Pricing of Index Options in Incomplete Markets -
Online Appendix

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Abstract

In Section 1 of this Online Appendix, we discuss in detail the estimation of minimum dispersion risk-neutral measures. Section 2 collects theoretical results and proofs that are not shown in the main paper. In Section 3, we report additional empirical results on the pricing of S&P 500 options given the underlying returns. Section 4 presents results for a robustness analysis.

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1 Estimation of Minimum Dispersion Risk-Neutral Measures

Consider the environment described in Section 2 of the main paper. We start with a general convex function \( \phi(.) \) to measure the dispersion between the physical measure \( P \) and the risk-neutral measure \( Q \). A minimum dispersion risk-neutral measure solves the following minimization in the space of admissible measures with \( I^\phi(Q, P) \equiv E \left[ \phi \left( \frac{dQ}{dP} \right) \right] < \infty \):

\[
Q^* = \arg \min_Q E \left[ \phi \left( \frac{dQ}{dP} \right) \right] = \int \phi \left( \frac{dQ}{dP} \right) \, dP, \text{ s.t. } E^Q \left[ R - R_f \right] = 0_K, \tag{1}
\]

where, under the restriction that \( Q \) is absolutely continuous with respect to \( P \), \( \frac{dQ}{dP} \) is a Radon-Nikodym derivative.\(^1\) The risk-neutral measure must also be nonnegative and integrate to one. By definition, we have that any admissible measure \( Q \) satisfies \( I^\phi(Q, P) \geq I^\phi(Q^*, P) \).

While at first glance the variational problem (1) might seem difficult to solve, we follow Kitamura (2006) and Almeida and Garcia (2017) and make use of results in Borwein and Lewis (1991) to show that a duality result comes to rescue:

**Theorem 1.** Consider the primal problem:

\[
\min_Q E \left[ \phi \left( \frac{dQ}{dP} \right) \right], \text{ s.t. } E^Q \left[ R - R_f \right] = 0_K, \ E \left( \frac{dQ}{dP} \right) = 1, \ \frac{dQ}{dP} \geq 0, \tag{2}
\]

and the dual problem:

\[
\max_{\alpha \in R, \lambda \in RK} \alpha - E \left[ \phi^{*,+} \left( \alpha + \lambda' \left( R - R_f \right) \right) \right] + \delta([\alpha \, \lambda] | \Lambda(R)), \tag{3}
\]

where \( \Lambda(R) = \{ \alpha \in R, \lambda \in RK : (\alpha + \lambda' \left( R - R_f \right)) \in \text{dom } \phi^{*,+} \} \),\(^2\) \( \delta(\cdot | C) \) is such that \( \delta(x | C) = 0 \) if \( x \in C \) and \( \infty \) otherwise, and \( \phi^{*,+} \) denotes the convex conjugate of \( \phi \):

\[
\phi^{*,+}(z) = \sup_{w \in [0, \infty) \cap \text{domain } \phi} zw - \phi(w). \tag{4}
\]

Absence of arbitrage implies that the values of the primal and the dual problems coin-

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\(^1\)A Radon-Nikodym derivative \( \frac{dQ}{dP} : \Omega \rightarrow [0, \infty) \) is a measurable function such that for any measurable set \( A \subseteq \Omega \), \( Q(A) = \int_A dQ \, dP \).

\(^2\)The domain of \( \phi^{*,+}(z) \) is defined as the values of \( z \) for which the function is finite \( (\phi^{*,+}(z) < \infty) \).
cide (with dual attainment). A sufficient condition allowing the unique minimum dispersion risk-neutral measure to be obtained from the solution of the dual optimization problem is that either \( d = \lim_{x \to \infty} \frac{\phi(x)}{x} = \infty \) or \( (d < \infty \text{ and } c = \lim_{x \to \infty} ((d - \phi'(x))x > 0) \). In such cases, the implied state-price density per unit of probability is obtained by:

\[
\frac{dQ^*}{dP} = \frac{\partial \phi^{*,+}(z)}{\partial z} \bigg|_{z = (\alpha^* + \lambda^*(R-R_f))},
\]

where \( \alpha^* \) and \( \lambda^* \) are the optimal Lagrange multipliers solving the dual problem (3).

**Proof.** Let \( \overline{m} \equiv m/\mathbb{E}(m) \) be the normalized version of a given SDF \( m \). When \( m \) is nonnegative, \( \overline{m} \) will correspond to a state-price density per unit of probability. In Theorem 2.4 of Borwein and Lewis (1991), let \( X \) be the space of normalized SDFs \( \overline{m} \) such that \( \mathbb{E}[\overline{m}(R - R_f)] = 0 \) and \( \mathbb{E}[\phi(\overline{m})] < \infty \), \( f(\overline{m}) = \mathbb{E}[\phi(\overline{m})] \), \( C = X^+ \) be the space of normalized nonnegative SDFs \( \overline{m} \) such that \( \mathbb{E}[\overline{m}(R - R_f)] = 0 \) and \( \mathbb{E}[\phi(\overline{m})] < \infty \), \( A \overline{m} = \mathbb{E}(\overline{m}((R - R_f)' 1') \), \( b = [0 \ 1]' \) and \( P = 0 \). Theorem 2.5 at page 327 in Borwein and Lewis (1991) allows us to conjugate \( \phi(\cdot) \) within the expectation to obtain \( g^* = \mathbb{E}(\phi^{*,+}) \).

In addition, we obtain \( \lambda' = \hat{\lambda}'((R - R_f)' 1') \) and \( P^+ = \mathbb{R}^K \). No-arbitrage guarantees the existence of at least one feasible point (a strictly positive normalized SDF such that \( \mathbb{E}[\overline{m}(R - R_f)] = 0 \)) in the quasi-relative interior of \( X^+ \). Therefore, rewriting \( \hat{\lambda} = [\alpha \ \lambda] \), the primal problem in Theorem 1 has a solution that coincides with that of the following dual problem:

\[
\max_{\lambda \in R^{K+1}} \alpha - \mathbb{E} \left[ \phi^{*,+}(\alpha + \lambda'(R - R_f)) + \delta(\hat{\lambda} | \Lambda(R)) \right]
\]

where \( \Lambda(R) = \{ \hat{\lambda} \in R^{K+1} : (\alpha + \lambda'(R - R_f)) \in \text{dom } \phi^{*,+} \} \). Furthermore, if either \( d = \lim_{x \to \infty} \frac{\phi(x)}{x} = \infty \) or \( (d < \infty \text{ and } c = \lim_{x \to \infty} ((d - \phi'(x))x > 0) \), Theorem 5.5 at page 335 of Borwein and Lewis (1991) guarantees that the unique primal optimal solution is obtained by differentiating the convex conjugate \( \phi^{*,+}(z) \) at the optimal (Lagrange multipliers) dual solution: \( \overline{m}^* = \frac{dQ^*}{dP} = \frac{\partial \phi^{*,+}(z)}{\partial z} \bigg|_{z = (\alpha^* + \lambda^*(R-R_f))} \).

The dual problem is a much simpler finite dimensional convex maximization problem. The absence of arbitrage implies the existence of an interior point in the space of admissible state-price densities per unit of probability. This is a fundamental condition to guarantee that the solutions of the primal and dual problems coincide. The vector \( \lambda \) and the scalar \( \alpha \) are Lagrange multipliers coming from the primal problem constraints given by the Euler equations for the basis assets and the restriction \( \mathbb{E}(dQ/dP) = 1 \), respectively. The primal problem nonnegativity constraint restricts the convex conjugate to be
calculated on the nonnegative real line, while the delta function \( \delta(., \Lambda(\mathbf{R})) \) restricts, for each vector of returns \( \mathbf{R} \) in the probability space, the optimization problem to a subset \( \Lambda(\mathbf{R}) \) of \( \mathbb{R}^K \) where the convex conjugate assumes finite values.

Theorem 1 provides a general result for obtaining a minimum discrepancy risk-neutral measure according to a convex function \( \phi \), which can be seen as a generalized minimum contrast estimation procedure (Bickel et al., 1993). This procedure falls into the category of methods that treat the data distribution nonparametrically in statistical estimation by comparing the empirical distribution of the data with the family of distributions implied by a statistical model. In our case, the statistical model is defined to be the set of all probability measures that satisfy the moment restrictions characterizing risk neutrality for the basis assets.

We focus on risk-neutral measures minimizing the Cressie and Read (1984) family of discrepancies, defined as:

\[
\phi_\gamma \left( \frac{dQ}{dP} \right) = \left( \frac{dQ}{dP} \right)^{\gamma+1} - 1 \gamma (\gamma + 1), \quad \gamma \in \mathbb{R}.
\]  

(7)

Kitamura (1996), Baggerly (1998) and Newey and Smith (2004) suggest using this comprehensive divergence family, that includes as particular cases several well-known measures of dispersion. For instance, the Euclidean divergence \( \gamma = 1 \), the KLIC \( \gamma \to 0 \), the Hellinger divergence \( \gamma = -1/2 \), the empirical likelihood \( \gamma \to -1 \) and Pearson’s Chi-Square \( \gamma = -2 \). Furthermore, Newey and Smith (2004) show that the minimum discrepancy Cressie-Read estimators are equivalent to the class of generalized empirical likelihood (GEL) estimators (Smith, 1997). That is, the GEL objective function is given by the dual problem of the minimum discrepancy problem, where \( \gamma \) indexes the particular estimator in this class. For instance, \( \gamma \to -1 \) yields the empirical likelihood (Owen, 1988), \( \gamma \to 0 \) the exponential tilting (Kitamura and Stutzer, 1997) and \( \gamma = 1 \) the continuous updating estimator (Hansen, Heaton and Yaron, 1996). These estimators are robust against distributional assumptions, possess desirable properties analogous to those of parametric likelihood procedures and have been used to improve on the small sample properties of GMM estimators.\(^3\)

By considering the Cressie-Read family, Theorem 1 produces the following corollary:

**Corollary 1.** Let the discrepancy function \( \phi \) in the minimization problem (2) belong to

\(^3\)See Kitamura (2006) for a review.
the family defined in (7), and assume absence of arbitrage. Then, letting \( \Lambda_\gamma(R) = \{ \lambda \in \mathbb{R}^K : (1 + \gamma \lambda' (R - R_f)) > 0 \} \):

i) if \( \gamma > 0 \), (3) specializes to:

\[
\alpha^*_\gamma = \frac{1}{\gamma}, \quad \lambda^*_\gamma = \arg\max_{\lambda \in \mathbb{R}^K} - \frac{1}{\gamma + 1} \mathbb{E} \left[ (1 + \gamma \lambda' (R - R_f))^{\frac{\gamma + 1}{\gamma}} I_{\Lambda_\gamma(R)}(\lambda) \right].
\] (8)

ii) if \( \gamma < 0 \), it specializes to:

\[
\alpha^*_\gamma = \frac{1}{\gamma}, \quad \lambda^*_\gamma = \arg\max_{\lambda \in \mathbb{R}^K} - \frac{1}{\gamma + 1} \mathbb{E} \left[ (1 + \gamma \lambda' (R - R_f))^{\frac{\gamma + 1}{\gamma}} - \delta(\lambda \mid \Lambda_\gamma(R)) \right].
\] (9)

iii) if \( \gamma = 0 \), the maximization is unconstrained:

\[
\alpha^*_0 = 1, \quad \lambda^*_0 = \arg\max_{\lambda \in \mathbb{R}^K} - \mathbb{E} \left[ e^{\lambda' (R - R_f)} \right],
\] (10)

where \( I_A(x) = 1 \) if \( x \in A \), and 0 otherwise.

**Proof.** We need to obtain the convex conjugate \( \phi^*_\gamma \) of \( \phi_\gamma \) belonging to the Cressie-Read family as in (7) to substitute in the dual problem (3). Given equation (4) defining the convex conjugate, we define an auxiliary function \( h_\gamma(z) = zw - \frac{w^{\gamma+1} - z^{\gamma+1}}{\gamma(\gamma+1)} \), whose domain is \( \text{dom}(h_\gamma) = [0, \infty) \cap \text{dom}(\phi_\gamma) \). Note that for \( \gamma > -1 \) and \( \gamma \neq 0 \), \( \text{dom}(h_\gamma) = [0, \infty) \), and for \( \gamma \leq -1 \) or \( \gamma = 0 \), \( \text{dom}(h_\gamma) = (0, \infty) \). In order to obtain the supremum in \( \phi^*_\gamma(z) = \sup_{w \in \text{dom}(h_\gamma)} h_\gamma(w) \), we differentiate \( h_\gamma(w) \) with respect to \( w \), leading to: \( \frac{dh_\gamma(z)}{dw} = z - \frac{w^{\gamma}}{\gamma} \).

Now we split the analysis in three cases: \( \gamma > 0 \), \( \gamma < 0 \), and \( \gamma = 0 \).

i) \( \gamma > 0 \).

