Online Appendix: Financial Intermediation and Capital Reallocation

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Appendix A. Aggregation in the fully dynamic model

In this subsection, we provide additional details that Proposition 1 applies to the dynamic model, thanks to the assumption that banks' net worth moves freely across islands at the end of every period. In particular, we show that if $\frac{z_{j+1}}{K_{j,t+1}+RA_{j,t+1}}$ is equalized across all firms within the same sector, then Proposition 1 applies and aggregate output can be represented as a function of (ϕ, u). We think of all firms that have the same realization of $\varepsilon_{j,t+1}$ as being in the same sector. Under this interpretation, the economy has two sectors, ε_H and ε_L. Because $\frac{z_{j+1}}{K_{j,t+1}+RA_{j,t+1}}$ is equalized on all islands in the same sector, the individual ratios must equal to the average ratio of the sector $\frac{E[z_{j,t+1}|\varepsilon_{j,t+1}=\varepsilon]}{E[K_{j,t+1}+RA_{j,t+1}|\varepsilon_{j,t+1}=\varepsilon]}$. For $\varepsilon = \varepsilon_H$, ε_L, the integrals in Equation (28) can therefore be written as

$$
\begin{aligned}
& \int_{\varepsilon_{j,t+1}=\varepsilon} \left(\frac{z_{j+1}}{K_{j,t+1}+RA_{j,t+1}}\right)^{(1-\xi)} \left(K_{j,t+1}+RA_{j,t+1}\right) dj \\
= & \int_{\varepsilon_{j,t+1}=\varepsilon} \left(\frac{z_{j+1}}{K_{j,t+1}+RA_{j,t+1}}\right)^{(1-\xi)} \left(K_{j,t+1}+RA_{j,t+1}\right) dj, \\
= & \left(\frac{E\left[z_{j,t+1}\middle|\,\varepsilon_{j,t+1}=\varepsilon\right]}{E\left[K_{j,t+1}+RA_{j,t+1}\middle|\,\varepsilon_{j,t+1}=\varepsilon\right]}\right)^{1-\xi} \int_{\varepsilon_{j,t+1}=\varepsilon} \left(K_{j,t+1}+RA_{j,t+1}\right) dj. \quad \text{(OA.E1)}
\end{aligned}
$$

Note that $E\left[z_{j,t+1}\middle|\,\varepsilon_{j,t+1}=\varepsilon\right] = E\left[z_{j,t}e^{\varepsilon_{j,t+1}}\middle|\,\varepsilon_{j,t+1}=\varepsilon\right] = e^{\varepsilon}$, as $\varepsilon_{j,t+1}$ is independent of $z_{i,t}$ and $E\left[z_{j,t}\right]=1$. Also, if we define $\phi_{t+1} = \frac{E[K_{j,t+1}+RA_{j,t+1}|\varepsilon_{j,t+1}=\varepsilon_H]}{E[K_{j,t+1}+RA_{j,t+1}|\varepsilon_{j,t+1}=\varepsilon_L]}$ as the ratio of the average size of firms in the two sectors, then $\frac{E[K_{j,t+1}+RA_{j,t+1}|\varepsilon_{j,t+1}=\varepsilon_H]}{u_{t+1}K_{t+1}} = \frac{\phi_{t+1}}{\pi\phi_{t+1}+(1-\pi)}$ as $u_{t+1}K_{t+1}$ is the average size of all firms in the economy. In addition, because the total measure of the ε_H sector is π,

$$
\begin{aligned}
\int_{\varepsilon_{j,t+1}=\varepsilon_H} \left(K_{j,t+1}+RA_{j,t+1}\right) dj &= \pi E\left[K_{j,t+1}+RA_{j,t+1}\middle|\,\varepsilon_{j,t+1}=\varepsilon_H\right] \\
&= \pi\frac{\phi_{t+1}}{\pi\phi_{t+1}+(1-\pi)} u_{t+1}K_{t+1}. \quad \text{(OA.E2)}
\end{aligned}
$$

