Optimal Capital Structure and Bankruptcy Choice: Dynamic Bargaining vs Liquidation: Online Appendix

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Roadmap

This Online Appendix provides proofs and calculations referenced in the paper.

In Section A, we calculate the coefficient θ referenced at the end of Section 2 of the paper. We then prove a technical lemma which we use throughout the appendix. Section A concludes with the proof of proposition 1.

In Section B, we calculate the value function for the social planner who chooses the efficient time to exit Chapter 11. Section B concludes with a proof of Proposition 2.

In Section C, we prove Proposition 3, and then calculate the bargaining value functions.

In Section D, we prove Propositions 4, 5, and 6. We also give a sufficient condition for Assumption 1 in the paper.

Finally, Section E surveys the empirical literature on the costs of bankruptcy and provides further evidence supporting our assumptions regarding the costs of Chapter 11.

A Solving Chapter 11 efficiently

First, we provide an expression for the constant θ. In the notation of Section 2, let

$$\begin{aligned}
p^1 &= \frac{r}{r-\mu}\frac{\psi-1}{\psi}[\frac{-\tau}{\psi(1-\tau)\alpha+(\psi-1)\tau}]^{\frac{-1}{\psi}} \\
p^2 &= \frac{\psi}{\psi-1}\frac{r-\mu}{r},
\end{aligned}$$

so $C^* = p^1\delta$ and $\delta_L = p^2 C^* = p^1 p^2 \delta$. Summing the values of equity and debt,

$$\begin{aligned}
E^L(\delta)+D^L(\delta) &= \delta^{\psi}\delta_L^{-\psi}(1-\tau)[\frac{C^*}{r}-\frac{\delta_L}{r-\mu}] \\
&+\frac{(1-\tau)}{r-\mu}\delta-\frac{(1-\tau)C^*}{r}+\frac{C^*}{r} \\
&+\delta^{\psi}\delta_L^{-\psi}[-\frac{C^*}{r}+(1-\alpha)(1-\tau)\frac{\delta_L}{r-\mu}] \\
&=\frac{1-\tau}{r-\mu}\delta+\frac{\tau C^*}{r}-\delta^{\psi}\delta_L^{-\psi}[\tau\frac{C^*}{r}+\alpha(1-\tau)\frac{\delta_L}{r-\mu}].
\end{aligned}$$

Evaluating at δ_0 and plugging in the above formulas, this is

$$\begin{aligned}
&=\frac{1-\tau}{r-\mu}\delta_0+\frac{\tau p^1\delta_0}{r}-\delta_0^{\psi}(p^1p^2\delta_0)^{-\psi}[\tau\frac{p^1\delta_0}{r}+\alpha(1-\tau)\frac{p^1p^2\delta_0}{r-\mu}] \\
&=[\frac{1-\tau}{r-\mu}+\frac{\tau p^1}{r}-(p^1p^2)^{-\psi}[\tau\frac{p^1}{r}+\alpha(1-\tau)\frac{p^1p^2}{r-\mu}]]\delta_0 \\
&=\theta\delta_0
\end{aligned}$$

Next, in many of the proofs in this appendix, we will need to apply dominated convergence. This next lemma allows us to do so under the assumption of $r > \mu$, whenever the function in question can be bounded by an affine function of δ, R.

Lemma A.1 *For any fixed constants $\delta_0, R_0, B_1, B_2 > 0$,*

$$\mathbb{E}^{(\delta_0,R_0)}[\sup_t e^{-rt}(B_1\delta_t - B_2R_t)] < \infty.$$

Proof of Lemma: Since $r > \mu$, for an arithmetic Brownian motion $Z_t = (-r + \mu - \frac{\sigma^2}{2})t + \sigma B_t$, the supremum of Z_t over all t has an exponential distribution with parameter $\hat{\lambda} = (2|-r+\mu-\frac{\sigma^2}{2}|)/(\sigma^2) > 1$ (see, for example, Graversen and Peskir (1998)). It follows that

$$\mathbb{E}^{\delta_0}[\sup_t e^{-rt}\delta_t] = \delta_0 \mathbb{E}[\sup_t e^{Z_t}] = \delta_0 \frac{\hat{\lambda}}{\hat{\lambda}-1} < \infty,$$

and, decomposing $r = r_1 + r_2$ with $r_1 > \mu$, $r_2 > 0$,

$$\begin{aligned}
\mathbb{E}^{\delta_0}[\sup_t e^{-rt}\int_0^t \delta_s ds] &\leq \mathbb{E}^{\delta_0}[\sup_t \int_0^t e^{-rs}\delta_s ds] \\
&\leq \mathbb{E}^{\delta_0}[\int_0^\infty e^{-rs}\delta_s ds] \\
&= \mathbb{E}^{\delta_0}[\int_0^\infty e^{-r_2 s}e^{-r_1 s}\delta_s ds] \\
&\leq \mathbb{E}^{\delta_0}[\int_0^\infty e^{-r_2 s}(\sup_s e^{-r_1 s}\delta_s) ds] \\
&= \mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s}\delta_s)\int_0^\infty e^{-r_2 s} ds] \\
&= \frac{1}{r_2}\mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s}\delta_s)],
\end{aligned}$$

which is similarly finite. Then putting everything together, we have that

$$\begin{aligned}
&\mathbb{E}^{(\delta_0,R_0)}[\sup_t e^{-rt}(B_1\delta_t - B_2 R_t)] \\
&= \mathbb{E}^{\delta_0}[\sup_t e^{-rt}(B_1\delta_t + B_2 h(1-\tau)\int_0^t \delta_s ds - B_2 R_0)] \\
&\leq \mathbb{E}^{\delta_0}[|\sup_t e^{-rt}(B_1\delta_t + B_2 h(1-\tau)\int_0^t \delta_s ds)|] \\
&\leq B_1 \mathbb{E}^{\delta_0}[\sup_t e^{-rt}\delta_t] + B_2 h(1-\tau)\mathbb{E}^{\delta_0}[\sup_t e^{-rt}\int_0^t \delta_s ds] \\
&< \infty,
\end{aligned}$$

completing the proof. As an immediate corollary, for any fixed δ, R,

$$
\begin{aligned}
V(\delta, R) = \sup_{T_R \in F^{\delta,R}} & \mathbb{E}^{(\delta,R)}[\mathbf{1}(T_R < T_c)e^{-rT_R}(\theta\delta_{T_R} - R_{T_R}) \\
& + \mathbf{1}(T_c < T_R)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})] < \infty
\end{aligned} \tag{1}
$$

since, letting $T = T_R \wedge T_c$, there exist B_1, B_2 such that the expression in the expectation is less than

$$
e^{-rT}(B_1\delta_T - B_2R_T + B_2R_0)
$$

with probability 1.