In this case, \( \text{dom}(h_\gamma) = [0, \infty) \). If \( z \leq 0 \), \( h_\gamma \) is a decreasing function of \( w \) and achieves its maximum at \( \hat{w} = 0 \). If \( z > 0 \), \( \hat{w} = (\gamma z)^{\frac{1}{\gamma}} \) will be the unique critical point where the function achieves its maximum. By combining these two solutions, we note that \( \text{dom}(\phi^*_\gamma) = \mathbb{R} \) (which implies that \( \Lambda_\gamma = \mathbb{R}^{K+1} \)), and, for an arbitrary \( z \), we obtain \( \hat{w} = (\gamma z)^{\frac{1}{\gamma}} I_{\{\gamma z > 0\}} \). Substituting \( \hat{w} \) in \( \phi^*_\gamma(z) = h_\gamma(\hat{w}) \), the convex conjugate becomes:

\[
\phi^*_\gamma(z) = \frac{(\gamma z)^{\frac{\gamma + 1}{\gamma}}}{\gamma + 1} I_{\{\gamma z > 0\}} + \frac{1}{\gamma(\gamma + 1)},
\] (11)

and the optimization problem becomes:

\[
\tilde{\lambda}^*_\gamma = \arg\max_{\alpha \in R, \lambda \in \mathbb{R}^K} \alpha - \mathbb{E} \left[ \frac{(\gamma (\alpha + \gamma (R - R_f)))^{\frac{\gamma + 1}{\gamma}}}{\gamma + 1} I_{\{\gamma (\alpha + \gamma (R - R_f)) > 0\}} \right] - \frac{1}{\gamma(\gamma + 1)}.
\] (12)
First, note that we can discard the constant \( \frac{1}{\gamma(\gamma+1)} \) from the maximization problem above. Now, as Kitamura (2006, page 12) notes, the convex conjugate \( \phi_\gamma^* \) is homogeneous, so we can concentrate out \( \alpha \) and re-define \( \lambda = \lambda/\alpha \) to obtain:

\[
\alpha - \gamma \left( \frac{\gamma + 1}{\gamma + 1} \right) E \left[ \left( 1 + \lambda \left( R - R_f \right) \right) \frac{\gamma + 1}{\gamma + 1} I_{\{\gamma(1+\lambda'\left( R - R_f \right)) > 0\}} \right].
\]  

**(13)**

Ignoring for now the indicator function inside the expectation, let \( \Gamma(\alpha) = \alpha - \frac{(\gamma\alpha)^{1+1}}{\gamma + 1} \), where the optimal concentrated \( \alpha \) is obtained by maximizing \( \Gamma \). From its first-order condition, we get:

\[
d \Gamma(\alpha) \over d\alpha = 0 \Rightarrow \alpha^* = \frac{1}{\gamma}.
\]

Since \( \gamma\alpha^* = 1 > 0 \), it does not affect the indicator function. Substituting \( \alpha^* \) in \( \lambda \) and (13), we have:

\[
\lambda^*_\gamma = \text{arg max}_{\lambda \in \mathbb{R}^K} \frac{1}{\gamma} - \frac{1}{\gamma + 1} E \left[ (1 + \gamma\lambda'\left( R - R_f \right)) \frac{\gamma + 1}{\gamma + 1} I_{\{1 + \gamma\lambda'\left( R - R_f \right) > 0\}} \right].
\]

**(14)**

Since the first term does not affect the maximization, we have the desired result.

**ii** \( \gamma < 0 \).

In this case, \( \text{dom}(h^*_\gamma) = [0, \infty) \) if \(-1 < \gamma < 0 \) and \( \text{dom}(h^*_\gamma) = (0, \infty) \) if \( \gamma \leq -1. \) If \( z \geq 0 \), \( h^*_\gamma \) is an increasing function of \( w \) and achieves its maximum at \( \hat{w} = \infty \). If \( z < 0 \), \( \hat{w} = (\gamma z)^{1/2} \) will be the unique critical point where the function achieves its maximum. The fact that \( \phi^*_\gamma \) is \( \infty \) for \( z \geq 0 \) and is finite otherwise directly implies that \( \text{dom}(\phi^*_\gamma) = (-\infty, 0) \). Combining these two solutions, the convex conjugate becomes:

\[
\phi^*_\gamma(z) = \frac{(\gamma z)^{1+1}}{\gamma + 1} + \delta(z|\{\hat{z} \in \mathbb{R} : \gamma \hat{z} > 0\}) + \frac{1}{\gamma(\gamma + 1)},
\]

**(15)**

and the optimization problem becomes:

\[
\hat{\lambda}^*_\gamma = \text{arg max}_{\alpha \in \mathbb{R}, \lambda \in \mathbb{R}^K} \alpha - E \left[ \left( \frac{(\alpha + \lambda'\left( R - R_f \right)) \frac{\gamma + 1}{\gamma + 1}}{\gamma + 1} \right) \delta([\alpha, \lambda])|\Lambda_\gamma(R) \right] - \frac{1}{\gamma(\gamma + 1)},
\]

**(16)**

where \( \Lambda_\gamma(R) = \{\alpha \in \mathbb{R}, \lambda \in \mathbb{R}^K : \gamma (\alpha + \lambda'\left( R - R_f \right)) > 0 \} \). Following the same procedure as in the previous case to concentrate out \( \alpha \), we obtain the desired result, with \( \Lambda_\gamma(R) \) simplifying to \( \Lambda_\gamma(R) = \{\lambda \in \mathbb{R}^K : (1 + \gamma\lambda'\left( R - R_f \right)) > 0 \} \).

**iii** \( \gamma = 0 \).

The limit \( \lim_{\gamma \to 0} E(\phi_\gamma(\overline{m})) = E(m \log(\overline{m})) \) coincides with the KLIC (Stutzer, 1995, page 375). Therefore, we need to obtain the convex conjugate \( \phi_0^* \) of \( \phi_0(m) = m \log(\overline{m}) \), whose domain is \( \text{dom}(\phi_0) = (0, \infty) \). Note that the corresponding auxiliary function
is \( h_\alpha(w) = zw - w \log(w) \), whose first derivative is \( \frac{dh_\alpha(w)}{dw} = z - 1 - \log(w) \). Since \( \text{dom}(h_\alpha) = (0, \infty) \), and in this range \( \log(w) \) covers the whole real line, for any value of \( z \) the only critical point will be \( \tilde{w} = e^{z-1} \), implying that \( \text{dom}(\phi^\star_\alpha) = \mathbb{R} \) (which in turn implies that \( \Lambda_\gamma = \mathbb{R}^{K+1} \)). Substituting \( \tilde{w} \) in \( h_\alpha(\tilde{w}) \), the convex conjugate becomes:

\[
\phi^\star_\alpha(z) = e^{z-1},
\]

and the optimization problem becomes:

\[
\tilde{\lambda}_\alpha^* = \arg \max_{\alpha \in \mathbb{R}, \lambda \in \mathbb{R}^K} \alpha - \mathbb{E} \left[ e^{(\alpha-1) + \lambda(R-R_f)} \right].
\]

To concentrate \( \alpha \) out of (18), we define \( \Gamma(\alpha) = \alpha - e^{\alpha-1} \) and obtain its first-order condition, to see that it is maximized at \( \alpha_0^* = 1 \). Substituting \( \alpha_0^* \) in (18) we obtain the desired result.

In the next corollary, we use Theorem 1 to identify the implied minimum dispersion state-price densities per unit of probability and to verify if the different discrepancies of the Cressie-Read family satisfy the regularity sufficient conditions that allow us to obtain the risk-neutral measures from the first derivative of the convex conjugate \( \phi^\star(\cdot) \).

**Corollary 2.** Let the discrepancy function \( \phi \) in the minimization problem (2) belong to the family defined in (7). For any \( \gamma \geq -1 \), at least one of the sufficient regularity conditions stated in Theorem 1 is satisfied by \( \phi_\gamma \) and the corresponding minimum dispersion implied state-price density per unit of probability will be given by:

\[
\begin{align*}
\frac{dQ^\star}{dP}(\gamma, \mathbf{R}) &= (1 + \gamma \lambda_\gamma^* (\mathbf{R} - \mathbf{R}_f))^{\frac{1}{\gamma}} I_{\Lambda_\gamma(R)}(\lambda_\gamma^*), \quad \gamma > 0, \\
\frac{dQ^\star}{dP}(\gamma, \mathbf{R}) &= (1 + \gamma \lambda_\gamma^* (\mathbf{R} - \mathbf{R}_f))^{\frac{1}{\gamma}}, \quad -1 \leq \gamma < 0, \\
\frac{dQ^\star}{dP}(0, \mathbf{R}) &= e^{\lambda_0^*(R-R_f)}, \quad \gamma = 0.
\end{align*}
\]

where for \( \gamma > 0 \), \( \lambda_\gamma^* \) solves (8), for \(-1 \leq \gamma < 0 \), \( \lambda_\gamma^* \) solves (9) and \( \lambda_0^* \) solves (10).

For \( \gamma < -1 \), none of the stated regularity conditions in Theorem 1 are satisfied. In this case, an alternative sufficient condition for the minimum dispersion implied state-price density per unit of probability to be given by the expression in (20), with \( \lambda_\gamma^* \) solving (9), is that:

\[
\inf_{\omega \in \Omega} \left( 1 + \gamma \lambda_\gamma^* (R(\omega) - R_f) \right) > 0.
\]
If the sample space has a finite number of states, the infimum becomes a minimum, and this condition is equivalent to \( \forall \omega \in \Omega : (1 + \gamma \lambda''^*(R(\omega) - R_f)) > 0 \).

Proof. i) \( \gamma > 0 \).

It is easy to see that \( d = \lim_{x \to \infty} \frac{\phi^*(x)}{x} = \lim_{x \to \infty} \frac{x^{1 - \frac{1}{\gamma}}}{\gamma(\gamma + 1)} = \infty \), implying that the first sufficient condition stated at Theorem 1 is satisfied.

ii) \( -1 < \gamma < 0 \).

Since, in this case, \( d = \lim_{x \to \infty} \frac{x^{\gamma - \frac{1}{\gamma}}}{\gamma(\gamma + 1)} = 0 \), we proceed to verify if \( c = \lim_{x \to \infty} ((d - \phi'(x))x) > 0 \). As \( \phi'(x) = \frac{x^\gamma}{\gamma} \), we obtain \( c = \lim_{x \to \infty} - \frac{x^{\gamma + 1}}{\gamma} = \infty > 0 \), implying that the second sufficient condition at Theorem 1 is satisfied.

iii) \( \gamma = -1 \).

Since \( \phi_{-1}(x) = -\log(x) \), we have \( \phi_{-1}'(x) = -\frac{1}{x} \) and \( c = \lim_{x \to \infty} \left(-\frac{1}{x}\right)x = 1 > 0 \), implying that the second sufficient condition at Theorem 1 is satisfied.

iv) \( \gamma < -1 \).

In this case, \( d = 0 \) and \( c = \lim_{x \to \infty} - \frac{x^{\gamma + 1}}{\gamma} = 0 \), implying that none of the two sufficient conditions appearing in Theorem 1 are satisfied. Therefore, we invoke Theorem 4.8 at page 334 of Borwein and Lewis (1991), which adapted to our problem states that if the optimal Lagrange multipliers in the dual problem satisfy:

\[
\inf_{\omega \in \Omega} \left(1 + \gamma \lambda''^*(R(\omega) - R_f)\right) > 0, \tag{23}
\]

the unique primal optimal solution can be obtained by differentiating the convex conjugate. If the sample space has a finite number of states, the infimum becomes a minimum, and (23) is equivalent to \( \forall \omega \in \Omega : (1 + \gamma \lambda''^*(R(\omega) - R_f)) > 0 \).

Now, we show that, whenever (5) is valid, differentiating the convex conjugate in (11), (15) and (17) with respect to \( z \) gives:

\[
\frac{\partial \phi^{*,+}(z)}{\partial z} = (\gamma z)^{\frac{1}{\gamma}} 1_{\{\gamma z \geq 0\}}, \quad \gamma > 0, \tag{24}
\]

\[
\frac{\partial \phi^{*,+}(z)}{\partial z} = (\gamma z)^{\frac{1}{\gamma}}, \quad \gamma < 0, \tag{25}
\]

\[
\frac{\partial \phi^{*,+}(z)}{\partial z} = e^{z-1}, \quad \gamma = 0. \tag{26}
\]
Substituting $z = (\alpha^*_\gamma + \lambda^*_\gamma (R - R_f))$, with $\alpha^*_\gamma = \frac{1}{\gamma}$ for $\gamma \neq 0$ and $\alpha^*_0 = 1$ for $\gamma = 0$, in (24), (25) and (26), we obtain expressions (19), (20) and (21).