We can use Equations (OA.E1) and (OA.E2) and write

$$
\begin{aligned}
\int_{\varepsilon_{j,t+1}=\varepsilon_H} z_{j+1}^{(1-\xi)} \left(K_{j,t+1}+RA_{j,t+1}\right)^{\xi} dj &= \pi\left(\frac{e^{\varepsilon_H}}{\frac{\phi_{t+1}}{\pi\phi_{t+1}+(1-\pi)}u_{t+1}K_{t+1}}\right)^{1-\xi} \frac{\phi_{t+1}}{\pi\phi_{t+1}+(1-\pi)}u_{t+1}K_{t+1}, \\
&= \pi e^{(1-\xi)\varepsilon_H}\left(\frac{\phi_{t+1}}{\pi\phi_{t+1}+(1-\pi)}\right)^{\xi} \left(u_{t+1}K_{t+1}\right)^{\xi}.
\end{aligned}
$$

We can simplify $\int_{\varepsilon_{j,t+1}=\varepsilon_L} z_{j+1}^{(1-\xi)} \left(K_{j,t+1} + RA_{j,t+1}\right)^{\xi} dj$ similarly and write Equation (28) as:

$$
\begin{aligned}
Y_{t+1} &= \bar{A}_{t+1} \left\{ \pi e^{(1-\xi)\varepsilon_H} \left(\frac{\phi_{t+1}}{\pi \phi_{t+1} + (1-\pi)} \right)^{\xi} + (1-\pi) e^{(1-\xi)\varepsilon_L} \left(\frac{1}{\pi \phi_{t+1} + (1-\pi)} \right)^{\xi} \right\}^{\frac{\alpha}{\xi}} \left(u_{t+1} K_{t+1}\right)^{\alpha}, \\
&= f\left(\phi_{t+1}\right) \bar{A}_{t+1} \left(u_{t+1} K_{t+1}\right)^{\alpha},
\end{aligned}
$$

where $f(\phi)$ is defined in Proposition 1.

Appendix B. Additional details for the proof of Proposition 2

Because aggregate output can be represented as a function of (ϕ, u) in the fully dynamic economy, our proof for Proposition 2 therefore applies to the fully dynamic economy as well. The policy functions for $\phi(A, s)$ and $u(A, s)$ are determined by conditions (24), (25), (26) and (27). We first introduce some notations to simplify our analysis. In what follows, subscript H and L are interpreted as high and low productivity islands, respectively, in the two-period model, and sectors with ε_H and ε_L shocks, respectively, in the fully dynamic model.

First, we compute the difference between the left-hand side of Equation (24) and that of Equation (25) as:

$$
\begin{aligned}
& (1-\theta) Q_L - \left[Q - (1-\theta) Q_L\right] \left(\frac{u}{\pi\phi + 1 - \pi} - 1 \right) \\
& - \left\{ (1-\theta) Q_H - \left[Q - (1-\theta) Q_H\right] \left(\frac{u\phi}{\pi\phi + 1 - \pi} - 1 \right) \right\}, \\
= \; & \frac{u}{\pi\phi + 1 - \pi} \left\{ (1-\theta) Q_L - (1-\theta)\phi Q_H + (\phi - 1) Q \right\}.
\end{aligned}
$$

As we will show below, the equilibrium prices of capital, Q_H, Q_L, and Q will be functions of (A, ϕ, u). Recall that we have defined $MPK(u) = Q(u) - 1 - \delta$. Therefore, we can write Δ as

$$
\begin{aligned}
\Delta(A, \phi, u) &= (1-\theta) Q_L(A, \phi, u) - (1-\theta)\phi Q_H(A, \phi, u) + (\phi - 1) Q(u), \\
&= (1-\theta) MPK_L(A, \phi, u) - (1-\theta)\phi MPK_H(A, \phi, u) + (\phi - 1) MPK(u) \\
&\quad + \theta(\phi - 1)(1 - \delta),
\end{aligned}
$$

where in the last line above, we use the market clearing condition for capital to replace $Q(u)$ with $MPK(u) + 1 - \delta$, for both sectors, ε_H and ε_L, as well as the price on the reallocation market.