Proof of Proposition 1: We introduce some simplifying notation. Let $O^* \equiv \{(\delta, R) : V(\delta, R) = \theta\delta - R\}$ be the set of values where the social planner's value function $V(\delta, R)$ equals the payoff $\theta\delta - R$. Fix a MPE with value functions E, D and equilibrium stopping time $\mathcal{T}$. It will be convenient to define the set $\mathcal{E} \equiv \cup_i O_i \times \{i\}$ so the game ends when $(\delta, R, s) \in \mathcal{E}$. From this point on, $\mathcal{T}$ is always defined as the first hitting time of $\mathcal{E}$. Let $V^e(\delta, R, s) \equiv E(\delta, R, s)$ and $V^d(\delta, R, s) \equiv D(\delta, R, s)$, and let $y \equiv \theta\delta - R$ and $z \equiv \zeta\delta - R$.

Now, the proof proceeds in three steps. First, from the definition of $J_i(\delta, R, s)$, we see that $\sum_i J_i(\delta, R, s) = y$. Given this, we have

$$
\begin{aligned}
& V^e(\delta, R, s) + V^d(\delta, R, s) \\
& = \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}] \\
& \leq V(\delta, R)
\end{aligned} \tag{2}
$$

by the definition of $V(\delta, R)$. Second, we claim that if $(\delta, R) \in O^*$, then $V^e(\delta, R, s) + V^d(\delta, R, s) = V(\delta, R)$. From the first observation, the leftside cannot be strictly greater. If it were strictly less, then letting $s' \neq s$,

$$
y - V^{s'}(\delta, R, s) = V(\delta, R) - V^{s'}(\delta, R, s) > V^s(\delta, R, s)
$$

where the first equality is the definition of O^*. It follows that player s would have a strictly profitable deviation to offer the other player their value function. This implies that if $(\delta, R) \in O^*$, then

$$\begin{aligned}
&\mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}] \qquad (3)\\
&= V^e(\delta, R, s) + V^d(\delta, R, s)\\
&= V(\delta, R)\\
&= \mathbb{E}^{(\delta,R)}[\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}]
\end{aligned}$$

where $\tau_* = \inf\{t : V(\delta_t, R_t) = y_t\}$ solves the optimal stopping problem. Third, any player can demand the game end at the maximum $\mathcal{T} \vee \tau_*$ with payoffs J_i. Specifically, any player i can deviate to making offers when $s_t = i$ and $(\delta, R) \in O_i \cap O^*$ and accepting offers from player j when $(\delta, R) \in O_j \cap O^*$. For this to be a MPE, this cannot be a profitable deviation for each player. Summing across i, we have

$$\begin{aligned}
&\sum_i V^i(\delta, R, s) \qquad (4)\\
&\geq \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} \vee \tau_* < T_c)e^{-r\mathcal{T}\vee\tau_*}y_{\mathcal{T}\vee\tau_*} + \mathbf{1}(T_c < \mathcal{T} \vee \tau_*)e^{-rT_c}z_{T_c}]\\
&= \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}]\\
&+ \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(\mathcal{T} < \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}].
\end{aligned}$$

Now, fix (δ_0, R_0, s_0). Let F_t be the filtration generated by (δ, R, s), which are jointly Markov. We have $F_0 \subset F_{\tau_*}$ where $\tau_*, \delta_{\tau_*}, R_{\tau_*}, s_{\tau_*}, \mathbf{1}(\mathcal{T} > \tau_*)$ are all F_{τ_*} measurable. Then

$$\begin{aligned}
&\mathbb{E}^{(\delta_0,R_0,s_0)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}]\\
&= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}|F_0]\\
&= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}|F_{\tau_*}\}|F_0].
\end{aligned}$$

Applying the Markov property, this equals

$$= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}^{(\delta_{\tau_*}, R_{\tau_*}, s_{\tau_*})}\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}|F_0],$$

and since $(\delta_{\tau_*}, R_{\tau_*}) \in O^*$ by definition, applying (3), this is

$$\begin{aligned}
&= \mathbb{E}[\mathbf{1}(\mathcal{T} > \tau_*)\mathbb{E}^{(\delta_{\tau_*}, R_{\tau_*})}\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}|F_0] \\
&= \mathbb{E}^{(\delta_0, R_0, s_0)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}].
\end{aligned}$$

Plugging this in to (4), we have that

$$\begin{aligned}
&\sum_i V^i(\delta, R, s) \\
&\geq \mathbb{E}^{(\delta, R, s)}[\mathbf{1}(\mathcal{T} > \tau_*)\{\mathbf{1}(\mathcal{T} < T_c)e^{-r\mathcal{T}}y_{\mathcal{T}} + \mathbf{1}(T_c < \mathcal{T})e^{-rT_c}z_{T_c}\}] \\
&+ \mathbb{E}^{(\delta, R, s)}[\mathbf{1}(\mathcal{T} < \tau_*)\{\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}\}] \\
&= \mathbb{E}^{(\delta, R, s)}[\mathbf{1}(\tau_* < T_c)e^{-r\tau_*}y_{\tau_*} + \mathbf{1}(T_c < \tau_*)e^{-rT_c}z_{T_c}] \\
&= V(\delta, R),
\end{aligned}$$

completing the proof.

B Efficient Chapter 11

Recall that, suppressing arguments, the HJB is

$$rV = -h(1-\tau)\delta V_R + \delta\mu V_\delta + \frac{\sigma^2}{2}\delta^2 V_{\delta\delta} + \iota[\zeta\delta - R - V],$$

where $\zeta = (1-\alpha)(1-\tau)/(r-\mu)$ such that $\zeta\delta - R$ is the liquidation value of the firm, and ιdt is the probability of liquidation per unit time.