The results so far focused on the population. In order to estimate minimum dispersion risk-neutral measures from data on basis assets returns, we consider the sample version of problem (2). In this case, the sample space $\Omega$ is finite and discrete, with states of nature $k = \{1, ..., n\}$, where $n > K$. Let $\{R_k\}_{k=1}^n$ be the observed gross returns of the $K$ basis assets, where each $R_k$ is independent and identically distributed according to $\mathbb{P}$. The unknown physical measure $\mathbb{P}$ can be replaced by the empirical measure $\mathbb{P}_n$ that gives weights $\pi_k = 1/n$ to the realization of each state of nature.\footnote{This is an optimal nonparametric estimator for $\mathbb{P}$. For more details, see Kitamura (2006).}

This allows us to exchange the expectation $\mathbb{E}$ with its sample counterpart $\frac{1}{n} \sum_{k=1}^n \equiv \sum_{k=1}^n \pi_k$. The state-price density will be given by $\pi_k^Q/\pi_k$, where $\pi_k^Q$ for $k = 1, ..., n$ is the empirical risk-neutral measure. In the following corollary, we summarize the results for the sample version of the problem of finding a minimum Cressie-Read dispersion risk-neutral measure:

**Corollary 3.** Consider the primal problem:

$$
\min_{\{\pi^Q_1, ..., \pi^Q_n\}} \sum_{k=1}^n \pi_k \left( \frac{\pi_k^Q/\pi_k}{\gamma+1} \right)^{1/\gamma},
$$

s.t. $\sum_{k=1}^n \pi_k^Q (R_k - R_f) = 0$, $\sum_{k=1}^n \pi_k^Q = 1$, $\pi_k^Q \geq 0 \ \forall \ k$. (27)

Absence of arbitrage in the observed sample implies that the value of the primal problem coincides (with dual attainment) with the value of the dual problem below:

i) if $\gamma > 0$:

$$
\lambda^*_\gamma = \arg \max_{\lambda \in \mathbb{R}^K} -\frac{1}{\gamma + 1} \sum_{k=1}^n \pi_k \left( 1 + \gamma \lambda' (R_k - R_f) \right)^{(\gamma+1)/\gamma} I_{\gamma}(R_k)(\lambda),
$$

(28)

ii) if $\gamma < 0$:

$$
\lambda^*_\gamma = \arg \max_{\lambda \in \Lambda_\gamma} -\frac{1}{\gamma + 1} \sum_{k=1}^n \pi_k \left( 1 + \gamma \lambda' (R_k - R_f) \right)^{(\gamma+1)/\gamma},
$$

(29)

iii) if $\gamma = 0$, the maximization is unconstrained:

$$
\lambda^*_0 = \arg \max_{\lambda \in \mathbb{R}^K} -\sum_{k=1}^n \pi_k e^{\lambda'(R_k - R_f)},
$$

(30)
where \( \Lambda_{\gamma} = \{ \lambda \in \mathbb{R}^K : \text{for } k = 1, \ldots, n, \ (1 + \gamma \lambda' (R_k - R_f)) > 0 \} \) and \( \Lambda_{\gamma}(R_k) = \{ \lambda \in \mathbb{R}^K : (1 + \gamma \lambda' (R_k - R_f)) > 0 \} \).

The minimum dispersion risk-neutral measure can then be recovered via the first derivative of the convex conjugate, or, equivalently, from the first-order conditions of (28), (29) and (30) with respect to \( \lambda \), evaluated at \( \lambda^*_\gamma \):

\[
\begin{align*}
\pi_{k}^{Q^*}(\gamma, R) &= \frac{(1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_k)}(\lambda^*_\gamma)}{\sum_{i=1}^{n} (1 + \gamma \lambda^*_\gamma'(R_i - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_i)}(\lambda^*_\gamma)}, \quad k = 1, \ldots, n; \quad \gamma > 0, \\
\pi_{k}^{Q^*}(\gamma, R) &= \frac{(1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}}}{\sum_{i=1}^{n} (1 + \gamma \lambda^*_\gamma'(R_i - R_f))^{\frac{1}{2}}} \pi, \quad k = 1, \ldots, n; \quad \gamma < 0, \\
\pi_{k}^{Q^*}(0, R) &= \frac{-e^{\lambda^*_\gamma(R_k - R_f)}}{\sum_{i=1}^{n} e^{\lambda^*_\gamma(R_i - R_f)}}, \quad k = 1, \ldots, n; \quad \gamma = 0.
\end{align*}
\]

**Proof.** The sample version is a particular case of the population version, so the duality result and the dual problem expressions (28), (29) and (30) follow directly from Theorem 1 and Corollary 1. Moreover, note that the \( \delta(.) \) function has been eliminated. This is because in the sample problem, the region \( \Lambda_{\gamma}(R) \) where we search for \( \lambda \) is simplified to depend on all observed returns at once (as in \( \Lambda_{\gamma} \)), instead of being a random region depending on each possible realization of returns \( R(\omega) \). Therefore, for \( \gamma < 0 \), we can directly constrain the maximization by restricting \( \lambda \) to \( \Lambda_{\gamma} \).

The minimum discrepancy risk-neutral measure can be recovered according to Corollary 2. Note that, in the sample version, we have the state-price density per unit of probability for each realization \( k \) of the nature:

\[
\frac{\pi_{k}^{Q^*}}{\pi_k} = (1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_k)}(\lambda^*_\gamma), \quad k = 1, \ldots, n; \quad \gamma > 0,
\]

which can be written as \( \pi_{k}^{Q^*} = \pi_k (1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_k)}(\lambda^*_\gamma) \). Note also that, since \( \sum_{k=1}^{n} \pi_k(\pi_{k}^{Q^*} / \pi_k) = 1 \Rightarrow \sum_{k=1}^{n} \pi_{k}^{Q^*} = 1 \Rightarrow \sum_{k=1}^{n} \pi_k(1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_k)}(\lambda^*_\gamma) = 1 \), we can write:

\[
\frac{\pi_{k}^{Q^*}}{\pi_k} = \frac{(1 + \gamma \lambda^*_\gamma'(R_k - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_k)}(\lambda^*_\gamma)}{\sum_{i=1}^{n} (1 + \gamma \lambda^*_\gamma'(R_i - R_f))^{\frac{1}{2}} I_{\Lambda_{\gamma}(R_i)}(\lambda^*_\gamma)}, \quad k = 1, \ldots, n; \quad \gamma > 0.
\]
which implies that:

$$
\pi_k^{Q^*} = \frac{(1 + \gamma \lambda^*_\gamma(k - R_f))^{\frac{1}{\gamma}} I_{\Lambda^*_\gamma(R_k)}(\lambda^*_\gamma)}{\sum_{i=1}^n (1 + \gamma \lambda^*_\gamma(R_i - R_f))^{\frac{1}{\gamma}} I_{\Lambda^*_\gamma(R_i)}(\lambda^*_\gamma)}, \ k = 1, \ldots, n; \ \gamma > 0.
$$

(36)

Following the same argument, we obtain expressions (32) and (33). That is, the sample version allows us to directly identify the risk-neutral measure weights. \qed

2 Theoretical Results and Proofs

2.1 Proof of Equivalence Between Dual Problem and Optimal Portfolio Problem

Proposition 1. Consider the class of HARA utility functions:

$$
u^\gamma(W) = -\frac{1}{\gamma + 1} (b - a \gamma W)^{\frac{\gamma + 1}{\gamma}},
$$

(37)

with $a > 0$ and $b - a \gamma W > 0$, which guarantees that the function $\nu^\gamma$ is well-defined, concave and strictly increasing. Now suppose a standard model of optimal portfolio choice, where an investor distributes her initial wealth $W_0$ putting $\tilde{\lambda}_j$ units of wealth on the risky asset $R_j$ and the remaining $W_0 - \sum_{j=1}^K \tilde{\lambda}_j$ in a risk-free asset paying $R_f$. Terminal wealth is then given by $W(\tilde{\lambda}) = W_0 R_f + \sum_{j=1}^K \tilde{\lambda}_j(R_j - R_f)$. The investor maximizes the expected HARA utility function in (37) solving one of the following optimal portfolio problems:

$$
\max_{\tilde{\lambda} \in R^K} \mathbb{E}\left[u^\gamma(W(\tilde{\lambda})) I_{\Lambda^\gamma(W)}(\tilde{\lambda})\right], \ \gamma > 0,
$$

(38)

$$
\max_{\tilde{\lambda} \in R^K} \mathbb{E}\left[u^\gamma(W(\tilde{\lambda})) - \delta(\tilde{\lambda} | \Lambda^\gamma(W))\right], \ \gamma < 0,
$$

(39)

$$
\max_{\tilde{\lambda} \in R^K} \mathbb{E}\left[u^0(W(\tilde{\lambda}))\right], \ \gamma = 0
$$

(40)

where $\Lambda^\gamma(W) = \{\lambda \in R^K : b - a \gamma W(\lambda) > 0\}$. Then, solving (38), (39) and (40) is equivalent to solving (8), (9), and (10), respectively. In particular, letting $\lambda^*_\gamma$ denote the dual problem solution, $\tilde{\lambda}^*_\gamma = -\lambda^*_\gamma(b - a \gamma W_0 R_f)/a$ if $\gamma \neq 0$ and $\tilde{\lambda}^*_\gamma = -\lambda^*_\gamma/a$ if $\gamma = 0$.  

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Proof. Substituting (37) in (38), we have:

$$\max_{\lambda \in \mathbb{R}^K} - \frac{1}{\gamma + 1} \mathbb{E} \left[ (b - a\gamma W_0 R_f - a\gamma \tilde{\lambda}(R - R_f))^{\frac{\gamma + 1}{\gamma}} 1_{\{b - a\gamma W > 0\}} \right],$$

which can be written as:

$$\max_{\lambda \in \mathbb{R}^K} - \frac{1}{\gamma + 1} \mathbb{E} \left[ ((b - a\gamma W_0 R_f)(1 + \gamma \lambda'(R - R_f))^{\frac{\gamma + 1}{\gamma}} 1_{\{b - a\gamma W_0 R_f(1 + \gamma \lambda'(R - R_f)) > 0\}} \right],$$

where \(\lambda = -\tilde{\lambda}a/(b - a\gamma W_0 R_f)\). Note that \(b - a\gamma W > 0 \Rightarrow b - a\gamma W_0 R_f > 0\), which implies that this term does not affect the indicator function. Furthermore, we can put this term in evidence outside the expectation (with exponent \(\gamma/(\gamma + 1)\)). Since it is positive, it does not affect the maximization, so we can ignore it. This implies that (42) is equivalent to:

$$\max_{\lambda \in \mathbb{R}^K} - \frac{1}{\gamma + 1} \mathbb{E} \left[ (1 + \gamma \lambda'(R - R_f))^{\frac{\gamma + 1}{\gamma}} 1_{\{1 + \gamma \lambda'(R - R_f) > 0\}} \right],$$

which is precisely the dual problem (8).

The argument is analogous for the equivalence between (38) and the dual problem (9).

In the case that \(\gamma = 0\), we take the limit \(\gamma \to 0\) of (40) to get:

$$\max_{\lambda \in \mathbb{R}^K} - \mathbb{E} \left[ e^{-aW} \right] \equiv \max_{\lambda \in \mathbb{R}^K} - \mathbb{E} \left[ e^{-aW_0 R_f - a\tilde{\lambda}'(R - R_f)} \right].$$

By taking \(e^{-aW_0 R_f}\) outside of the expectation, note that it does not affect the maximization, so we can ignore it. By letting \(\lambda = -a\tilde{\lambda}\), we have the desired result. \(\square\)

### 2.2 Proof of Proposition 1 in Main Paper

**Proposition 2.** Consider the minimum dispersion primal problem (27) and the corresponding dual problems in the usual case where there is risk premia in the economy. Then, neither the primal nor the dual problems have a solution when \(\gamma \to -\infty\) or \(\gamma \to \infty\).

**Proof.** Consider first the primal problem (27). One of its constraints is that \(\sum_{k=1}^{n} \pi_k^Q = 1\), which implies that \(\frac{1}{n} \sum_{k=1}^{n} \pi_k^Q = \frac{1}{n} = \pi_k\), that is, on average the risk-neutral measure equals \(\pi_k\). Therefore, since \(\pi_k^Q \neq \pi_k\) for all \(k\), \(\pi_k^Q / \pi_k\) will always have values smaller and larger than 1. This implies that when \(\gamma \to -\infty\) (\(\gamma \to \infty\)), the values smaller than 1 (larger than 1) will diverge to infinity, faster than the denominator, in the objective.
function. Therefore, in the limits the primal problem is not well-defined and does not have a solution.

Consider now the dual problem for $\gamma < 0$. If $\gamma \to -\infty$, we have $(1 + \gamma \lambda^*_\gamma(R_k - R_f))^{1/2} \to 1$, so the risk-neutral measure converges to the physical measure: $\pi_k^{Q^*} \to \pi_k = 1/n$. Since there is risk premia in the economy, this measure does not price the basis assets, hence not satisfying the first-order conditions of the dual problem, implying that the dual problem does not have a solution.

Now, consider the dual problem for $\gamma > 0$. If $\gamma \to \infty$, we have $(1 + \gamma \lambda^*_\gamma(R_k - R_f))^{1/2} \to 1$. Let’s analyze the behavior of the indicator function in the limit. For states of nature $k$ such that $\lambda^*_\gamma(R_k - R_f) \geq 0$ in the limit, the indicator function will be one, otherwise it will be zero. This can be seen by rewriting, for a finite $\gamma$, the condition for the indicator function to be one as $\lambda^*_\gamma(R_k - R_f) > -1/\gamma$, and noticing that in the limit it becomes $\lambda^*_\gamma(R_k - R_f) \geq 0$. Therefore, the limit of the risk-neutral measure when $\gamma \to \infty$ is $1/\beta$ for states of nature such that $\lambda^*_\gamma(R_k - R_f) \geq 0$, where $\beta$ is the number of such states, and 0 otherwise. This measure does not price the basis assets, implying that the dual problem does not have a solution.

2.3 Proof of Proposition 2 in Main Paper

Proposition 3. Let $\lambda^*_\gamma$ be the solution for a given $\gamma$ to the dual problem in Corollary 3 when there is just one risky basis asset.

(i) Let $\gamma < 0$ in a neighborhood of zero. If $\mathbb{E}(R - R_f) > 0$ ($\mathbb{E}(R - R_f) < 0$), we have $\lambda^*_\gamma < 0$ ($\lambda^*_\gamma > 0$) and $\gamma \lambda^*_\gamma > 0$ ($\gamma \lambda^*_\gamma < 0$). As $\gamma$ becomes more negative, $\gamma \lambda^*_\gamma$ increases (decreases). The dual problem has a solution while $\gamma \lambda^*_\gamma < -1/\min\{R_k - R_f\}$ ($\gamma \lambda^*_\gamma > -1/\max\{R_k - R_f\}$). When this condition can no longer be satisfied, there is no longer a solution to the dual problem.