Based on the tightness of the constraints, we have three cases:

- Only the constraint on high productivity island (24) binds $\Longleftrightarrow \Delta > 0$.

- Both constraints (24) and (25) bind $\Longleftrightarrow \Delta = 0$.
- Only the constraint on low productivity island (25) binds $\Longleftrightarrow \Delta < 0$.

It is also convenient to define the left-hand side of constraint on the high productivity island as a function of (A, u, ϕ):

$$\begin{aligned}\Psi(A,\phi,u) &= (1-\theta) Q_H(A,\phi,u) - [Q(u) - (1-\theta) Q_H(A,\phi,u)] \frac{RA_H}{K}, \\ &= (1-\theta)[MPK_H(A,\phi,u) + (1-\delta)] \\ &\quad - [MPK(u) - (1-\theta) MPK_H(A,\phi,u) + \theta(1-\delta)] \left(\frac{u\phi}{\pi\phi + 1 - \pi} - 1\right).\end{aligned} \tag{OA.E1}$$

Below, we derive the functional form of $\phi(A,s)$ and $u(A,s)$ for all three cases.

First best case, no constraint binds If none of the constraints in (24) and (25) binds, then equations (26) and (27) imply

$$Q_H(A,\phi,u) = Q_L(A,\phi,u) = Q(u) = \alpha u^{\alpha-1} A + 1 - \delta. \tag{OA.E2}$$

The optimal capital utilization Equation (6) implies

$$\alpha u^{\alpha-1} A + 1 - \delta = b_0 (1-u)^{\nu-1}, \tag{OA.E3}$$

where given our choice of the functional form of the storage technology (42), $Q(u) = b_0(1-u)^{\nu-1}$. Equation (OA.E3) defines the capital utilization rate as a function of productivity, $\hat{u}(A)$. We use (OA.E2) to define $\hat{Q}$ as the price of capital in the first best case given the productivity A:

$$\hat{Q}(A) = \alpha \hat{u}(A)^{\alpha-1} A + 1 - \delta.$$

Also define $\hat{s}(A)$ to be the highest level of s such that there is no capital misallocation:

$$\hat{s}(A) = \Psi\left(A, \hat{\phi}, \hat{u}(A)\right).$$

In the above equation, $\hat{\phi}$ is defined as $\hat{\phi} = e^{\varepsilon_H - \varepsilon_L}$, which is the first best capital reallocation ratio that equalizes the marginal products of capital across all islands.

Using the above construction of prices, we can simplify the expression (OA.E1) and obtain:

$$\Psi\left(A, \hat{\phi}, \hat{u}(A)\right) = \frac{\hat{Q}(A)}{\pi\hat{\phi} + 1 - \pi} \left\{(1 - \theta\hat{u}(A))\hat{\phi} - (1-\pi)\left(\hat{\phi} - 1\right)\right\}.$$

The policy functions $\phi(A,s)$ and $u(A,s)$ are given in the following claim.

Claim 1. *If $s \leq \hat{s}(A)$ then the optimal policy is given by:*

$$\phi(A,s) = \hat{\phi}, \quad u(A,s) = \hat{u}(A), \tag{OA.E4}$$

where the function $\hat{u}(A)$ is defined by equation (OA.E3).

Proof. We need to show that Equations (24), (25), (26) and (27) are satisfied with appropriate choices of the Lagrangian multipliers. Under the proposed policies and prices, the left-hand side of Equation (24) is

$$\begin{aligned} & (1-\theta)\left[MPK_H\left(A,\hat{\phi},\hat{u}(A)\right)+(1-\delta)\right] \\ & -\left[\begin{array}{c} MPK_H\left(A,\hat{\phi},\hat{u}(A)\right) \\ -(1-\theta)\,MPK_H\left(A,\hat{\phi},\hat{u}(A)\right)+\theta(1-\delta) \end{array}\right]\left(\frac{\hat{u}(A)\,\hat{\phi}}{\pi\hat{\phi}+(1-\pi)}-1\right), \\ = \; & \Psi\left(A,\hat{\phi},\hat{u}(A)\right)=\hat{s}(A)\geq s. \end{aligned}$$