We will solve this PDE by using a change of variables. Define $v \equiv V/\delta$ and $x \equiv R/\delta$. Note this means we expect exercise at low values of x. Straightforward calculus shows $v' = V_R$, $-xv' = V_\delta - v$, $v''x^2 = \delta V_{\delta\delta}$. Then dividing by δ and substituting, we get

$$(r+\iota-\mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2}x^2 v'' + \iota(\zeta - x).$$

B.1 General solution of the homogeneous equation

To start, consider the homogeneous equation

$$(r+\iota-\mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2}x^2 v''.$$

Conjecture a solution $v = x^\beta w(x)$ for some function w and constant β. This implies derivatives

$$\begin{aligned} v' &= \beta x^{\beta-1} w + x^\beta w' \\ v'' &= \beta(\beta-1)x^{\beta-2} w + 2\beta x^{\beta-1} w' + x^\beta w'' \end{aligned}$$

Plugging this conjecture in, we get

$$\begin{aligned} (r+\iota-\mu)v = &-(\mu x + h(1-\tau))[\beta x^{\beta-1} w + x^\beta w'] \\ &+ \frac{\sigma^2}{2}x^2[\beta(\beta-1)x^{\beta-2} w + 2\beta x^{\beta-1} w' + x^\beta w'']. \end{aligned}$$

First, gather v terms:

$$0 = v[-(r+\iota-\mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)].$$

We define β such that this equals 0. That is, β is a positive or negative root of

$$0 = [-(r+\iota-\mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)].$$

Let κ be the positive root and γ be the negative root, and for now let β be a placeholder for either root. Plugging in this β, we're left with

$$0 = -h(1-\tau)\beta x^{\beta-1} w - (\mu x + h(1-\tau))x^\beta w' + \frac{\sigma^2}{2}x^2[2\beta x^{\beta-1} w' + x^\beta w''].$$

Multiply through by $x^{-\beta+2}$:

$$0 = -h(1-\tau)\beta x w - (\mu x + h(1-\tau))x^2 w' + \frac{\sigma^2}{2}[2\beta x^3 w' + x^4 w'']$$

Finally, conduct a second change of variables to $z \equiv [-2h(1-\tau)]/[\sigma^2 x]$ and $f(z) = w(x)$. Then $w' = f'[2h(1-\tau)]/[\sigma^2 x^2]$ and

$$w'' = -2f'\frac{2h(1-\tau)}{\sigma^2 x^3} + f''\frac{4(h(1-\tau))^2}{\sigma^4 x^4},$$

where combining implies

$$\begin{aligned} w'' &= -2\frac{w'}{x} + f''\frac{4(h(1-\tau))^2}{\sigma^4 x^4} \\ x^4 w'' &= -2x^3 w' + f''\frac{4(h(1-\tau))^2}{\sigma^4} \end{aligned}$$

Plugging in,

$$\begin{aligned} 0 = &-h(1-\tau)\beta x f - (\mu x + h(1-\tau))\frac{2h(1-\tau)}{\sigma^2}f' \\ &+ \frac{\sigma^2}{2}[f''\frac{4(h(1-\tau))^2}{\sigma^4} + 2(\beta-1)xf'\frac{2h(1-\tau)}{\sigma^2}]. \end{aligned}$$

Multiplying by $-1/[h(1-\tau)x]$ and rearranging,

$$0 = \beta f + ((-2(\beta-1) + \frac{2\mu}{\sigma^2}) - z)f' + zf'',$$

which is Kummer's ODE, with general solutions

$$f(z) = M(-\beta, -2(\beta-1) + \frac{2\mu}{\sigma^2}, z)$$

$$f(z) = U(-\beta, -2(\beta-1) + \frac{2\mu}{\sigma^2}, z).$$

Thus for either root $\beta = \kappa, \gamma$, and either solution f to Kummer's ODE, we get a solution

$$A_3 x^\beta f(\frac{-2h(1-\tau)}{\sigma^2 x})$$

for some constant A_3.

B.2 Applying boundary conditions

Since $x = R/\delta$, and R can go negative, x starts out large and positive, then declines over time. We conjecture that the option is exercised before x becomes negative (which we verify shortly). Then we should be looking for a solution on a positive domain of x. Also, as $x = R/\delta \to \infty$, the cost of exercise is large and the payoff is small, so the value should not explode. We now impose this condition.

As $x \to \infty$, $M(a, b, -2h(1-\tau)/[\sigma^2 x])$ converges to $M(a, b, 0) = 1$. Thus $x^\beta M$ works as a solution for the negative root γ, but will not work for the positive root κ since then $x^\beta \to \infty$.

As $x \to \infty$, applying the positive root $\beta = \kappa$ we get $U(-\beta, -2(\beta-1) + 2\mu/\sigma^2, 0)$ is a finite constant. But then $x^\beta U$ goes to infinity. Applying the negative root $\beta = \gamma$, $U(-\beta, -2(\beta-1) + 2\mu/\sigma^2, z)$ explodes faster than x^β goes to 0, violating the boundary condition.

In conclusion, the homogeneous solution must take the form

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x})$$

for some constant A_3.

B.3 Finishing the value function

Finally, we must add back in the risk of liquidation to the single agent optimization. Recall after the change of variables, the HJB may be written

$$(r + \iota - \mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2} x^2 v'' + \iota(\zeta - x).$$

As discussed above, the only solution to the homogeneous equation satisfying the necessary boundary conditions is

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x}),$$

where γ is the negative root of

$$0 = [-(r+\iota-\mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta-1)].$$

The relevant particular solution including the last nonhomogeneous term is

$$\frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r+\iota-\mu} - \frac{\iota x}{r+\iota},$$

leading to a solution

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h}{\sigma^2 x}) + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r+\iota-\mu} - \frac{\iota x}{r+\iota}$$

for some constant A_3. At exercise, the firm receives $\theta\delta - R$ so this should smooth paste on $\theta - x$. Conjecturing exercise occurs at a lower barrier $\bar{x}$, the smooth pasting and value matching conditions are

$$v(\bar{x}) = \theta - \bar{x}$$
$$v'(\bar{x}) = -1.$$

Note that

$$\frac{d}{dz}M(a,b,z) = \frac{a}{b}M(a+1, b+1, z),$$

so

$$\begin{aligned} v'(x) = {} & A_3\gamma x^{\gamma-1} M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x}) \\ & - \frac{\iota}{r+\iota} + A_3 x^\gamma \frac{-\gamma}{-2(\gamma-1)+\frac{2\mu}{\sigma^2}} \frac{2h(1-\tau)}{\sigma^2 x^2} \\ & \times M(-\gamma+1, -2(\gamma-1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1-\tau)}{\sigma^2 x}). \end{aligned}$$

Solving for this v, we have $V(\delta, R) = \delta v(R/\delta)$ for $\delta \le R/\bar{x}$ and $V(\delta, R) = \theta\delta - R$ for $\delta \ge R/\bar{x}$. We now prove that this V is the value function for the social planner's problem, and the optimal policy is reorganize when $R/\delta \le \bar{x}$, or when $\delta \ge R/\bar{x}$.