(ii) Let $\gamma > 0$ in a neighborhood of zero. If $\mathbb{E}(R - R_f) > 0$ ($\mathbb{E}(R - R_f) < 0$), we have $\lambda^*_\gamma < 0$ ($\lambda^*_\gamma > 0$) and $\gamma \lambda^*_\gamma < 0$ ($\gamma \lambda^*_\gamma > 0$). As $\gamma$ increases, $\gamma \lambda^*_\gamma$ decreases (increases). While $\gamma \lambda^*_\gamma > -1/\max\{R_k - R_f\}$ ($\gamma \lambda^*_\gamma < -1/\min\{R_k - R_f\}$), the implied risk-neutral measure is strictly positive. When $\gamma \lambda^*_\gamma \leq -1/\max\{R_k - R_f\}$ ($\gamma \lambda^*_\gamma \geq -1/\min\{R_k - R_f\}$), the measure is not strictly positive anymore. Let also $\max_i$ ($\min_i$) denote the $i^{th}$ highest (smallest) value and $\min_{>0}$ ($\max_{<0}$) the minimum positive (maximum negative) value. The $i^{th}$ zero in the measure will appear when $\gamma \lambda^*_\gamma \leq -1/\max_i\{R_k - R_f\}$ ($\gamma \lambda^*_\gamma \geq -1/\min_i\{R_k - R_f\}$).
When the last positive (negative) return is set to zero, i.e., $\gamma \lambda_\gamma^* \leq -1/\min_{>0}\{R_k - R_f\}$ $(\gamma \lambda_\gamma^* \geq -1/\max_{<0}\{R_k - R_f\})$, the dual problem does not have a solution anymore.

Proof. It is a well known result that, for the portfolio optimization problem in Proposition 1 with one risky asset and any utility function with $u'(\cdot) > 0$ and $u''(\cdot) < 0$, we have, denoting the solution $\tilde{\lambda}^*$:

$$\tilde{\lambda}^* > 0 \iff \mathbb{E}(R - R_f) > 0,$$
$$\tilde{\lambda}^* = 0 \iff \mathbb{E}(R - R_f) = 0,$$
$$\tilde{\lambda}^* < 0 \iff \mathbb{E}(R - R_f) < 0.$$

In general, $\mathbb{E}(R - R_f) > 0$ and this implies that the investor will always hold a positive amount of the asset. However, it is possible that $\mathbb{E}(R - R_f) < 0$, in the case of assets indexed on catastrophe events, for instance. Therefore, we prove our results considering both possibilities. Proposition 1 shows that, denoting $\lambda^*$ to be the dual problem solution, we have that $\tilde{\lambda}^* > 0$ implies $\lambda^* < 0$ and that $\tilde{\lambda}^* < 0$ implies $\lambda^* > 0$.

In order to study the solutions of the dual problem, note that it is strictly concave, implying that there exists a solution if and only if the first-order condition (FOC) with respect to $\lambda$ is satisfied:

$$\sum_{k=1}^{n} \left( 1 + \gamma \lambda_\gamma^*(R_k - R_f) \right)^{\frac{1}{\gamma}} (R_k - R_f) = 0, \text{ if } \gamma < 0, \quad (45)$$
$$\sum_{k=1}^{n} \left( 1 + \gamma \lambda_\gamma^*(R_k - R_f) \right)^{\frac{1}{\gamma}} I_{\Lambda_\gamma}(R_k)(\lambda_\gamma^*)(R_k - R_f) = 0, \text{ if } \gamma > 0. \quad (46)$$

In addition, a solution for the dual problem when $\gamma < 0$ must also satisfy $\Lambda_\gamma$. Note also that the FOC imposes that the solution of the dual problem satisfies the risk neutrality constraint of the primal problem.

(i) Let $\gamma < 0$. We are going to analyze how the expression given by the FOC (45) changes with respect to $\gamma$ and $\gamma \lambda_\gamma^*$. By the implicit function theorem, given a solution $(\gamma^*, \lambda^*)$ it is always possible to obtain a $\lambda$ satisfying the FOC in an open interval containing $\gamma^*$. Whenever $\gamma$ varies, $\gamma \lambda_\gamma^*$ will change in response in order to continue to set the FOC equal to zero. Therefore, we can look at the variations of the FOC with respect to $\gamma \lambda_\gamma^*$ and $\gamma$ separately. Being so, let $u \equiv \gamma \lambda_\gamma^*$. Recall that when $\gamma < 0$, if $\lambda_\gamma^* < 0$ ($\lambda_\gamma^* > 0$), we have
\( u > 0 \) \((u < 0)\). Rewriting the FOC:

\[
\sum_{k=1}^{n} (1 + u(R_k - R_f))^{\frac{1}{\gamma}} (R_k - R_f) = 0.
\]

Start from a negative \(\gamma\) close to zero for which the dual problem has a solution by the implicit function theorem, since for \(\gamma \to 0\) the problem is unconstrained and there is a solution. Then, \(\sum_{k=1}^{n} (1 + u(R_k - R_f))^{\frac{1}{\gamma}} (R_k - R_f) = 0\) and \(u\) satisfies \(\Lambda_\gamma\). Taking the derivative of the expression in the FOC with respect to \(\gamma\), if \(\lambda^* \gamma < 0\) \((\lambda^* \gamma > 0)\), we have:

\[
- \sum_{k=1}^{n} \frac{1}{\gamma^2} (1 + u(R_k - R_f))^{\frac{1}{\gamma}} \ln(1 + u(R_k - R_f))(R_k - R_f) < 0 \quad (> 0),
\]

because \((1 + u(R_k - R_f)) > 0\), since it is a solution and \(u\) satisfies \(\Lambda_\gamma\), and \(\ln(1 + u(R_k - R_f))\) has the same sign as \((R_k - R_f)\) if \(u > 0\) (the opposite sign if \(u < 0\)), so that the product inside the sum for each \(k\) is positive (negative). This implies that when we vary \(\gamma\) to be more negative, the FOC increases if \(\lambda^* \gamma < 0\), and decreases if \(\lambda^* \gamma > 0\). Now we take the derivative of the FOC with respect to \(u\):

\[
\sum_{k=1}^{n} \frac{1}{\gamma} (1 + u(R_k - R_f))^{\frac{1}{\gamma} - 1} (R_k - R_f)^2 < 0.
\]

That is, when \(u\) increases (decreases), the FOC decreases (increases). Therefore, when we are at a solution of the dual problem and we make \(\gamma\) slightly more negative, if \(\lambda^* \gamma < 0\) \((\lambda^* \gamma > 0)\), the FOC increases (decreases) and in order to continue having a solution \(u\) must increase (decrease) to compensate. Remember that a solution of the dual problem must also satisfy:

\[
1 + u(R_k - R_f) > 0 \quad \forall \ k \iff 1 + u \min\{R_k - R_f\} > 0 \iff u < \frac{-1}{\min\{R_k - R_f\}}, \quad \text{if} \ u > 0,
\]

\[
1 + u(R_k - R_f) > 0 \quad \forall \ k \iff 1 + u \max\{R_k - R_f\} > 0 \iff u > \frac{-1}{\max\{R_k - R_f\}}, \quad \text{if} \ u < 0.
\]

Thus, if \(u > 0\) \((u < 0)\), there is an upper (lower) bound for \(u\) to be a solution. Therefore, the problem can not compensate a smaller \(\gamma\) with a larger \(u\) indefinitely, in the case that \(u > 0\), or with a smaller \(u\) indefinitely if \(u < 0\). The dual problem breaks down when we can not make \(u\) larger or smaller in order to continue to have a solution.
More specifically, if $\lambda^*_\gamma < 0$ ($\lambda^*_\gamma > 0$), when $u \geq \frac{-1}{\min\{R_k - R_f\}}$ ($u \leq \frac{-1}{\max\{R_k - R_f\}}$) the dual problem does not have a solution anymore.

(ii) When $\gamma > 0$, if $\lambda^*_\gamma < 0$ ($\lambda^*_\gamma > 0$), we have $u < 0$ ($u > 0$). Rewriting the FOC (46):

$$\sum_{k=1}^{n} (1 + u(R_k - R_f))^\frac{1}{2} I_{\Lambda^*_\gamma(R_k)}(\lambda^*_\gamma)(R_k - R_f) = 0.$$

Start from a positive $\gamma$ close to zero for which the dual problem has a solution by the implicit function theorem, since for $\gamma \rightarrow 0$ the problem is unconstrained and there is a solution. Then, $\sum_{k=1}^{n} (1 + u(R_k - R_f))^\frac{1}{2} I_{\Lambda^*_\gamma(R_k)}(\lambda^*_\gamma)(R_k - R_f) = 0$. Taking the derivative of the expression in the FOC with respect to $\gamma$, if $\lambda^*_\gamma < 0$ ($\lambda^*_\gamma > 0$), we have:

$$-\sum_{k=1}^{n} \frac{1}{\gamma^2} (1 + u(R_k - R_f))^\frac{1}{2} I_{\Lambda^*_\gamma(R_k)}(\lambda^*_\gamma)\ln(1 + u(R_k - R_f))(R_k - R_f) > 0 \quad (< 0),$$

because $(1 + u(R_k - R_f)) > 0$, otherwise the indicator function would be set to zero, and $\ln(1 + u(R_k - R_f))$ has the same sign as $(R_k - R_f)$ if $u > 0$ (the opposite sign if $u < 0$), so that the product inside the sum for each $k$ is positive (negative) or zero if there is an active indicator function. This implies that when we increase $\gamma$, the FOC increases if $\lambda^*_\gamma < 0$, and decreases if $\lambda^*_\gamma > 0$. Now we take the derivative of the FOC with respect to $u$:

$$-\sum_{k=1}^{n} \frac{1}{\gamma^2} (1 + u(R_k - R_f))^\frac{1}{2} I_{\Lambda^*_\gamma(R_k)}(\lambda^*_\gamma)(R_k - R_f)^2 < 0.$$
functions are active. If $\lambda^*_i < 0$ ($\lambda^*_i > 0$), when $u \leq \max\{R_k - R_f\}$ ($u \geq \min\{R_k - R_f\}$), the risk-neutral measure sets the maximum (minimum) return to zero. From that point on, the implied measure is not strictly positive anymore. However, the dual problem will continue to have a solution, because it is unconstrained and the measures will be able to continue to zero out the positive (negative) returns in order to satisfy the FOC. Let $\max_i$ and $\min_{>0}$ ($\min_i$ and $\max_{<0}$) denote the $i^{th}$ highest (smallest) value and the minimum positive (maximum negative) value, respectively. The second zero in the measure will appear when $u \leq \frac{-1}{\max_2\{R_k - R_f\}}$ ($u \geq \frac{-1}{\min_2\{R_k - R_f\}}$), and so on. This will continue in this fashion until the last positive (negative) return is set to zero, i.e., when $u \leq \frac{-1}{\min_{>0}\{R_k - R_f\}}$ ($u \geq \frac{-1}{\max_{<0}\{R_k - R_f\}}$). From this point on, the dual problem does not have a solution anymore, because the measure puts weights only on negative (positive) returns.

2.4 Constructing Minimum Dispersion Price Bounds from $K$ Basis Assets

In the main paper, we focus on obtaining option price bounds from the returns of a single risky asset (the underlying). In this subsection, we show how to construct minimum dispersion price bounds for any non-redundant asset from returns on $K$ basis assets. The next proposition provides guidance on how to identify the set of admissible minimum dispersion measures in this case.

Proposition 4. Consider the dual problem as in Corollary 3.

(i) Let $\gamma < 0$. Restriction $\Lambda_\gamma$ is equivalent to $0 < \max_k\{\lambda^*_\gamma (R_k - R_f)\} < -1/\gamma$. The dual problem has a solution while this restriction is satisfied. As $\gamma \to -\infty$, the dual problem does not satisfy the constraint and does not have a solution anymore.

(ii) Let $\gamma > 0$. The indicator function $I_{\Lambda_\gamma\{R_k\}}$ is equal to one for all $k$ if $0 > \min_k\{\lambda^*_\gamma (R_k - R_f)\} > -1/\gamma$. A strictly positive solution is guaranteed while this restriction is satisfied. As $\gamma \to \infty$, the implied risk-neutral measure will not be strictly positive anymore.

Proof. Consider the first-order conditions for the dual problem for $\gamma < 0$ (it will be analogous for $\gamma > 0$):

$$\sum_{k=1}^{n} (1 + \gamma \lambda^*_\gamma (R_k - R_f)) \hat{z} (R_k^j - R_f) = 0, \quad j = 1, \ldots, K. \quad (47)$$
This implies that the risk-neutral measure must price all the basis assets and any portfolio based on the basis assets. In particular, it must price the optimal portfolio:

$$\sum_{k=1}^{n} (1 + \gamma \lambda_k^*(R_k - R_f))^{\frac{1}{\gamma}} (\lambda_k^*(R_k - R_f)) = 0.$$  

(48)

The condition above has no solution if \(\lambda_k^*(R_k - R_f)\) does not alternate in sign, which would imply that the portfolio has a return larger than the risk-free asset in all states of nature, providing an arbitrage. Therefore, it has to alternate signs. This implies that

$$\max_k \{\lambda_k^*(R_k - R_f)\} > 0 \quad \text{and} \quad \min_k \{\lambda_k^*(R_k - R_f)\} < 0.$$  

Note also that \(\lambda_k^* \neq 0\), because otherwise the risk-neutral measure would be equal to 1 in all states of nature, not pricing the basis assets in the presence of risk premia in the economy.