Also, because the marginal product of capital is equalized everywhere,

$$\begin{aligned} \Delta &= (1-\theta)\,MPK_L\left(A,\hat{\phi},\hat{u}(A)\right)-(1-\theta)\,\phi MPK_H\left(A,\hat{\phi},\hat{u}(A)\right)+(\phi-1)MPK\left(A,\hat{\phi},\hat{u}(A)\right), \\ &= \theta(\phi-1)\,MPK_H\left(A,\hat{\phi},\hat{u}(A)\right)>0. \end{aligned}$$

Therefore, both Equations (24) and (25) are satisfied. Finally, note that Equation (OA.E4) implies that $MPK_H\left(A,\hat{\phi},\hat{u}(A)\right)=MPK_L\left(A,\hat{\phi},\hat{u}(A)\right)=MPK\left(\hat{u}(A)\right)=\alpha\hat{u}(A)^{\alpha-1}A$, and therefore $\xi_H\left(A,\hat{\phi},\hat{u}(A)\right)=\xi_L\left(A,\hat{\phi},\hat{u}(A)\right)=0$. As a result, the Kuhn-Tucker conditions (26) and (27) for optimality are satisfied. ■

Only the constraint on high-productivity islands binds: We first construct the prices and then show that these prices satisfy the corresponding Kuhn-Tucker conditions. In the case where only the constraint on high-productivity islands (or the ε_H sector in the dynamic model) bind,

$$Q_H(A,\phi,u) > Q_L(A,\phi,u) = Q(u)$$

where we define

$$\begin{aligned} Q_H(A,\phi,u) &= \alpha A u^{\alpha-1} f(\phi)\frac{\pi\phi+1-\pi}{\pi\hat{\phi}^{1-\xi}\phi^{\xi}+1-\pi}+1-\delta, \\ Q_L(A,\phi,u) &= \alpha A u^{\alpha-1} f(\phi)\frac{\pi\phi+1-\pi}{\pi\hat{\phi}^{1-\xi}\phi^{\xi}+1-\pi}\left(\frac{\hat{\phi}}{\phi}\right)^{1-\xi}+1-\delta. \end{aligned}$$

The optimality condition for capital utilization implies (using the fact that $Q(u) =$

$b_0 (1-u)^{\nu-1}$):

$$\alpha A u^{\alpha-1} f(\phi) \frac{\pi\phi + 1 - \pi}{\pi\hat{\phi}^{1-\xi}\phi^{\xi} + 1 - \pi} + (1-\delta) = b_0 (1-u)^{\nu-1}. \tag{OA.E5}$$

The above equation defines the capital utilization rate as a function of (A, ϕ), which we will denote as $u_L(A, \phi)$. Let $\overline{\phi}(A)$ be the unique solution to

$$\Delta\left(A, \overline{\phi}(A), u_L\left(A, \overline{\phi}(A)\right)\right) = 0,$$

and define $\bar{s}(A)$ to be the highest level of s such that only the constraint on high productivity islands binds

$$\bar{s} = \Psi\left(A, \overline{\phi}(A), u_L\left(A, \overline{\phi}(A)\right)\right).$$

Given the definition of $u_L(A, \phi)$, we can show that

$$\begin{aligned}
\Psi(A, \phi, u_L(A, \phi)) &= \frac{MPK_L(A, \phi, u_L(A, \phi))}{\pi\phi + 1 - \pi} \\
&\quad \left\{ \begin{array}{c} (1-\theta) u_L(A, \phi) \hat{\phi}^{1-\xi}\phi^{\xi} \\ -\left[(\phi - 1)(1-\pi) - \phi(1 - u_L(A, \phi))\right] \end{array} \right\} \\
&\quad + \frac{1-\delta}{\pi\phi + 1 - \pi} \left\{ (1-\theta) u_L(A, \phi)\phi - \left[\begin{array}{c} (\phi - 1)(1-\pi) \\ -\phi(1 - u_L(A, \phi)) \end{array} \right] \right\},
\end{aligned}$$

and

$$\begin{aligned}
\Delta(A, \phi, u_L(A, \phi)) &= MPK_L(A, \phi, u_L(\phi)) \left[(\phi - 1) - (1-\theta)\left(\hat{\phi}^{1-\xi}\phi^{\xi} - 1\right) \right] \\
&\quad + (1-\delta)\theta(\phi_1 - 1).
\end{aligned}$$