B.4 Proof of Proposition 2

Define an operator $\mathcal{A}$ that maps smooth functions V of δ, R to

$$-h(1-\tau)\delta V_R + \delta\mu V_\delta + \frac{\sigma^2}{2}\delta^2 V_{\delta\delta} + \iota[\zeta\delta - R - V].$$

By construction, $V(\delta, R) = \delta v(R/\delta)$ is smooth since it smooth pastes at $\delta = R/\bar{x}$. Also by construction, $\mathcal{A}V = rV$ for $\delta \leq R/\bar{x}$. For $\delta \geq R/\bar{x}$ we have $V = \theta\delta - R$, so in this region

$$\begin{aligned}\mathcal{A}V &= h(1-\tau)\delta + \delta\mu\theta + \iota(\zeta - \theta)\delta \\ &= [h(1-\tau) + \mu\theta + \iota(\zeta-\theta)]\delta,\end{aligned}$$

and thus in this region, $-rV + \mathcal{A}V \leq 0$ if and only if

$$\begin{aligned}& -r(\theta\delta - R) + [h(1-\tau) + \mu\theta + \iota(\zeta-\theta)]\delta \leq 0 \\ \iff & \frac{h(1-\tau) + \mu\theta + \iota(\zeta-\theta) - r\theta}{r}\delta \leq -R \\ \iff & -\frac{h(1-\tau) + \mu\theta + \iota(\zeta-\theta) - r\theta}{r} \geq x,\end{aligned}$$

which is guaranteed by the first condition of Proposition 2,

$$-\frac{h(1-\tau) + \mu\theta + \iota(\zeta-\theta) - r\theta}{r} \geq \bar{x}.$$

By construction of V, we have $V(\delta, R) = \theta\delta - R$ when $\delta \geq R/\bar{x}$, and by condition 2 of Proposition 2,

$$V(\delta, R) = \delta v(\frac{R}{\delta}) \geq \delta(\theta - x) = \delta\theta - R,$$

so putting this together, under the conditions of Proposition 2, our candidate value function is smooth and satisfies the variational inequality

$$\max(\theta\delta - R - V, -rV + \mathcal{A}V) = 0.$$

Next, we show that there is a constant C such that $x \geq \bar{x} \Rightarrow v(x) \leq C$. To see this, we can use the fact that $\gamma < 0$ to write $M(-\gamma, -2(\gamma-1) + 2\mu/\sigma^2, -2h(1-\tau)/[\sigma^2 x]) = M(a, b, z)$ in its integral representation

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu} u^{a-1} (1-u)^{b-a-1} du.$$

For $u \in [0, 1]$ we have $e^{z_1 u} \leq e^{z_2 u}$ for $z_1 \leq z_2$ and all terms are positive, so $M(a, b, z)$ is positive and monotonically increasing in z. Then $M(-\gamma, -2(\gamma-1) + 2\mu/\sigma^2, -2h(1-\tau)/[\sigma^2 x])$ is monotonically increasing in x for $x \geq \bar{x}$. Since the expression converges to $M(a, b, 0) = 1$ as $x \to \infty$, it follows that $M \in (0, 1)$ for $x \geq \bar{x}$. Then since $\gamma < 0$, we have $x^\gamma M \leq x^\gamma \leq \bar{x}^\gamma$ for $x \geq \bar{x}$. It follows that

$$\begin{aligned} v(x) &= A_3 x^\gamma M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h}{\sigma^2 x}) + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r+\iota-\mu} - \frac{\iota x}{r+\iota} \\ &\leq A_3 \bar{x}^\gamma + \frac{\iota\zeta + \frac{h(1-\tau)\iota}{r+\iota}}{r+\iota-\mu} - \frac{\iota\bar{x}}{r+\iota} = C \end{aligned} \tag{5}$$

whenever $x \geq \bar{x}$. Thus $V(\delta, R)$ is bounded above by $C\delta$ whenever $R \geq \delta\bar{x}$, while $V(\delta, R) = \theta\delta - R$ when $R \leq \delta\bar{x}$. Combining this, it is clear we can bound $V(\delta, R)$ from above by the affine function $B\delta - R + R_0$ as long as $B > \max(\theta, C)$ and $R \leq R_0$, where the latter inequality holds almost everywhere by definition of R_t.

We are now ready to finish the verification. Fix δ_0, R_0 and define $Y_t \equiv \mathbf{1}(t < T_c) e^{-rt} V(\delta_t, R_t) + \mathbf{1}(t \geq T_c) e^{-rt}(\zeta\delta_t - R_t)$. By Ito's lemma for semimartingales, for $t < T_c$,

$$Y_t = V(\delta_0, R_0) + \int_0^t e^{-rs}[-rV(\delta_s, R_s) + \mathcal{A}V(\delta_s, R_s)]ds + M_t \tag{6}$$

for a local martingale M_t with $M_0 = 0$. Since V satisfies the variational inequality, for $t < T_c$,

$$Y_t = e^{-rt} V(\delta_t, R_t) \leq V(\delta_0, R_0) + M_t,$$

where V is bounded below so M_t is a supermartingale. Take an arbitrary stopping time T and let T_n be a sequence of stopping times increasing to $T \wedge T_c$.[1] Applying optional sampling[2] for the bounded below supermartingale M_t at T_n:

$$\mathbb{E}^{(\delta_0,R_0)}[Y_{T_n}] \leq V(\delta_0, R_0).$$

Since $V(\delta, R) \geq \theta\delta - R$, it follows that

$$\mathbb{E}^{(\delta_0,R_0)}[e^{-rT_n}\mathbf{1}(T_n < T_c)(\theta\delta_{T_n} - R_{T_n}) + e^{-rT_c}\mathbf{1}(T_n \geq T_c)(\zeta\delta_{T_c} - R_{T_c})] \leq V(\delta_0, R_0).$$

Taking n to infinity and using the bound $V(\delta, R) \leq B\delta - R + R_0$ along with the lemma of Appendix A to apply dominated convergence,

$$\mathbb{E}^{(\delta_0,R_0)}[e^{-rT}\mathbf{1}(T < T_c)(\theta\delta_T - R_T) + e^{-rT_c}\mathbf{1}(T \geq T_c)(\zeta\delta_{T_c} - R_{T_c})] \leq V(\delta_0, R_0).$$