(i) Consider the case with \(\gamma < 0\). A solution to the dual problem satisfies:

$$1 + \gamma \lambda_k^*(R_k - R_f) > 0 \quad \forall \ k,$$

which is equivalent to:

$$1 + \gamma \max_k \{\lambda_k^*(R_k - R_f)\} > 0,$$

(50)

because \(\gamma < 0\) and \(\max_k \{\lambda_k^*(R_k - R_f)\} > 0\), making \(\gamma \max_k \{\lambda_k^*(R_k - R_f)\} < 0\) the minimum value across the \(n\) states of nature. Combining (50) with the fact that \(\max_k \{\lambda_k^*(R_k - R_f)\} > 0\), we have:

$$0 < \max_k \{\lambda_k^*(R_k - R_f)\} < -\frac{1}{\gamma}.$$  

(51)

For \(\gamma \to 0\), the problem is unconstrained and there will be a solution. When \(\gamma \to -\infty\), condition (51) breaks down, and the dual problem does not have a solution. It remains to show that, as \(\gamma\) decreases, there will be a solution while (51) is satisfied. By the maximum theorem, the function \(h(\gamma)\) giving the \(argmax\) of the dual problem is continuous. Moreover, by the implicit function theorem, given a solution \((\gamma^*, \lambda^*)\) it is always possible to obtain a \(\lambda\) satisfying the first-order condition in an open interval containing \(\gamma^*\). Therefore, as \(\gamma\) decreases, \(\lambda\) can change to continue satisfying the first-order condition, where \(h(\gamma)\) changes continuously until it is not possible to satisfy condition (51).

(ii) Consider now that \(\gamma > 0\). The dual problem is unconstrained, but contains an indicator function that sets to zero weights to states of nature where \(1 + \gamma \lambda_k^*(R_k - R_f) \leq 0\).
Notice that $1 + \gamma \min_k \{\lambda^*_\gamma(R_k - R_f)\} > 0$ implies that $1 + \gamma \lambda^*_\gamma(R_k - R_f) > 0$ for all $k$, because since $\gamma > 0$, $\gamma \min_k \{\lambda^*_\gamma(R_k - R_f)\} < 0$ is the minimum value across states of nature. Therefore, there will be no zero weights as long as the following holds:

$$0 > \min_k \{\lambda^*_\gamma(R_k - R_f)\} > -\frac{1}{\gamma}. \quad (52)$$

For $\gamma \to 0$, (52) will hold. For $\gamma \to \infty$, condition (52) breaks down, and the risk-neutral measure is not strictly positive. Starting from $\gamma$ close to zero, as $\gamma$ increases, the implicit function theorem guarantees strictly positive solutions of the dual problem while (52) is satisfied, where the first-order condition is differentiable. When $\min_k \{\lambda^*_\gamma(R_k - R_f)\} \leq -1/\gamma$, the guaranteed solutions will not be strictly positive anymore.

The proposition above suggests an algorithm to find the set of admissible minimum dispersion risk-neutral measures. In practice, the results will kick in way before $\gamma$ approaches minus or plus infinity, where, as $\gamma$ becomes more negative, the constraint will be satisfied until some negative $\gamma$, while as $\gamma$ becomes larger, the measures will be strictly positive until a given $\gamma$ is reached. Therefore, one can solve the dual problem for a grid of $\gamma$'s becoming more negative and more positive, while calculating $\max_k \{\lambda^*_\gamma(R_k - R_f)\}$ and $\min_k \{\lambda^*_\gamma(R_k - R_f)\}$. The negative $\gamma$ where $\max_k \{\lambda^*_\gamma(R_k - R_f)\}$ approaches $-1/\gamma$ and positive $\gamma$ where $\min_k \{\lambda^*_\gamma(R_k - R_f)\} \leq -1/\gamma$ define the interval $[\gamma_\gamma, \gamma]$ of strictly positive minimum dispersion risk-neutral measures. This set is endogenously determined by the basis assets returns. It is also possible to include measures with zeros in some states of nature by allowing for $\gamma$'s greater than $\gamma$.

Since $[\gamma, \gamma]$ explicitly identifies all the measures within the set, it is possible to calculate the implied price of a non-redundant asset by each measure, given a grid. Moreover, because the risk-neutral measures are continuous functions of $\gamma$, the implied prices will change continuously with $\gamma$ and reach a maximum and minimum in the interval. One can then calculate upper and lower price bounds for the non-redundant asset by taking the maximum and minimum prices implied by the measures in $[\gamma, \gamma]$. More specifically, let $x$ be the non-redundant payoff we want to price. The lower minimum dispersion price

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6For $\gamma < 0$, the problem is constrained by $\Lambda \gamma$. Computationally, this constraint is imposed and will not be violated. Instead, the dual problem stops having a solution in practice when $\max_k \{\lambda^*_\gamma(R_k - R_f)\}$ gets arbitrarily close to $-1/\gamma$, up to the fifth decimal place. Solutions beyond this point will be associated to non-negligible pricing errors for the basis assets, which is the computational counterpart of the dual problem not having a solution anymore. For $\gamma > 0$, $\min_k \{\lambda^*_\gamma(R_k - R_f)\} > -1/\gamma$ is not a constraint to the problem, so it will become $\min_k \{\lambda^*_\gamma(R_k - R_f)\} \leq -1/\gamma$ at some point. An appropriate grid of $\gamma$'s is with 0.1 spacing. In general, the $\gamma$'s defining the pricing interval will be between -20 and 20.
bound solves:

\[ C = \min_{(\gamma^Q)} \frac{1}{R_f} \sum_{k=1}^{n} \pi^Q x_k, \text{ s.t. } \gamma \in [\gamma, \gamma], \]  

(53)

where the upper bound \(\overline{C}\) solves the corresponding maximum. If, \textit{a posteriori}, the asset price \(C\) is observed and \(C \in [\underline{C}, \overline{C}]\), one can recover the risk-neutral measure and the implied \(\gamma\) that correctly price the asset.

### 2.5 Proof of Proposition 3 in Main Paper

**Proposition 5.** Suppose a Black-Scholes economy with drift \(\mu\), risk-free rate \(r\) and volatility \(\sigma\). Then, there is an optimal Cressie-Read discrepancy indexed by \(\gamma^*\) for which the implied minimum dispersion option price equals the Black-Scholes price, given by:

\[ \gamma^* = -\frac{\sigma^2}{\mu - r}. \]  

(54)

**Proof.** Rubinstein (1976) proves that if the underlying stock price and the SDF are jointly lognormal, the Black-Scholes option pricing formula obtains without dynamic trading. A sufficient condition for the lognormality condition is that the SDF comes from a CRRA utility function, i.e., \(m = R^{-\eta}\) with \(\eta > 0\), where \(R\) is the compound gross rate of return of the underlying asset through the expiration \(t\) of the option. The parameter \(\eta\) can be obtained from the stochastic process of \(R\). From the pricing equation \(1 = \mathbb{E}[m R]\), where \(m\) is an admissible SDF, we have \(1 = \text{cov}(R, m) + \mathbb{E}[R]\mathbb{E}[m]\). Since \(\mathbb{E}[m] = 1/R_f\), we can write this equation as:

\[ R_f = \frac{\text{Cov}(R, m)}{\mathbb{E}[m]} + \mathbb{E}[R], \]  

(55)

where \(R_f = (1 + r)^t\), \(r = e^{r_c} - 1\) and \(r_c\) is the continuous time risk-free rate of the model. Substituting \(m = R^{-\eta}\), we have:

\[ (1 + r)^t = \frac{\text{Cov}(R, R^{-\eta})}{\mathbb{E}[R^{-\eta}]} + \mathbb{E}[R], \]  

(56)

\[ \Rightarrow (1 + r)^t = \frac{\mathbb{E}[R^{1-\eta}]}{\mathbb{E}[R^{-\eta}]]. \]  

(57)

Under lognormality of \(R\), \(R^{1-\eta}\) and \(R^{-\eta}\) are lognormal, implying that:

\[ (1 + r)^t = \frac{\exp[(1 - \eta)\mathbb{E}(\ln R) + \frac{1}{2}(1 - \eta)^2\text{Var}(\ln R)]}{\exp[(-\eta)\mathbb{E}(\ln R) + \frac{1}{2}\eta^2\text{Var}(\ln R)]}, \]  

(58)
which can be simplified to:

$$\eta = \frac{\mathbb{E}(\ln R) - t \ln(1 + r)}{\text{Var}(\ln R)} + \frac{1}{2}. \quad (59)$$

Under the Black-Scholes economy, $\ln R \sim N((\mu - \sigma^2/2)t, \sigma^2t)$. Using this in (59), we have:

$$\eta = \frac{\mu - r_c}{\sigma^2}. \quad (60)$$

From Proposition 1, when $b = 0$ and $a = -1/\gamma$, the minimum dispersion risk-neutral measure is equivalent to the one implied by CRRA utility. Comparing with $m = R^{-\eta}$, we have $\gamma = -1/\eta$. Therefore, the Cressie-Read discrepancy that generates the risk-neutral measure for which the Black-Scholes option pricing equation is valid is given by:

$$\gamma^* = -\frac{\sigma^2}{\mu - r_c}. \quad (61)$$

### 2.6 Proof of Proposition 4 in Main Paper

We follow Cerny (2003) in measuring the attractiveness of a self-financing investment by the certainty equivalent of the resulting wealth $W$ relative to the wealth of a riskless investment. The value of the best deal in the market with excess returns $R$, denoted $\alpha(R)$, is defined implicitly as:

$$u^\gamma(w_0 R_f + \alpha(R)) \equiv \max \mathbb{E}[u^\gamma(w_0 R_f + \lambda'(R - R_f))]. \quad (62)$$

The next proposition derives the SDF moment restrictions implied by the class of HARA utility functions.

**Proposition 6.** Consider the class of HARA utility functions as in (37) and the environment described in Proposition 1. Then, the following SDF moment restrictions hold for $\gamma \in (-\infty, \infty)$:

$$\left(1 - \gamma A^\gamma(w_0)\gamma_{\text{basis}}\right)^{-(\gamma + 1)} \leq \mathbb{E}[m^{\gamma + 1}] \leq \left(1 - \gamma A^\gamma(w_0)\gamma_{\text{basis}}\right)^{-(\gamma + 1)}, \quad (63)$$

s.t. $\mathbb{E}[m(R - R_f)] = 0$, $m \geq 0$.

---

7The limiting cases of $\gamma \to 0$ and $\gamma \to -1$ are proved in Cerny (2003).
where \( w_0 = W_0R_f \), \( A^\gamma(w_0) = a/(b - \gamma aw_0) \), \( \sigma_\gamma \geq \alpha_{\gamma_{\text{basis}}} \) and \( \alpha_{\gamma_{\text{basis}}} \), the certainty equivalent of the best deal attainable in the market containing only the basis assets, is given by:

\[
\alpha_{\gamma_{\text{basis}}} = \min_m \alpha_\gamma(m), \quad \text{s.t. } \mathbb{E}[m(R - R_f)] = 0, \quad m \geq 0.
\]  

(64)

The restrictions above can be interpreted as reward-for-risk measures as follows:

\[
(1 + h_{\gamma_{\text{basis}}}^2)^{\frac{\gamma+1}{2}} \leq \mathbb{E}[m^{\gamma+1}] \leq \left(1 + h_{\gamma}^2\right)^{\frac{\gamma+1}{2}},
\]

\[
\text{s.t. } \mathbb{E}[m(R - R_f)] = 0, \quad m \geq 0,
\]

(65)

where \( h_{\gamma}^2 = 2A^\gamma(w_0)\alpha_\gamma \) is the generalized Sharpe ratio.