Using the above expressions, we can prove that $\Psi(A, \phi, u_L(A, \phi))$ is strictly decreasing in ϕ and $\Delta(A, \phi, u_L(A, \phi))$ is strictly increasing functions of ϕ. As a result, i) $\phi \geq \overline{\phi}(A)$ if and only if $\Delta(A, \phi, u_L(A, \phi)) \geq 0$; ii) $\phi \geq \overline{\phi}(A)$ if and only if $\Psi(A, \phi, u_L(A, \phi)) \leq \Psi\left(A, \overline{\phi}, u_L\left(A, \overline{\phi}\right)\right)$. We can now prove the second part of Proposition 2 by verifying the following claim.

Claim 2. *If $\hat{s}(A) \leq s \leq \bar{s}(A)$ then the optimal policy $\phi(A, s)$ is implicitly defined by the unique solution to*

$$\Psi(A, \phi, u_L(A, \phi)) = s. \tag{OA.E6}$$

Given $\phi(A, s)$, the optimal policy $u(A, s)$ is given by

$$u(A, s) = u_L(A, \phi(A, s)), \tag{OA.E7}$$

where $u_L(A,\phi)$ is the implicit function defined by (OA.E5).

Proof. First, by construction, $\Psi(A,\phi,u_L(A,\phi)) = s$ and constraint (24) holds with equality. Also, the assumption that $s \leq \bar{s}(A)$ implies $\phi \geq \bar{\phi}(A)$ and $\Delta(A,\phi,u_L(A,\phi)) \geq 0$; therefore, (24) is satisfied. Finally, condition (OA.E7) implies $MPK(A,s) = MPK_L(A,\phi(A,s))$ and $\xi_L(A,s) = 0$; therefore, the Kuhn-Tucker conditions (26) and (27) are satisfied. ■

Both constraints bind: In the case where both constraints are binding, from $\Delta = 0$, $MPK(u) = Q(u) - (1-\delta) = b_0(1-u)^{\nu-1} - (1-\delta)$ must satisfy the following condition:

$$MPK(u) = (1-\theta)\, MPK_L(A,\phi,u)\, \frac{\hat{\phi}^{1-\xi}\phi^{\xi} - 1}{\phi - 1} - \theta(1-\delta)\,. \qquad \text{(OA.E8)}$$

Equation (OA.E8) defines u as a function of (A,ϕ), which we will denote as $u_{HL}(A,\phi)$. The fact that the constraint for H type island binds implies

$$\Psi(A,\phi,u_{HL}(A,\phi)) = s.$$

Part 3 of Proposition 2 can therefore be proved as the result of the following claim.

Claim 3. *For $s > \bar{s}(A)$, then the optimal policy $\{\phi(A,s), u(A,s)\}$ are jointly determined by:*

$$\Psi(A,\phi,u_{HL}(A,\phi)) = s, \qquad u(A,s) = u_{HL}(A,\phi(A,s))\,, \qquad \text{(OA.E9)}$$

where the function $u_{HL}(A,\phi)$ is implicitly defined in (OA.E8).

Proof. Clearly, by construction, both constrains (24) and (25) hold with equality. Using the definition of $u_{HL}(A,\phi)$ in (OA.E9) and the definition of $u_L(A,\phi)$ in (OA.E5) we can show that $u_{HL}(A,\phi) < u_L(A,\phi)$. Compare equation (OA.E8) with equations (OA.E3) and (OA.E5), $u_{HL}(A,\phi) < u_L(A,\phi)$ implies $MPK(u) < MPK_L(A,\phi,u) < MPK_H(A,\phi,u)$. As a result, the Kuhn-Tucker conditions (26) and (27) are satisfied with and $\zeta_j(A,\phi,u_{HL}(A,\phi)) > 0$ for $j = H, L$. ■