Now, define $T_R \equiv \inf\{t : R_t/\delta_t \leq \bar{x}\}$ and fix R_0, δ_0 such that $T_R > 0$.[3] Then by definition of V, $-rV + \mathcal{A}V = 0$ for $t < T_R$, so applying Ito's lemma as before gives

$$Y_t = V(R_0, \delta_0) + M_t.$$

let Q_n be a sequence of stopping times increasing to $T_R \wedge T_c$, let τ_n be the localizing sequence of stopping times for the local martingale M_t, and let $T_n = Q_n \wedge \tau_n \wedge n$. Applying optional sampling,

$$\mathbb{E}^{(\delta_0,R_0)}[Y_{T_n}] = V(\delta_0, R_0).$$

Taking n to infinity and using the bound $V(\delta, R) \leq B\delta - R + R_0$ along with the lemma of Appendix A to apply dominated convergence,

[1] For example, $T_n = \max(0, T \wedge T_c - \frac{1}{n})$

[2] By an application of Fatou's lemma, optional sampling for bounded below supermartingales holds for arbitrary stopping times.

[3] If $T_R = 0$, the following conclusion is immediate.

$$
\begin{aligned}
\mathbb{E}^{(\delta_0,R_0)}[Y_{T_R\wedge T_c}] &= \mathbb{E}^{(\delta_0,R_0)}[\mathbf{1}(T_R < T_c)e^{-rT_R}V(\delta_{T_R}, R_{T_R}) + \mathbf{1}(T_R \geq T_c)e^{-rT_C}(\zeta\delta_{T_C} - R_{T_C})] \\
&= \mathbb{E}^{(\delta_0,R_0)}[\mathbf{1}(T_R < T_c)e^{-rT_R}(\theta\delta_{T_R} - R_{T_R}) + \mathbf{1}(T_R \geq T_c)e^{-rT_C}(\zeta\delta_{T_C} - R_{T_C})] \\
&= V(\delta_0, R_0),
\end{aligned}
$$

where the penultimate equality follows from the definition of T_R and V, completing the verification.

C Proof of Proposition 3, calculating equilibrium

Proof of Proposition 3: Let $V^e(\delta, R, s) \equiv E(\delta, R, s)$ and $V^d(\delta, R, s) \equiv D(\delta, R, s)$. The proof proceeds in two steps. First, we show it is without loss of generality to assume the offer strategy $\omega_i(\delta, R) = V^j(\delta, R, i)$ is optimal. If there were an alternate strategy $(\hat{\omega}, \hat{A}, \hat{O})$ that performed strictly better than the proposed strategy and $\hat{\omega}_i(\delta, R) > V^j(\delta, R, i)$ for some δ, R, i, j, then another strategy $(\tilde{\omega}, \hat{A}, \hat{O})$ does even better by setting $\tilde{\omega}_i(\delta, R) = V^j(\delta, R, i)$ in those cases. In words, offering more than necessary to make the opponent accept is wasteful, since there is complete information and offers cannot change future behavior according to stationary strategies. Likewise, if there were an alternate strategy $(\hat{\omega}, \hat{A}, \hat{O})$ that performed strictly better than the proposed strategy and $\hat{\omega}_i(\delta, R) < V^j(\delta, R, i)$ for some δ, R, i, j, then another strategy $(\tilde{\omega}, \hat{A}, \tilde{O})$ does just as well where $\tilde{\omega}_i(\delta, R) = V^j(\delta, R, i)$ and those cases are removed from the offer region. In words, if player i makes an offer that they know will be rejected, they do just as well by not making the offer. Therefore, when we consider profitable deviations, it is sufficient to consider deviations of $\hat{A}, \hat{O}$ where the alternate offer function is still $\omega_i(\delta, R) = V^j(\delta, R, i)$.

Second, we show that the equilibrium time $\mathcal{T}$ solves

$$
\sup_{T_i \in F^{(\delta,R,s)}} \mathbb{E}^{(\delta,R,s)}[\mathbf{1}(T_i < T_c)e^{-rT_i}J_i(\delta_{T_i}, R_{T_i}, s_{T_i}) + \mathbf{1}(i = d)\mathbf{1}(T_c \leq T_i)e^{-rT_c}(\zeta\delta_{T_c} - R_{T_c})] \tag{7}
$$

with associated value function $V^i(\delta, R, s)$. Since each player tries to optimize this quantity subject to constraints imposed by the opponent's strategy, and the equilibrium time $\mathcal{T}$ solves the unconstrained problem, this implies each player acts optimally in the MPE.

To show this, define $N_t = \mathbf{1}(t \geq T_c)$ and for notational convenience define an operator $\mathcal{H}_s$ mapping appropriately differentiable functions $f(\delta, R, s, N)$ to[4]

$$- h(1-\tau)\delta f_R(\delta, R, s, N) + \mu\delta f_\delta(\delta, R, s, N) + \frac{\sigma^2\delta^2}{2} f_{\delta\delta}(\delta, R, s, N) + \lambda_s[f(\delta, R, s', N) - f(\delta, R, s, N)]$$
$$+ \iota[f(\delta, R, s, 1) - f(\delta, R, s, 0)].$$

Fix $N_0 = 0$. Defining $U^i(\delta, R, s, 0) = V^i(\delta, R, s)$ and $U^i(\delta, R, s, 1) = \mathbf{1}(i = d)(\zeta\delta - R)$, by construction U^i solves

$$-rU^i + \mathcal{H}_s U^i = 0$$

except possibly when $(\delta, R, s) \in O^* \times \{i\}$. By assumption, we have $U^i + U^j = V(\delta, R)$.[5] Also by construction, when $(\delta, R, s) \in O_i^* \times \{i\}$ we have $-rU^j + \mathcal{H}_s U^j = 0$ and $-rV(\delta, R) + \mathcal{H}_s V(\delta, R) \leq 0$ by Proposition 2. By the linearity of $\mathcal{H}_s$, it follows that

$$(-r + \mathcal{H}_s)U^i = (-r + \mathcal{H}_s)[V(\delta, R) - U^j] \leq 0.$$

For $t < T_c$, applying Ito's lemma for semimartingales (see, for example, Duffie (2010)) to U^i gives

$$\begin{aligned} e^{-rt}U^i(\delta_t, R_t, s_t, N_t) &= U^i(\delta_0, R_0, s_0, N_0) \\ &+ \int_0^t e^{-ru}[(-r + \mathcal{H}_{s_u})U^i(\delta_u, R_u, s_u, N_u)]du + M_t \end{aligned}$$

for a local martingale M_t with $M_t = 0$. Applying an identical argument to that used in the proof of Proposition 2 (Appendix B.4) gives that for an arbitrary stopping time T,

[4]The fact that N_t does not transition from 1 to 0 is irrelevant.