Proof. Suppose that the market containing basis assets is complete and the state prices are given by a unique state-price density \( m \). We aim to find the maximum certainty equivalent gain \( \alpha(m) \) in this market. To that end, we search for the optimal distribution of wealth, subject to the budget constraint imposed by \( m \):

\[
\max_\lambda \mathbb{E}[u^\gamma(w_0 + \lambda'(R - R_f))] = \max_W \mathbb{E}[u^\gamma(W)] \quad \text{s.t. } \mathbb{E}[mW] = w_0,
\]

(66)

where \( w_0 = W_0R_f \), which implies:

\[
u^\gamma(w_0 + \alpha_\gamma(m)) = \max_W \mathbb{E}[u^\gamma(W)] \quad \text{s.t. } \mathbb{E}[mW] = w_0.
\]

(67)

Following Cerny (2003), problem (67) can be solved using unconstrained maximization separately in each state with a Lagrange multiplier \( \delta \):

\[
\max_{W(\omega), \omega \in \Omega} u^\gamma(W(\omega)) - \delta m(\omega)W(m).
\]

(68)

The first-order conditions give \( a(b - a\gamma W)^{\frac{1}{\gamma}} = \delta m \), which can be written as:

\[
W = \left(\frac{-1}{a\gamma}\right) \left(\frac{\delta m}{a}\right)^\gamma + \frac{b}{a\gamma}.
\]

(69)

We can substitute (69) in the restriction \( \mathbb{E}[mW] = w_0 \) to recover \( \delta \):

\[
\mathbb{E}[m^{\gamma+1}] \left(\frac{\delta}{a}\right)^\gamma \left(\frac{-1}{a\gamma}\right) + \mathbb{E}[m] \frac{b}{a\gamma} = w_0,
\]
\[
\Rightarrow \delta^\gamma = \frac{a^\gamma(b - a^\gamma w_0)}{\mathbb{E}[m^{\gamma+1}]}.
\] (70)

Substituting back in (69), we have:

\[
W = \left(w_0 - \frac{b}{a^\gamma}\right) \frac{m^\gamma}{\mathbb{E}[m^{\gamma+1}]} + \frac{b}{a^\gamma},
\] (71)

\[
\Rightarrow u^\gamma(W) = -\frac{1}{\gamma + 1}\left[b - a^\gamma \left(w_0 - \frac{b}{a^\gamma}\right) \frac{m^\gamma}{\mathbb{E}[m^{\gamma+1}]} - b\right]^{\frac{\gamma+1}{\gamma}},
\]

\[
\Rightarrow \mathbb{E}[u^\gamma(W)] = -\frac{1}{\gamma + 1}(b - a^\gamma w_0)^{\frac{\gamma+1}{\gamma}} \mathbb{E}[m^{\gamma+1}]^{-\frac{1}{\gamma}}.
\] (72)

To recover the certainty equivalent of the optimal risky investment, we use (72) in (67):

\[
-\frac{1}{\gamma + 1}(b - a^\gamma \alpha_\gamma - a^\gamma w_0)^{\frac{\gamma+1}{\gamma}} = -\frac{1}{\gamma + 1}(b - a^\gamma w_0)^{\frac{\gamma+1}{\gamma}} \mathbb{E}[m^{\gamma+1}]^{-\frac{1}{\gamma}},
\]

\[
\alpha_\gamma(m) = \frac{(b - a^\gamma w_0) - a^\gamma \mathbb{E}[m^{\gamma+1}]^{-\frac{1}{\gamma}}}{a^\gamma} + \frac{(b - a^\gamma w_0)}{a^\gamma}.
\] (73)

Noting that \( A^\gamma(w_0) = a/(b - a^\gamma w_0) \) is the absolute risk aversion in initial wealth, we have:

\[
\alpha_\gamma(m) = -\frac{1}{\gamma A^\gamma(w_0)} \mathbb{E}[m^{\gamma+1}]^{-\frac{1}{\gamma+1}} + \frac{1}{\gamma A^\gamma(w_0)}.
\] (74)

The link between the complete and incomplete market is provided by the extension theorem, which asserts that any incomplete market without good deals can be embedded in a complete market that has no good deals. Denote by \( \alpha_{\gamma,\text{basis}} \) the certainty equivalent of the best deal attainable in the market containing only basis assets. Cerny (2003) shows that the extension theorem implies that:

\[
\alpha_{\gamma,\text{basis}} = \min_m \alpha_\gamma(m),
\] (75)

where \( m \) must price correctly all basis assets. Suppose that we want to find all prices of a non-redundant asset that do not provide good deals of size \( \alpha_\gamma \) in the enlarged market. From the extension theorem, all such prices must be supported by pricing kernels for which \( \alpha_\gamma(m) \leq \alpha_{\gamma,\text{basis}} \). This is the dual no-good-deal discount factor restriction. Therefore, by re-writing (74), the no-good-deal SDF restrictions are:

\[
(1 - \gamma A^\gamma(w_0) \alpha_{\gamma,\text{basis}})^{-\gamma+1} \leq \mathbb{E}[m^{\gamma+1}] \leq (1 - \gamma A^\gamma(w_0) \alpha_{\gamma})^{-\gamma+1}.
\] (76)
Now we interpret the restrictions (76) as reward-for-risk measures that are close in spirit to the Sharpe ratio. Cerny (2003) shows that there is an asymptotic relationship between the certainty equivalent $\alpha$ and the Sharpe ratio $h$: $A^\gamma(w_0)\alpha_\gamma = h_\gamma^2 / 2$. However, this transformation is not unique. For all $\kappa$ we have:

$$
(1 - \gamma A^\gamma(w_0)\alpha_\gamma)^{-(\gamma+1)} = \left(1 - \frac{h_\gamma^2}{2}\right)^{-(\gamma+1)} = \left(1 - \kappa \frac{h_\gamma^2}{2}\right)^{-(\gamma+1)}.
$$

The ambiguity is resolved by maintaining the consistency with the arbitrage-adjusted Sharpe ratio, for which the duality, as shown by Cerny (2003), is $E[m^2] = 1 + h_A^2$. The HARA utility function obtains the truncated quadratic utility when $\gamma = 1$. Therefore, by choosing $\kappa = -2/\gamma$, we obtain:

$$
(1 + h_\gamma^2)^{\frac{\gamma(\gamma+1)}{2}} = 1 + h_\gamma^2 = 1 + h_A^2.
$$

The restrictions in terms of generalized Sharpe ratios then become:

$$
(1 + h_{\gamma_{basis}}^2)^{\frac{\gamma(\gamma+1)}{2}} \leq E[m^{\gamma+1}] \leq (1 + h_\gamma^2)^{\frac{\gamma(\gamma+1)}{2}}.
$$

3 Additional Empirical Results

3.1 Summary Statistics - Berkeley Options Database

Summary statistics for the S&P 500 index option data after applying our filtering to the Berkeley Options Database are reported in Table 1. The average call price ranges from $2.29 for short-term OTM to $38.66 for long-term ITM, while the average put price is between $2.64 for short-term OTM and $29.43 for long-term ITM. For both calls and puts, the majority of options are ATM. Moreover, short-term options represent approximately 50% of the total sample. Table 1 also reports the average implied volatilities in each moneyness ($S/X$) and maturity category. The implied volatilities of short-term call and put options present a smile in moneyness. For medium and long-term options, implied volatility is increasing in moneyness for calls and decreasing for puts. Implied volatilities are also mostly increasing in time to expiration. The differences in implied volatilities are more pronounced for short-term options.
3.2 Option Price Bounds and Implied $\gamma$ - Berkeley Options Database

We calculate price bounds from the underlying returns for each option in our sample. Table 2 reports, for the Berkeley Options Database, the percentage of times that observed option prices are contained in the bounds, the percentage of upper and lower bound violations and the average tightness of the bounds, for each option category. Focusing first on the aggregate results, almost all option prices (97.68% of the calls and 94.40% of the puts) are consistent with the SSD bounds. Most of the observed prices are also contained in the tighter MD bounds (95.42% of the calls and 91.71% of the puts). The larger number of violations than for the OptionMetrics database can be explained by the presence of the 1987 crash in the Berkeley data. Before the crash, there were more violations as investors were pricing options using the Black-Scholes formula, ignoring information from the underlying returns. Even so, the vast majority of option prices still lies within the MD and SSD bounds, confirming the evidence that option prices are mostly consistent with underlying returns when we consider incomplete markets.

Detailing the analysis at the level of option categories, the options that most violate the bounds are short-term ITM calls and ITM puts, where 87.82% and 73.66% of these options are contained by the MD bounds, respectively. Similarly to the OptionMetrics database, this may be due to the fact that ITM options are less liquid and may present some unreliable prices. On the other hand, 95.45% of the short-term OTM puts are consistent with the MD bounds. This contradicts the common notion that it is hard to reconcile the left tail of the risk-neutral distribution. As the time to maturity increases for medium- and long-term options, the percentage of options contained in the bounds increases. That is, the longer the maturity, the easier it is to reconcile the information from option prices and underlying returns.

We also calculate the MDNA bounds, identifying prices consistent with strictly positive risk-neutral measures. The MDNA bounds capture 86.62% of the calls and 82.77% of the puts, leaving unexplained some of the option prices. However, a specific pattern appears when we look at the categories of options. ITM calls and OTM puts can be explained by strictly positive measures, with the much tighter MDNA bounds capturing most of the observed prices. In contrast, OTM calls and ITM puts clearly require risk-neutral measures with zeros in some states of nature to be priced, as the number of MDNA lower bound violations are considerably large. This indicates that to recon-
cile the right tail of the risk-neutral distribution, risk-neutral measures identified from the physical distribution of underlying returns need to decrease the probability mass in positive returns to make them compatible with option prices.

Table 3 reports the average implied $\gamma$ over the 1987-1995 sample for each option category. On average, the implied $\gamma$ of call options is decreasing in moneyness ($S/X$), while it is increasing for put options. That is, option prices present a smirk of the implied $\gamma$, indicating the existence of heterogeneous marginal investors in a segmented option market. OTM puts (and ITM calls) are priced by investors with positive prudence, convex marginal utilities and aversion to downside risk. On the other hand, OTM calls (and ITM puts) require large positive $\gamma$'s to be priced, associated to concave marginal utilities (except for long-term options, for which the implied $\gamma$'s are positive but smaller than one). The heterogeneity of marginal investors is more pronounced for short-term options, decreasing as the maturity increases.

Overall, the results for the Berkeley Options Database are qualitatively similar to those for the OptionMetrics data, providing additional robustness to our analysis.

### 3.3 Marginal Risk-Neutral Measures and Implied $\gamma$ - Medium- and Long-Term Options

Figures 1 and 2 plot, for 1996-2019 and 1987-1995, respectively, the two-month moving average of the implied $\gamma$ and the $\bar{\gamma}$ and $\tilde{\gamma}$ defining the MDNA bounds, for medium- and long-term calls and puts with different moneyness. As can be seen, the plots follow closely the same patterns carefully discussed in the main paper. The main difference is that the heterogeneity of marginal minimum dispersion risk-neutral measures gets smaller as the maturity increases. Even so, the heterogeneity is time-varying and mainly driven by the implied $\gamma$ of OTM calls and ITM puts.

### 3.4 Deep Out-of-the-Money and In-the-Money Options

In this section, we show that our results are robust to extending the range of moneyness considered in the main paper to deep out-of-the-money (DOTM) and deep in-the-money (DITM) options. In particular, we group options in two intervals: DOTM call (DITM put) if $S/X \in [0.80, 0.90]$ and DITM call (DOTM put) if $S/X \in (1.10, 1.60]$. Although these options have not been considered in most of the literature cited in the
paper, they have been increasingly traded in recent years, so that it is worth to consider them as a robustness check. We report the results for the OptionMetrics Database, which includes the recent sample where DOTM and DITM options are more liquid.

Table 4 presents the summary statistics for the S&P 500 DOTM and DITM options. The average call price ranges from $5.85 for short-term DOTM to $344.08 for long-term DITM, while the average put price is between $5.41 for short-term DOTM and $227.20 for long-term DITM. The number of observations for DOTM puts is much higher than for DITM puts, while it is similar for DOTM and DITM calls. Table 4 also reports the average implied volatilities in each moneyness-maturity category. The average implied volatilities of DOTM and DITM calls and puts are decreasing in time to maturity, indicating that the implied volatility smile is more pronounced for short-term options.

We calculate price bounds from the underlying returns for each of the DOTM and DITM options. Table 5 reports, for each deep moneyness-maturity category, the percentage of times that observed option prices are contained in the bounds, the percentage of upper and lower bound violations and the average tightness of the bounds. The options that most violate the bounds are short-term DITM calls and DITM puts, where 84.97% and 75.79% of these options are contained by the MD bounds, respectively. This is due to the fact that DITM options are much less liquid and may present unreliable prices. On the other hand, 94.38% and 91.53% of the short-term DOTM calls and puts are consistent with the MD bounds, respectively. In particular, the 8.47% of violations for short-term DOTM puts are basically all MD upper bound violations. The SSD bounds are able to reconcile these prices, but with much wider bounds. As the time to maturity increases for medium- and long-term options, the percentage of options contained in the bounds increases. As for the tighter MDNA bounds, they capture most of the DITM calls and DOTM puts. That is, these options are reconciled by strictly positive risk-neutral measures. Overall, the results show that DOTM and DITM options can be reconciled with underlying returns in incomplete markets.

Table 6 provides information on how DOTM and DITM options are reconciled by reporting the average implied $\gamma$ over the 1996-2019 sample for each deep moneyness-maturity category. As expected, on average DOTM calls (and DITM puts) are reconciled by non-prudent investors, while DOTM puts (and DITM calls) by prudent investors. Moreover, the results further highlight the differences in skew patterns associated to the implied $\gamma$ and the implied volatility. From the comparison of Table 4 in this Online
Appendix with Table 1 in the main paper, we can see that DOTM calls and DOTM puts are relatively more expensive than OTM calls and OTM puts, respectively, in terms of implied volatility. This is in contrast with the comparison of Table 6 in this Online Appendix with Table 3 in the main paper. First, DOTM calls are relatively cheaper than OTM calls (larger positive $\gamma$'s) when we take into account physical underlying returns with the implied $\gamma$. Second, DOTM puts are relatively more expensive than OTM puts, but only at a marginal level. In particular, the average implied $\gamma$ of DOTM short-term puts is approximately the same of that of OTM short-term puts, confirming that the implied $\gamma$ curve is usually flat for OTM puts.

