of the event $t_1(\pi^{(1)}) > \tau$ in the game with n agents. Let $\psi(\pi) = \int_\pi^{\overline{\pi}} \Gamma(x, \pi_0)h(x)\,\mathrm{d}x$. Since $t_1(\pi^{(1)}) > \tau$ is equivalent to $\psi(\pi^{(1)}) > r\tau(n-1)^{-1}$, we obtain $G_n(\tau) = \{F(\psi^{-1}(\frac{r\tau}{n-1}))\}^n$ for sufficiently large n.[58] By L'Hôpital's rule, $\log G_n(\tau) = \{\log F(\psi^{-1})\}/n^{-1}$ converges pointwise to

$$\lim_{n\to\infty} \frac{r\tau h(\psi^{-1})}{\psi'}\left(\frac{n}{n-1}\right)^2 = \lim_{n\to\infty} \frac{r\tau h(\psi^{-1})}{-h(\psi^{-1})\Gamma(\psi^{-1}, \pi_0)} \cdot \left(\frac{n}{n-1}\right)^2 = -\lambda\tau$$

with $\lambda = r/\Gamma(\overline{\pi}, \pi_0)$. Therefore, $1 - G_n(\tau)$, the probability of $t_1(\pi^{(1)}) \leq \tau$, converges in distribution to the exponential distribution $1 - e^{-\lambda\tau}$.

C.17 Proof of Proposition 12

By Corollary 3, the absolute action time $T_k(\pi^{(k)}; \pi^{(1,\dots,k-1)})$ is non-increasing in $\pi^{(\ell)}$ for all k and ℓ. Therefore, the number of actions at a fixed absolute time never decreases when $\pi^{(\ell)}$ increases.

[58]The value of $r\tau(n-1)^{-1}$ may exceed the domain of ψ^{-1}, but the value shrinks and stays within the domain as n increases with fixed τ.

C.18 Proof of Proposition 13

We know the slowest case is $\pi^{(1)} = \cdots = \pi^{(n)} = \underline{\pi}$ due to Corollary 3. The comparative statics with respect to π_0 is due to Proposition 8.

C.19 Proof of Proposition 14

By differentiating $\Psi(\Delta) = 1/\Gamma(\pi + \Delta, \Pi + k\Delta)$, we obtain

$$\Psi'(0) = \frac{2u_0\pi^2 + \Pi - u_0\Pi^2}{\pi^2} + k \cdot \frac{2u_0\Pi + 3u_0\pi - 1}{\pi} = \frac{\Omega(k, \pi, \Pi)}{\pi^2}.$$

The function Ω is increasing in k because $u_0\Pi \geq u_0\pi_0 > 1$. Define $\pi_\Omega(k, \Pi)$ as the root of $\Omega(k, \pi, \Pi) = 0$. Such π exists in $(0, \infty)$ because $\Omega(k, 0, \Pi) = \Pi(1 - u_0\Pi) < 0$ and $\Omega \to \infty$ as $\pi \to \infty$.

Arrival of new information increases k by 1 and Π by π. The value of Ω increases after this operation because

$$\Omega(k + 1, \pi, \Pi + \pi) - \Omega(k, \pi, \Pi) = 2(k + 2)u_0\pi^2$$

This in turn increases $\partial\lambda_k^*/\partial\Delta$ because $(n - k)/(n - k - 1)$ is increasing in k.

C.20 Proof of Proposition 15

By differentiating $\Phi_k(\pi) = 1/\Gamma(\pi, \pi_0 + k\pi)$, we obtain

$$\Phi_k'(\pi) = \frac{1}{\pi^2}\left((k^2 + 3k + 2)u_0\pi^2 - \left(u_0\pi_0^2 - \pi_0\right)\right) = (k^2 + 3k + 2)u_0 - \left(\frac{\pi_0}{\pi}\right)^2 u(\pi_0).$$

Since this value is increasing in k, $\Phi_k'(\pi) > 0$ for all k if and only if $\Phi_0'(\pi) > 0 \Leftrightarrow 2u_0 \cdot \pi^2 > \pi_0^2 \cdot u(\pi_0)$. Similarly, $\Phi_k'(\pi) < 0$ for all $k \leq n - 2$ if and only if $\Phi_{n-2}'(\pi) < 0 \Leftrightarrow n(n-1)u_0 \cdot \pi^2 < \pi_0^2 \cdot u_0(\pi_0)$.

C.21 Proof of Corollary 4

By Proposition 14, the value of $\Omega(k, \pi, \Pi_k)$ is non-negative for all $k \in \{0, \ldots, n-2\}$, all $\pi \in [\overline{\pi}, \underline{\pi}]$, all possible realizations of $\pi^{(1,\ldots,k)}$ if and only if $\Omega(0, \underline{\pi}, \pi_0) \equiv 2u_0\underline{\pi}^2 + \pi_0 - u_0\pi_0^2 \geq 0$ $\Leftrightarrow \underline{\pi} \geq \pi_0\sqrt{u(\pi_0)/(2u_0)}$. Similarly, the value of $\Omega(k, \pi, \Pi_k)$ is always non-positive with the identical quantifiers if and only if

$$\Omega(n-2, \overline{\pi}, \pi_0 + (n-2)\overline{\pi}) \equiv n(n-1)u_0\overline{\pi}^2 - \pi_0^2 u(\pi_0) \leq 0 \quad \Leftrightarrow \quad \overline{\pi} \leq \pi_0\sqrt{\frac{u(\pi_0)}{n(n-1)u_0}}.$$