[5]In a slight abuse of notation, we sometimes view V as a trivial function of s that equals $\zeta\delta - R$ when $N = 1$.

$$
\begin{aligned}
&\mathbb{E}^{(\delta_0,R_0,s_0,0)}[U^i(\delta_{T\wedge T_c},R_{T\wedge T_c},s_{T\wedge T_c},N_{T\wedge T_c})]\\
&=\mathbb{E}^{(\delta_0,R_0,s_0)}[\mathbf{1}(T<T_c)e^{-rT}V^i(\delta_T,R_T,s_T)+\mathbf{1}(i=d)\mathbf{1}(T_c\le T)e^{-rT_c}(\zeta\delta_{T_c}-R_{T_c})]\\
&\le U^i(\delta_0,R_0,s_0,0)\\
&=V^i(\delta_0,R_0,s_0). \qquad (8)
\end{aligned}
$$

For the equilibrium time $\mathcal{T}$, we have $t<\mathcal{T}$ implies[6] $(\delta_t,R_t,s_t)\notin O_i^*\times\{i\}$ which implies $(-r+\mathcal{H}_s)U^i=0$, so an argument identical to that used in the proof of Proposition 2 (Appendix B.4) gives that

$$
\begin{aligned}
&\mathbb{E}^{(\delta_0,R_0,s_0,0)}[U^i(\delta_{\mathcal{T}\wedge T_c},R_{\mathcal{T}\wedge T_c},s_{\mathcal{T}\wedge T_c},N_{\mathcal{T}\wedge T_c})]\\
&=\mathbb{E}^{(\delta_0,R_0,s_0)}[\mathbf{1}(\mathcal{T}<T_c)e^{-r\mathcal{T}}V^i(\delta_{\mathcal{T}},R_{\mathcal{T}},s_{\mathcal{T}})+\mathbf{1}(i=d)\mathbf{1}(T_c\le\mathcal{T})e^{-rT_c}(\zeta\delta_{T_c}-R_{T_c})]\\
&=U^i(\delta_0,R_0,s_0,0)\\
&=V^i(\delta_0,R_0,s_0). \qquad (9)
\end{aligned}
$$

Note that $V(\delta,R)\ge\theta\delta-R$, combined with the assumption that $\sum_{i=e,d}V^i(\delta,R,s)=V(\delta,R)$, implies that

$$
\begin{aligned}
J_i(\delta,R,s)&=\mathbf{1}(s=i)[\theta\delta-R-V^j(\delta,R,i)]+\mathbf{1}(s=j)V^i(\delta,R,j)\\
&\le\mathbf{1}(s=i)[V(\delta,R)-V^j(\delta,R,i)]+\mathbf{1}(s=j)V^i(\delta,R,j)\\
&=V^i(\delta,R,s)
\end{aligned}
$$

so $V^i(\delta,R,s)\ge J_i(\delta,R,s)$, and by definition $V^i(\delta_{\mathcal{T}},R_{\mathcal{T}},s_{\mathcal{T}})=J_i(\delta_{\mathcal{T}},R_{\mathcal{T}},s_{\mathcal{T}})$. Plugging this into (8, 9) completes the proof.

[6]The conclusion is trivial when $\mathcal{T}=0$.

C.1 Equilibrium value functions

Following Appendix B, general solutions of the homogeneous equation

$$(r + \iota - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2} x^2 v''$$

take the form

$$Gx^\beta f(\frac{-2h(1 - \tau)}{\sigma^2 x})$$

for a constant G, where f is either of the solutions to Kummer's ODE:

$$f(z) = M(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z)$$

$$f(z) = U(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z)$$

and β is either root of

$$0 = [-(r + \iota - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

First, consider general solutions in the range $x \leq 0$. As $x \to -\infty$, each player's value function should be bounded above by $\theta - x$, such that $V^i = \delta v$ is bounded above by the value of immediate exercise with all proceeds going to one player. It turns out that none of the general solutions satisfy this with $G \neq 0$,[7] so the general solution in this region is 0.

Next, consider general solutions in the range $x \in [0, \bar{x}]$. As $x \to 0$ from above, $z = -2h(1 - \tau)/[\sigma^2 x] \to -\infty$, and except for some devious corner cases $M(a, b, z)$ is asymptotically proportional to $(-z)^{-a}$. Thus for either the positive or negative root β, the product

$$Gx^\beta M(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z)$$

is finite at $x = 0, z = -\infty$. The Tricomi U function, evaluated at a negative z, is complex valued and cannot be multiplied by a constant Z to have all real values, so we rule this function

[7] We state this without proof, but by showing the conditions of Proposition 3 are met the final solution must be the value function.

out.[8] Thus the general solution in this region is

$$G_1 x^\gamma M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, z) + G_2 x^\kappa M(-\kappa, -2(\kappa-1) + \frac{2\mu}{\sigma^2}, z)$$

where κ (γ) is the positive (negative) root β of the above quadratic.

We are now ready to solve the system of equations (28)-(31) from the paper characterizing the value functions for $(\delta, R) \notin O^*$. Recall these are

$$rE(\delta,R,e) = \mathcal{L}E(\delta,R,e) + \lambda_e[E(\delta,R,d) - E(\delta,R,e)] + \iota[0 - E(\delta,R,e)] \tag{10}$$

$$rE(\delta,R,d) = \mathcal{L}E(\delta,R,d) + \lambda_d[E(\delta,R,e) - E(\delta,R,d)] + \iota[0 - E(\delta,R,d)] \tag{11}$$

$$rD(\delta,R,e) = \mathcal{L}D(\delta,R,e) + \lambda_e[D(\delta,R,d) - D(\delta,R,e)] + \iota[\zeta\delta - R - D(\delta,R,e)] \tag{12}$$

$$rD(\delta,R,d) = \mathcal{L}D(\delta,R,d) + \lambda_d[D(\delta,R,e) - D(\delta,R,d)] + \iota[\zeta\delta - R - D(\delta,R,d)], \tag{13}$$

where

$$\mathcal{L}f = \delta\mu f_\delta + \frac{\sigma^2}{2}\delta^2 f_{\delta\delta} - (1-\tau)h\delta f_R. \tag{14}$$