4 Robustness Analysis

4.1 Implied $\gamma$, Implied Volatility and Sampling Uncertainty

The implied $\gamma$ represents an alternative to the implied volatility for interpreting the structure of option prices, i.e., for drawing relative value comparisons between options in the cross-section. An implied volatility smile denotes deviations from the lognormal SPD associated to the Black-Scholes model with ATM implied volatility. That is, options are relatively more expensive in terms of implied volatility if they require a lognormal distribution with higher volatility to be priced. In contrast, an implied $\gamma$ smile denotes deviations from the minimum dispersion SPD according to the ATM implied $\gamma$. Options are relatively more expensive in terms of implied $\gamma$ if they need a minimum dispersion SPD associated to a smaller implied $\gamma$ to be priced, which means they require larger probability weights in extreme underlying returns from the physical distribution. Therefore, the implied $\gamma$ provides relative value comparisons using information from the physical distribution, while the implied volatility draws comparisons assuming that underlying returns are lognormal.

In practice, the physical distribution used to compute the implied $\gamma$ must be estimated from data on underlying returns. Hence, the implied $\gamma$ measures the relative option expensiveness given the estimated physical distribution. While estimating the conditional physical distribution of S&P 500 returns is a challenging task and there are different methods available to use, one empirical stylized fact is that the distribution is not lognormal. For instance, it is well-known that the S&P 500 return distribution is negatively
skewed and has fat tails. Therefore, any sensible estimation of the physical distribution incorporating the empirical patterns of skewness and kurtosis makes the implied \( \gamma \) more reliable than the implied volatility in terms of providing relative option expensiveness given physical underlying returns.

One advantage of implied volatility is that its computation is not subject to sampling uncertainty, while the implied \( \gamma \) depends on the sample of the estimated conditional physical distribution. In order to assess the sampling uncertainty associated to the computation of the implied \( \gamma \), we propose to construct confidence intervals using bootstrap. The bootstrap was introduced by Efron (1979) and is a widely used method for estimating the distribution of an estimator by resampling the data. Under mild regularity conditions, the bootstrap yields an accurate approximation to the distribution of an estimator and confidence intervals (see Horowitz, 2001).

We propose a standard nonparametric bootstrap procedure to calculate confidence intervals for the implied \( \gamma \), as follows. For a given date \( t \) and time to maturity \( T \), estimate the conditional physical distribution as described in the main paper. Then, apply our method to obtain the implied \( \gamma_j \) for each option \( j \) with maturity \( T \) in the cross-section. To calculate the bootstrap confidence intervals, generate 1000 bootstrap underlying return samples by sampling from the estimated conditional return distribution randomly with replacement. For each bootstrap sample, compute the implied \( \gamma_j^* \) for each option and calculate \( \delta_j^* = \gamma_j^* - \gamma_j \). Then, a 95% confidence interval for the implied \( \gamma \) can be calculated as \([\gamma_j - \delta_{j,0.975}, \gamma_j - \delta_{j,0.025}]\), where \( \delta_{j,0.975} \) and \( \delta_{j,0.025} \) are the 97.5th and 2.5th percentiles of the sorted list of \( \delta_j^* \) across bootstrap samples, respectively.

We illustrate the sampling uncertainty associated to the computation of the implied \( \gamma \) by calculating 95% bootstrap confidence intervals for implied \( \gamma \)'s obtained from the options with time to maturity closest to two months on January 28, 2015. The sampling uncertainty patterns for this date are representative of the rest of the sample. Figure 3 depicts the results. As can be seen, the confidence intervals are fairly tight around implied \( \gamma \) estimates, especially for negative implied \( \gamma \)'s associated to OTM put options. Most importantly, Figure 3 shows that sampling uncertainty is rather small and does not account for the implied \( \gamma \) smirk pattern. That is, the market segmentation in terms of heterogeneous marginal investors in the cross-section, where marginal investors in OTM puts (OTM calls) are prudent (non-prudent), is statistically significant.

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8See, for instance, Bakshi, Kapadia and Madan (2003).
4.2 Option Price Bounds, Implied $\gamma$ and Moments of the Return Distribution

In order to grasp why the approach we use to estimate the conditional physical distribution is appropriate, it is useful to understand how the different moments of the return distribution affect the estimation of the option price bounds and implied $\gamma$. As detailed in the main paper, we first estimate the unconditional return distribution using a long sample, in order to estimate the tails of the physical distribution as reliably as possible. For the mean, we impose the economic restriction of a 5% lower bound on the annualized equity premium. Then, we make the distribution conditional by adjusting for the conditional volatility at time $t$. We estimate the conditional volatility in a forward looking manner by discounting a premium from the ATM implied volatility. To illustrate the effects of the estimated moments on the price bounds and implied $\gamma$'s, we again consider the options with time to maturity closest to two months on January 28, 2015, for which the discussed patterns are representative.

We start by analyzing the sensitivity of the option price bounds and implied $\gamma$ curve to the mean of the physical return distribution. Figure 4 depicts in blue the MDNA bounds (solid lines) and the MD lower bound (dashed line) in the left panel (converted to implied volatilities) and the implied $\gamma$ curve in the right panel, obtained from the original estimated physical distribution. The annualized return mean at January 28, 2015 was 7.91%. The sensitivity results in red (yellow) are obtained by decreasing (increasing) the annualized mean of the distribution by 2%. As can be seen, changing the mean has little effect on the lower bounds, while an increase (decrease) in the mean widens (tightens) the upper bound. Consequently, implied $\gamma$'s of OTM calls are mostly not affected, as they mainly consist of positive $\gamma$'s that are more sensitive to changes in the lower bounds. On the other hand, the smaller the return distribution mean the more negative are the implied $\gamma$'s of OTM puts, as the upper bound gets tighter. Therefore, positive implied $\gamma$'s are more robust to changes in the return mean. Even so, the effect for negative implied $\gamma$'s is rather small. More importantly, the mean does not affect the overall shape of the implied $\gamma$ curve. This illustrates how our results are mostly robust to the mean of the physical distribution.\footnote{It also provides motivation for the 5% lower bound on the equity premium. For the few dates where the annualized equity premium is below 5%, we demean the returns and reintroduce a 5% premium. This avoids option price bounds that are unreasonably tight. Even so, results are practically unchanged by this restriction.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Option price bounds and implied $\gamma$ curve for different mean values.}
\end{figure}
Figure 5 depicts how the volatility of the physical return distribution affects the option price bounds and implied $\gamma$ curve. Results in blue are obtained from the original estimated conditional distribution, for which the estimated conditional (annualized) physical volatility is 16.80%. The sensitivity results in red (yellow) are obtained by decreasing (increasing) the annualized volatility of the distribution by 2%. As can be noted in the left panel, the underlying return physical volatility affects the level of the price bounds in terms of implied volatilities. This underscores the importance of properly accounting for the conditional volatility in the estimation of the physical distribution. For instance, an unconditional return distribution (with unconditional volatility) does not account for the current state of the economy. This can result in option price bounds misaligned with observed option prices, which by nature depend on investors’ expectations at time $t$ of future return volatility. That is, the level of option price bounds can be dislocated from the level of the implied volatility curve. A common approach is to use GARCH models to estimate the conditional volatility at time $t$. However, GARCH models are subject to model misspecification and are inherently backward-looking, as they rely only on past underlying returns. This can again result in mismatch between the level of price bounds and the level of option prices. In practice, investors use all information available to them to form expectations about future volatility.

The approach we use to estimate the conditional volatility avoids the issues discussed above. The ATM implied volatility is by nature forward-looking, reflecting market’s expectation at time $t$ about uncertainty regarding future underlying returns. Since it contains a volatility risk premium, we estimate the physical volatility by discounting a premium using the simulations in Section 4 of the main paper. In this manner, our estimate of the conditional return distribution anchors the option prices bounds at the appropriate conditional level at time $t$. That is, it properly accounts for investors’ expectations about future physical volatility. Other papers in the literature also use the ATM implied volatility to estimate physical volatility. For instance, Dew-Becker, Giglio and Kelly (2021) use the ATM implied volatility directly as a proxy for the physical volatility, assuming that risk premium is quantitatively small. They show that implied volatilities provide very good summaries of the available information in the data for forecasting future volatility, driving out other standard uncertainty measures from forecasting regressions. We are more conservative and opt to discount a premium to obtain the conditional physical volatility.
The right panel of Figure 5 illustrates how the mismatch between the level of the option price bounds and the level of option prices can affect the implied \( \gamma \) curve. With a too high return volatility (in yellow), option prices are too close to the MD lower bound (in terms of implied volatilities), making OTM calls and ATM options be reconciled by larger implied \( \gamma \)'s. In contrast, if return volatility is too low (in red), OTM call and ATM option prices are far from the MD lower bound, being reconciled by smaller implied \( \gamma \)'s. Given the slope of the MD bounds in the moneyness region \([1.03, 1.10]\), the implied \( \gamma \)'s of OTM puts are less affected and, thus, are more robust to changes in the return volatility.

In order to assess how the skewness, kurtosis and tail probabilities of the return distribution affect the estimation of the option price bounds and implied \( \gamma \), we compare in Figure 6 the results obtained from our original estimated physical distribution with those from a lognormal distribution with the same mean and volatility.\(^{10}\) While the estimated physical distribution is negatively skewed and has fat tails, log-returns from the lognormal distribution are Gaussian, with skewness and kurtosis equal to zero and three, respectively. Therefore, this comparison isolates the effects of higher moments in the return distribution. Figure 6 shows that higher moments determine the slope of the option price bounds. The physical distribution estimated with our approach preserves the empirical patterns of skewness, kurtosis and tail probabilities, which makes the bounds generate pronounced implied volatility smirks, with slope similar to the observed smirk in the option cross-section. In contrast, the smirks generated by the bounds arising from the lognormal distribution are much less pronounced and nearly flat.

The slope of the option price bounds has important implications for the implied \( \gamma \) curve in the right panel of Figure 6. Under the lognormal distribution, OTM calls are cheaper (larger implied \( \gamma \)'s) and OTM puts are more expensive (smaller implied \( \gamma \)'s), in relative terms, as compared to under the estimated physical distribution. This is because the right (left) tail of the lognormal is thicker (thinner) than that of the estimated distribution, implying that less (more) probability mass in extreme positive (negative) underlying returns is necessary to price the OTM calls (OTM puts). That is, the relative option expensiveness patterns are affected by the higher moments of the return distribution. For instance, when we use our approach that properly accounts for the negative skewness and fat tails in the data, we can see that the implied \( \gamma \) curve for

\(^{10}\)That is, from a lognormal distribution with mean and volatility equal to the mean and volatility of the estimated physical return distribution, we sample the same number of observations that we have for the estimated physical distribution, and use them to calculate the option price bounds and implied \( \gamma \)'s.
OTM puts is nearly flat, that is, these options are generally not too expensive. Failing to account for the empirical higher moments with the lognormal distribution leads to a decreasing implied $\gamma$ curve in the moneyness region $[1.00, 1.10]$, i.e., it leads to relative expensiveness conclusions that are similar to those obtained via the implied volatility (which denotes deviations from a lognormal distribution).

In sum, the discussion above motivates the use of our approach for estimating the conditional physical distribution of underlying returns. First, it shows that our results are fairly robust to changes in the return distribution mean. Second, it indicates that our conditioning approach, where we discount a premium from the ATM implied volatility, properly accounts for the current state of the economy and investors’ expectation at time $t$ about future physical volatility, which in turn anchors the option price bounds at the appropriate level. Third, it reveals the importance of preserving the empirical patterns of skewness, kurtosis and tail probabilities in order to suitably obtain relative option expensiveness patterns given the physical distribution with the implied $\gamma$’s.
References


Cowles Foundation Discussion Papers 1569, Yale University.


<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Call Options</th>
<th>Put Options</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Short</td>
<td>Medium</td>
</tr>
<tr>
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<td>$2.29</td>
<td>$6.21</td>
</tr>
<tr>
<td></td>
<td>14.63%</td>
<td>15.09%</td>
</tr>
<tr>
<td></td>
<td>14.63%</td>
<td>15.09%</td>
</tr>
<tr>
<td>[0.97, 1.03)</td>
<td>$9.32</td>
<td>$16.27</td>
</tr>
<tr>
<td></td>
<td>14.19%</td>
<td>15.57%</td>
</tr>
<tr>
<td></td>
<td>14.19%</td>
<td>15.57%</td>
</tr>
<tr>
<td>[1.03, 1.10]</td>
<td>$27.16</td>
<td>$32.48</td>
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<td></td>
<td>19.04%</td>
<td>17.67%</td>
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<tr>
<td></td>
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<tr>
<td>Subtotal</td>
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</tr>
<tr>
<td></td>
<td>15.93%</td>
<td>16.08%</td>
</tr>
<tr>
<td></td>
<td>15.93%</td>
<td>16.08%</td>
</tr>
</tbody>
</table>

This table presents summary statistics of the S&P 500 index option data after applying our filtering to the Berkeley Options Database. The sample ranges from January 2, 1987 to December 29, 1995. The columns Short, Medium and Long refer to the maturity categories. For each moneyness ($S/X$) and maturity category, the first row depicts the average option price, the second row the average implied volatility and the third row the number of observations (in braces). The average of the daily values of the S&P 500 index and the (annualized) risk-free rate in the sample period were 383.92 and 5.51%, respectively.
Table 2: Option Price Bounds for S&P 500 Options (1987-1995)