C.22 Proof of Lemma 1

First note $a^{(k)} - \mu = \Pi_k^{-1} \sum_{i=1}^{k} \pi^{(i)}(s^{(i)} - \mu)$. Thus the mean of $a^{(k)}$ is μ and the covariance in question is given by

$$\frac{1}{\Pi_k \Pi_\ell} \left(\sum_{i=1}^{k} \pi^{(i)} \sum_{j=1}^{\ell} \pi^{(j)} \operatorname{Var}(v) + \sum_{j=1}^{\ell} (\pi^{(i)})^2 \operatorname{Var}(\varepsilon_i) \right)$$
$$= \frac{1}{\Pi_k \Pi_\ell} \left((\Pi_k - \pi_0)(\Pi_\ell - \pi_0) \frac{1}{\pi_0} + (\Pi_k - \pi_0) \right) = \frac{\Pi_k - \pi_0}{\pi_0 \Pi_k} = \frac{1}{\pi_0} - \frac{1}{\Pi_k}.$$

C.23 Proof of Proposition 16

To simplify notations, let $X = \Pi_k - \pi_0$ and $Y = \Pi_\ell - \Pi_k$. The squared correlation is given by

$$Q = \left(\frac{1}{\pi_0} - \frac{1}{\pi_0 + X} \right) \left(\frac{1}{\pi_0} - \frac{1}{\pi_0 + X + Y} \right)^{-1} = \frac{\pi_0 + X + Y}{\pi_0 + X} \cdot \frac{X}{X + Y} = \frac{\Pi_\ell}{\Pi_k} \cdot \frac{\Pi_k - \pi_0}{\Pi_\ell - \pi_0}.$$

By differentiating the above expression, we obtain

$$\frac{\partial Q}{\partial \pi_0} = -\frac{XY}{(\pi_0 + X)^2 (X + Y)} < 0$$
$$\frac{\partial Q}{\partial X} = \frac{\pi_0 Y (\pi_0 + 2X + Y)}{(\pi_0 + X)^2 (X + Y)^2} > 0$$
$$\frac{\partial Q}{\partial Y} = -\frac{\pi_0 X}{(\pi_0 + X)(X + Y)^2} < 0.$$

Therefore, the squared correlation Q and the correlation itself are increasing in $\pi^{(1)}, \ldots, \pi^{(k)}$ and decreasing in $\pi^{(k+1)}, \ldots, \pi^{(\ell)}$, and π_0.

C.24 Proof of Proposition 17

By Lemma 1, the expected boldness is

$$\mathbb{E}\left[(a^{(k)} - a^{(k-1)})^2\right] = \operatorname{Var}(a^{(k)}) + \operatorname{Var}(a^{(k-1)}) - 2 \operatorname{Cov}(a^{(k)}, a^{(k-1)}) = \frac{1}{\Pi_{k-1}} - \frac{1}{\Pi_k}.$$

Here, all operations that involve expectation are implicitly conditional on $\pi^{(1,\ldots,n)}$.

C.25 Proof of Proposition 18

It is clear whether $\Pi_{\ell-1} - \Pi_k$ and $\pi^{(\ell)}$ have positive or negative effects on $B_k - B_\ell$ and $(B_k - B_\ell)/B_k$ because B_k does not depend on these two variables. We thus focus on partial derivatives with respect to Π_{k-1} and $\pi^{(k)}$. By differentiating $B_k - B_\ell$, we obtain

$$\frac{\partial(B_k - B_\ell)}{\partial \Pi_{k-1}} = -\Pi_{k-1}^{-2} + \Pi_k^{-2} + \Pi_{\ell-1}^{-2} - \Pi_\ell^{-2}$$
$$\frac{\partial(B_k - B_\ell)}{\partial \pi^{(k)}} = \Pi_k^{-2} + \Pi_{\ell-1}^{-2} - \Pi_\ell^{-2} > 0$$

Note that $\partial(B_k - B_\ell)/\partial \Pi_{k-1}$ increases in $\pi^{(\ell)}$ and decreases in $\Pi_{\ell-1} - \Pi_k$. Hence

$$\begin{aligned}\frac{\partial(B_k - B_\ell)}{\partial \Pi_{k-1}} &\leq -\Pi_{k-1}^{-2} + (\Pi_{k-1} + \pi^{(k)})^{-2} + (\Pi_{k-1} + \pi^{(k)})^{-2} - (\Pi_{k-1} + 2\pi^{(k)})^{-2} \\ &= -\frac{2(\pi^{(k)})^2(3\Pi_{k-1}^2 + 6\Pi_{k-1}\pi^{(k)} + 2(\pi^{(k)})^2)}{\Pi_{k-1}^2(\Pi_{k-1} + \pi^{(k)})^2(\Pi_{k-1} + 2\pi^{(k)})^2} < 0.\end{aligned}$$

To determine the signs of $\nabla[(B_k - B_\ell)/B_k]$, we instead calculate

$$\frac{\partial \log(B_k/B_\ell)}{\partial \Pi_{k-1}} = (\Pi_{\ell-1}^{-1} - \Pi_{k-1}^{-1}) + (\Pi_\ell^{-1} - \Pi_k^{-1}) < 0$$
$$\frac{\partial \log(B_k/B_\ell)}{\partial \pi^{(k)}} = (\pi^{(k)})^{-1} - \Pi_k^{-1} + \Pi_{\ell-1}^{-1} + \Pi_\ell^{-1} > (\pi^{(k)})^{-1} - \Pi_k^{-1} > 0.$$

The counterparts of $(B_k - B_\ell)/B_k$ have the same signs. The above calculations follow from $\log(B_k/B_\ell) = \log \pi^{(k)} + \log \Pi_{\ell-1} + \log \Pi_\ell - \log \Pi_{k-1} - \log \Pi_k - \log \pi^\ell$.