Start with equity values: letting $\hat{r} \equiv r + \iota$ and rearranging (10,11), we can use the linearity of the operator $\mathcal{L}$ to write

$$\begin{bmatrix} \hat{r}+\lambda_e & -\lambda_e \\ -\lambda_d & \hat{r}+\lambda_d \end{bmatrix} \begin{bmatrix} E(\delta,R,e) \\ E(\delta,R,d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} E(\delta,R,e) \\ E(\delta,R,d) \end{bmatrix}.$$

The matrix $\begin{bmatrix} \hat{r}+\lambda_e & -\lambda_e \\ -\lambda_d & \hat{r}+\lambda_d \end{bmatrix}$ has eigendecomposition

$$\begin{bmatrix} \hat{r}+\lambda_e & -\lambda_e \\ -\lambda_d & \hat{r}+\lambda_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r}+\lambda_e+\lambda_d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1}.$$

Define

[8] Again, all that matters in the end is that the conditions of Proposition 3 are met.

$$\begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix}$$

Then $\hat{E}$ follows the delinked system of HJBs

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{E}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix}.$$

Define

$$\xi(x, \gamma) \equiv x^{\gamma} M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x}).$$

As before, let γ be the negative root of

$$0 = [-(\hat{r} - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)],$$

and let ν be the negative root of

$$0 = [-(\hat{r} + \lambda_e + \lambda_d - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

Then as shown in Appendix B.3, in the region where $x \geq \bar{x}$, these equations have solutions $\hat{E}(\delta, R, e) = K_1 \delta \xi(\frac{R}{\delta}, \gamma)$ and $\hat{E}(\delta, R, d) = K_2 \delta \xi(\frac{R}{\delta}, \nu)$ for some constants K_1, K_2. Multiplying by $\begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda_d}{\lambda_e} \end{bmatrix}$ delivers

$$\begin{bmatrix} E(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \begin{bmatrix} K_1 \delta \xi(\frac{R}{\delta}, \gamma) + K_2 \delta \xi(\frac{R}{\delta}, \nu) \\ K_1 \delta \xi(\frac{R}{\delta}, \gamma) - \frac{\lambda_d}{\lambda_e} K_2 \delta \xi(\frac{R}{\delta}, \nu) \end{bmatrix}.$$

Given these value functions, we can define $D(\delta, R, s) \equiv V(\delta, R) - E(\delta, R, s)$, and by linearity of the operator $\mathcal{L}$,

$$\begin{aligned}
(r-\mathcal{L})D(\delta,R,s) &= (r-\mathcal{L})(V(\delta,R)-E(\delta,R,s)) \\
&= (r-\mathcal{L})V(\delta,R)-(r-\mathcal{L})E(\delta,R,s) \\
&= \iota[\zeta\delta-R-V(\delta,R)]-[\lambda_s[E(\delta,R,s')-E(\delta,R,s)]-\iota E(\delta,R,s)] \\
&= \iota[\zeta\delta-R-(V(\delta,R)-E(\delta,R,s))]+\lambda_s[V(\delta,R)-E(\delta,R,s')-(V(\delta,R)-E(\delta,R,s))] \\
&= \iota[\zeta\delta-R-D(\delta,R,s)]+\lambda_s[D(\delta,R,s')-D(\delta,R,s)].
\end{aligned}$$

So $D(\delta,R,s)$ satisfies (12, 13) as desired. Thus we have determined the value functions for $(\delta,R)\notin O^*$ up to two constants K_1,K_2. Now, in the region where $(\delta,R)\in O^*$, we will solve for the value functions while receiving offers, $E(\delta,R,d)$ and $D(\delta,R,d)$. Recall these must satisfy the HJBs

$$\begin{aligned}
rE(\delta,R,d) &= \mathcal{L}E(\delta,R,d)+\lambda_d[E(\delta,R,e)-E(\delta,R,d)]+\iota[0-E(\delta,R,d)] \\
rD(\delta,R,e) &= \mathcal{L}D(\delta,R,e)+\lambda_e[D(\delta,R,d)-D(\delta,R,e)]+\iota[\zeta\delta-R-D(\delta,R,e)],
\end{aligned}$$

and since offers are made in equilibrium in this region,

$$\begin{aligned}
E(\delta,R,e) &= \theta\delta-R-D(\delta,R,e) \\
D(\delta,R,d) &= \theta\delta-R-E(\delta,R,d).
\end{aligned}$$

Plugging this in and rearranging, this is

$$\begin{bmatrix}\hat{r}+\lambda_e & \lambda_e \\ \lambda_d & \hat{r}+\lambda_d\end{bmatrix}\begin{bmatrix}D(\delta,R,e) \\ E(\delta,R,d)\end{bmatrix}=\mathcal{L}\begin{bmatrix}D(\delta,R,e) \\ E(\delta,R,d)\end{bmatrix}+\begin{bmatrix}\lambda_e \\ \lambda_d\end{bmatrix}(\theta\delta-R)+\begin{bmatrix}\iota \\ 0\end{bmatrix}(\zeta\delta-R).$$

The matrix $\begin{bmatrix}\hat{r}+\lambda_e & \lambda_e \\ \lambda_d & \hat{r}+\lambda_d\end{bmatrix}$ has eigendecomposition

$$\begin{bmatrix} \hat{r}+\lambda_e & \lambda_e \\ \lambda_d & \hat{r}+\lambda_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix} \begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r}+\lambda_e+\lambda_d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1}.$$

Define

$$\begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix}$$

Then $\hat{E}, \hat{D}$ follow the delinked system of HJBs

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r}+\lambda_e+\lambda_d \end{bmatrix} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_e \\ \lambda_d \end{bmatrix} (\theta\delta - R) + \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} \begin{bmatrix} \iota \\ 0 \end{bmatrix} (\zeta\delta - R).$$

Note

$$\begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix}^{-1} = \frac{\lambda_e}{\lambda_e+\lambda_d} \begin{bmatrix} \frac{\lambda_d}{\lambda_e} & -1 \\ 1 & 1 \end{bmatrix},$$

so this is

$$\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r}+\lambda_e+\lambda_d \end{bmatrix} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} = \mathcal{L} \begin{bmatrix} \hat{D}(\delta, R, e) \\ \hat{E}(\delta, R, d) \end{bmatrix} + \begin{bmatrix} \frac{\iota\lambda_d(\zeta\delta - R)}{\lambda_e+\lambda_d} \\ \lambda_e(\theta\delta - R) + \frac{\iota\lambda_e(\zeta\delta - R)}{\lambda_e+\lambda_d} \end{bmatrix}.$$