<table>
<thead>
<tr>
<th>Category</th>
<th>MDNA Bounds</th>
<th></th>
<th>MD Bounds</th>
<th></th>
<th>SSD Bounds</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>In</td>
<td>Upper</td>
<td>Lower</td>
<td>In</td>
<td>Upper</td>
<td>Lower</td>
</tr>
<tr>
<td>Short OTM</td>
<td>47.94%</td>
<td>9.58% (1.36)</td>
<td>42.48% (0.85)</td>
<td>88.09%</td>
<td>9.58% (1.56)</td>
<td>2.33% (0.37)</td>
</tr>
<tr>
<td>Short ATM</td>
<td>89.41%</td>
<td>3.76% (1.18)</td>
<td>6.83% (0.93)</td>
<td>95.95%</td>
<td>3.76% (1.19)</td>
<td>0.28% (0.76)</td>
</tr>
<tr>
<td>Short ITM</td>
<td>85.10%</td>
<td>9.33% (1.05)</td>
<td>5.57% (0.96)</td>
<td>87.82%</td>
<td>9.33% (1.05)</td>
<td>2.85% (0.94)</td>
</tr>
<tr>
<td>Medium OTM</td>
<td>75.81%</td>
<td>1.37% (1.38)</td>
<td>22.83% (0.88)</td>
<td>98.41%</td>
<td>1.37% (1.50)</td>
<td>0.22% (0.45)</td>
</tr>
<tr>
<td>Medium ATM</td>
<td>99.48%</td>
<td>0.14% (1.18)</td>
<td>0.37% (0.90)</td>
<td>99.85%</td>
<td>0.14% (1.18)</td>
<td>0.01% (0.76)</td>
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<tr>
<td>Medium ITM</td>
<td>97.62%</td>
<td>0.34% (1.08)</td>
<td>2.04% (0.94)</td>
<td>98.85%</td>
<td>0.34% (1.08)</td>
<td>0.81% (0.90)</td>
</tr>
<tr>
<td>Long OTM</td>
<td>94.31%</td>
<td>0.69% (1.36)</td>
<td>5.00% (0.84)</td>
<td>99.23%</td>
<td>0.69% (1.39)</td>
<td>0.07% (0.51)</td>
</tr>
<tr>
<td>Long ATM</td>
<td>99.89%</td>
<td>0.05% (1.19)</td>
<td>0.05% (0.88)</td>
<td>99.94%</td>
<td>0.06% (1.19)</td>
<td>0.00% (0.75)</td>
</tr>
<tr>
<td>Long ITM</td>
<td>98.60%</td>
<td>0.05% (1.11)</td>
<td>1.35% (0.91)</td>
<td>99.12%</td>
<td>0.05% (1.11)</td>
<td>0.83% (0.86)</td>
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<tr>
<td>All Calls</td>
<td>86.62%</td>
<td>3.64% (1.19)</td>
<td>9.74% (0.91)</td>
<td>95.42%</td>
<td>3.64% (1.23)</td>
<td>0.94% (0.72)</td>
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<tr>
<td>Short OTM</td>
<td>92.07%</td>
<td>2.49% (1.90)</td>
<td>5.44% (0.70)</td>
<td>95.45%</td>
<td>2.49% (1.91)</td>
<td>2.06% (0.50)</td>
</tr>
<tr>
<td>Short ATM</td>
<td>85.02%</td>
<td>3.31% (1.20)</td>
<td>11.67% (0.91)</td>
<td>92.33%</td>
<td>3.31% (1.20)</td>
<td>4.35% (0.77)</td>
</tr>
<tr>
<td>Short ITM</td>
<td>40.59%</td>
<td>21.24% (1.02)</td>
<td>38.17% (0.98)</td>
<td>73.66%</td>
<td>21.24% (1.03)</td>
<td>5.10% (0.95)</td>
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<tr>
<td>Medium OTM</td>
<td>97.10%</td>
<td>2.36% (1.49)</td>
<td>0.54% (0.68)</td>
<td>97.43%</td>
<td>2.36% (1.49)</td>
<td>0.20% (0.49)</td>
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<tr>
<td>Medium ATM</td>
<td>96.80%</td>
<td>2.52% (1.20)</td>
<td>0.68% (0.86)</td>
<td>97.20%</td>
<td>2.52% (1.20)</td>
<td>0.28% (0.69)</td>
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<tr>
<td>Medium ITM</td>
<td>74.54%</td>
<td>4.52% (1.07)</td>
<td>20.94% (0.96)</td>
<td>94.76%</td>
<td>4.52% (1.08)</td>
<td>0.72% (0.88)</td>
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<tr>
<td>Long OTM</td>
<td>93.34%</td>
<td>6.55% (1.41)</td>
<td>0.10% (0.61)</td>
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<td>6.55% (1.41)</td>
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<tr>
<td>Long ATM</td>
<td>94.73%</td>
<td>5.02% (1.24)</td>
<td>0.24% (0.80)</td>
<td>94.92%</td>
<td>5.02% (1.24)</td>
<td>0.05% (0.64)</td>
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<tr>
<td>Long ITM</td>
<td>89.95%</td>
<td>5.69% (1.12)</td>
<td>4.37% (0.92)</td>
<td>94.03%</td>
<td>5.69% (1.12)</td>
<td>0.28% (0.81)</td>
</tr>
<tr>
<td>All Puts</td>
<td>82.77%</td>
<td>6.22% (1.35)</td>
<td>11.00% (0.82)</td>
<td>91.71%</td>
<td>6.22% (1.33)</td>
<td>2.07% (0.69)</td>
</tr>
</tbody>
</table>

This table presents empirical results for the minimum dispersion no-arbitrage (MDNA), minimum dispersion (MD) and stochastic dominance (SSD) option price bounds for S&P 500 options. For each category, column In reports the percentage of prices inside the bounds. The column Upper reports the percentage of violations of the upper bounds and, in parenthesis, the average ratio of the upper bound over the observed option price for the prices inside the bounds. Analogously, Lower reports the percentage of violations of the lower bounds and, in parenthesis, the average ratio of the lower bound over the observed option price for the prices inside the bounds. The sample ranges from January 2, 1987 to December 29, 1995.
<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Call Options</th>
<th>Put Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.90, 0.97)</td>
<td>6.62 3.13 0.36</td>
<td>7.71 2.99 0.17</td>
</tr>
<tr>
<td>[0.97, 1.03)</td>
<td>0.48 -0.69 -0.83</td>
<td>0.76 -0.88 -1.00</td>
</tr>
<tr>
<td>[1.03, 1.10]</td>
<td>-0.61 -0.83 -0.94</td>
<td>-0.08 -1.15 -1.23</td>
</tr>
</tbody>
</table>

This table presents the average implied $\gamma$ of S&P 500 index options for each moneyness ($S/X$) and maturity category. The sample ranges from January 2, 1987 to December 29, 1995. The columns Short, Medium and Long refer to the maturity categories.
Table 4: **Summary Statistics of S&P 500 Index Options (1996-2019) - Wider Range of Moneyness**

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Call Options</th>
<th>Put Options</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Short</td>
<td>Medium</td>
</tr>
<tr>
<td>[0.80, 0.90)</td>
<td>22.86%</td>
<td>17.58%</td>
</tr>
<tr>
<td>{3737}</td>
<td>{7288}</td>
<td>{15932}</td>
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<td>$249.60</td>
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<td>$344.08</td>
</tr>
<tr>
<td>(1.10, 1.60]</td>
<td>29.78%</td>
<td>25.46%</td>
</tr>
<tr>
<td>{20279}</td>
<td>{8841}</td>
<td>{8111}</td>
</tr>
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</table>

This table presents summary statistics of S&P 500 options from OptionMetrics Database with a wider range of moneyness than considered in the main paper. The sample ranges from January 4, 1996 to June 28, 2019. The columns Short, Medium and Long refer to the maturity categories. For each moneyness \((S/X)\) and maturity category, the first row depicts the average option price, the second row the average implied volatility and the third row the number of observations (in braces).
Table 5: Option Price Bounds for S&P 500 Options (1996-2019) - Wider Range of Moneyness

<table>
<thead>
<tr>
<th>Category</th>
<th>MDNA Bounds</th>
<th>MD Bounds</th>
<th>SSD Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>In</td>
<td>Upper</td>
<td>Lower</td>
</tr>
<tr>
<td>Panel A: Calls</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short DOTM</td>
<td>23.23%</td>
<td>2.03% (1.28)</td>
<td>74.74% (0.88)</td>
</tr>
<tr>
<td>Short DITM</td>
<td>83.06%</td>
<td>12.11% (1.02)</td>
<td>4.83% (0.98)</td>
</tr>
<tr>
<td>Medium DOTM</td>
<td>35.26%</td>
<td>1.36% (1.77)</td>
<td>63.38% (0.79)</td>
</tr>
<tr>
<td>Medium DITM</td>
<td>91.66%</td>
<td>5.25% (1.05)</td>
<td>3.09% (0.97)</td>
</tr>
<tr>
<td>Long DOTM</td>
<td>50.34%</td>
<td>1.50% (1.85)</td>
<td>48.16% (0.80)</td>
</tr>
<tr>
<td>Long DITM</td>
<td>94.45%</td>
<td>3.30% (1.08)</td>
<td>2.24% (0.95)</td>
</tr>
<tr>
<td>Panel B: Puts</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Short DOTM</td>
<td>91.09%</td>
<td>8.39% (2.79)</td>
<td>0.52% (0.44)</td>
</tr>
<tr>
<td>Short DITM</td>
<td>19.88%</td>
<td>11.08% (1.01)</td>
<td>69.04% (0.99)</td>
</tr>
<tr>
<td>Medium DOTM</td>
<td>92.35%</td>
<td>7.10% (2.30)</td>
<td>0.55% (0.41)</td>
</tr>
<tr>
<td>Medium DITM</td>
<td>35.49%</td>
<td>1.47% (1.02)</td>
<td>63.04% (0.99)</td>
</tr>
<tr>
<td>Long DOTM</td>
<td>95.47%</td>
<td>3.95% (2.01)</td>
<td>0.58% (0.44)</td>
</tr>
<tr>
<td>Long DITM</td>
<td>46.36%</td>
<td>2.40% (1.05)</td>
<td>51.24% (0.98)</td>
</tr>
</tbody>
</table>

This table presents empirical results for the minimum dispersion no-arbitrage (MDNA), minimum dispersion (MD) and stochastic dominance (SSD) option price bounds for S&P 500 options. For each category, column In reports the percentage of prices inside the bounds. The column Upper reports the percentage of violations of the upper bounds and, in parenthesis, the average ratio of the upper bound over the observed option price for the prices inside the bounds. Analogously, Lower reports the percentage of violations of the lower bounds and, in parenthesis, the average ratio of the lower bound over the observed option price for the prices inside the bounds. The sample ranges from January 4, 1996 to June 28, 2019.
Table 6: S&P 500 Options Implied $\gamma$ (1996-2019) - Wider Range of Moneyness

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Call Options</th>
<th>Put Options</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Short</td>
<td>Medium</td>
</tr>
<tr>
<td>[0.80, 0.90]</td>
<td>12.52</td>
<td>11.43</td>
</tr>
<tr>
<td>(1.10, 1.60]</td>
<td>-0.73</td>
<td>-0.64</td>
</tr>
</tbody>
</table>

This table presents the average implied $\gamma$ of S&P 500 index options for each moneyness ($S/X$) and maturity category. The sample ranges from January 4, 1996 to June 28, 2019. The columns Short, Medium and Long refer to the maturity categories.

Figure 1: This figure plots the 2-month moving averages of the mean implied $\gamma$ for OTM, ATM and ITM options and the mean $\gamma$ and $\bar{\gamma}$ defining the MDNA bounds, for medium- and long-term calls and puts. Shaded areas depict NBER recession dates. The sample ranges from January 4, 1996 to June 28, 2019.
Figure 2: This figure plots the 2-month moving averages of the mean implied $\gamma$ for OTM, ATM and ITM options and the mean $\gamma$ and $\bar{\gamma}$ defining the MDNA bounds, for medium- and long-term calls and puts. Shaded areas depict NBER recession dates and the vertical dashed line corresponds to the October 1987 market crash. The sample ranges from January 2, 1987 to December 29, 1995.

**Implied $\gamma$ and Confidence Intervals**

Figure 3: This figure plots the implied $\gamma$’s for call and put options with time to maturity closest to 60 days on January 28, 2015 and the corresponding 95% bootstrap confidence intervals.
Option Price Bounds, Implied $\gamma$ and Distribution Mean

Figure 4: This figure plots the MDNA and MD bounds and observed S&P 500 option prices converted to implied volatilities in the left panel and the corresponding implied $\gamma$ in the right panel, for the options with time to maturity closest to 60 days on January 28, 2015. The plots in blue are obtained from the original estimated physical distribution. The plots in red (yellow) are obtained by decreasing (increasing) the annualized mean of the return distribution by 2%.

Option Price Bounds, Implied $\gamma$ and Distribution Volatility

Figure 5: This figure plots the MDNA and MD bounds and observed S&P 500 option prices converted to implied volatilities in the left panel and the corresponding implied $\gamma$ in the right panel, for the options with time to maturity closest to 60 days on January 28, 2015. The plots in blue are obtained from the original estimated physical distribution. The plots in red (yellow) are obtained by decreasing (increasing) the annualized volatility of the return distribution by 2%.
Figure 6: This figure plots the MDNA and MD bounds and observed S&P 500 option prices converted to implied volatilities in the left panel and the corresponding implied $\gamma$ in the right panel, for the options with time to maturity closest to 60 days on January 28, 2015. The plots in blue are obtained from the original estimated physical distribution. The plots in red are obtained from a lognormal distribution with mean and volatility equal to the original estimated physical distribution.