Let κ be the positive root of

$$0 = [-(\hat{r} - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)]$$

and let ϕ be the positive root of

$$0 = [-(\hat{r} + \lambda_e + \lambda_d - \mu) - \mu\beta + \frac{\sigma^2}{2}\beta(\beta - 1)].$$

Then by the previous discussion, in the region where $\delta\bar{x} \geq R$ and $R \geq 0$ (so $x \in [0, \bar{x}]$), the homogeneous ODEs associated with this system (i.e., ignoring δ, R terms) have general solutions

$$\hat{D}(\delta, R, e) = K_3\delta\xi(\frac{R}{\delta}, \gamma) + K_4\delta\xi(\frac{R}{\delta}, \kappa)$$
$$\hat{E}(\delta, R, d) = K_5\delta\xi(\frac{R}{\delta}, \nu) + K_6\delta\xi(\frac{R}{\delta}, \phi).$$

Given constants q, c, d, one can show the particular solution to

$$(q-\mu)v = -(\mu x + h(1-\tau))v' + \frac{\sigma^2}{2}x^2 v'' + cx + d$$

takes the form

$$v = \frac{c}{q}x + \frac{(-\frac{ch(1-\tau)}{q} + d)}{q-\mu}$$
$$V = \delta v = \frac{c}{q}R + \frac{(-\frac{ch(1-\tau)}{q} + d)}{q-\mu}\delta.$$

After carrying out the matrix multiplication, the relevant parameters for $\hat{E}(\delta, R, d)$ are

$$c = -\lambda_e - \frac{\lambda_e}{\lambda_e + \lambda_d}\iota$$
$$d = \lambda_e\theta + \frac{\lambda_e}{\lambda_e + \lambda_d}\iota\zeta$$
$$q = \hat{r} + \lambda_d + \lambda_e,$$

while for $\hat{D}(\delta, R, e)$ they are

$$c = \frac{-\lambda_d}{\lambda_e + \lambda_d}\iota$$
$$d = \frac{\lambda_d}{\lambda_e + \lambda_d}\iota\zeta$$
$$q = \hat{r}.$$

Plugging this in, the relevant particular solutions are

$$\hat{D} = \frac{\frac{-\lambda_d}{\lambda_e+\lambda_d}\iota}{\hat{r}}R + \frac{\left(\frac{\frac{\lambda_d}{\lambda_e+\lambda_d}\iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e+\lambda_d}\iota\zeta\right)}{\hat{r}-\mu}\delta$$
$$\hat{E} = \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e+\lambda_d}\iota}{\hat{r}+\lambda_d+\lambda_e}R + \frac{\left(\frac{[\lambda_e+\frac{\lambda_e}{\lambda_e+\lambda_d}\iota]h(1-\tau)}{\hat{r}+\lambda_d+\lambda_e} + [\lambda_e\theta + \frac{\lambda_e}{\lambda_e+\lambda_d}\iota\zeta]\right)}{\hat{r}+\lambda_d+\lambda_e-\mu}\delta$$

or, adding back the general solutions,

$$\begin{bmatrix}\hat{D}(\delta,R,e)\\ \hat{E}(\delta,R,d)\end{bmatrix} = \begin{bmatrix}K_3\delta\xi(\frac{R}{\delta},\gamma) + K_4\delta\xi(\frac{R}{\delta},\kappa)\\ K_5\delta\xi(\frac{R}{\delta},\nu) + K_6\delta\xi(\frac{R}{\delta},\phi)\end{bmatrix}$$
$$+ \begin{bmatrix}\frac{\frac{-\lambda_d}{\lambda_e+\lambda_d}\iota}{\hat{r}}R + \frac{\left(\frac{\frac{\lambda_d}{\lambda_e+\lambda_d}\iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e+\lambda_d}\iota\zeta\right)}{\hat{r}-\mu}\delta\\ \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e+\lambda_d}\iota}{\hat{r}+\lambda_d+\lambda_e}R + \frac{\left(\frac{[\lambda_e+\frac{\lambda_e}{\lambda_e+\lambda_d}\iota]h(1-\tau)}{\hat{r}+\lambda_d+\lambda_e} + [\lambda_e\theta + \frac{\lambda_e}{\lambda_e+\lambda_d}\iota\zeta]\right)}{\hat{r}+\lambda_d+\lambda_e-\mu}\delta\end{bmatrix}.$$

Multiplying by $\begin{bmatrix}1 & 1\\ -1 & \frac{\lambda_d}{\lambda_e}\end{bmatrix}$ gives

$$\begin{bmatrix}D(\delta,R,e)\\ E(\delta,R,d)\end{bmatrix} = \begin{bmatrix}K_3\delta\xi(\frac{R}{\delta},\gamma) + K_4\delta\xi(\frac{R}{\delta},\kappa) + K_5\delta\xi(\frac{R}{\delta},\nu) + K_6\delta\xi(\frac{R}{\delta},\phi)\\ -K_3\delta\xi(\frac{R}{\delta},\gamma) - K_4\delta\xi(\frac{R}{\delta},\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\delta\xi(\frac{R}{\delta},\nu) + K_6\delta\xi(\frac{R}{\delta},\phi)]\end{bmatrix} + \begin{bmatrix}c_1\delta + c_2R\\ c_3\delta + c_4R\end{bmatrix},$$

where

$$\begin{bmatrix}c_2\\ c_4\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & \frac{\lambda_d}{\lambda_e}\end{bmatrix}\begin{bmatrix}\frac{\frac{-\lambda_d}{\lambda_e+\lambda_d}\iota}{\hat{r}}\\ \frac{-\lambda_e - \frac{\lambda_e}{\lambda_e+\lambda_d}\iota}{\hat{r}+\lambda_d+\lambda_e}\end{bmatrix}$$
$$\begin{bmatrix}c_1\\ c_3\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & \frac{\lambda_d}{\lambda_e}\end{bmatrix}\begin{bmatrix}\frac{\left(\frac{\frac{\lambda_d}{\lambda_e+\lambda_d}\iota h(1-\tau)}{\hat{r}} + \frac{\lambda_d}{\lambda_e+\lambda_d}\iota\zeta\right)}{\hat{r}-\mu}\\ \frac{\left(\frac{[\lambda_e+\frac{\lambda_e}{\lambda_e+\lambda_d}\iota]h(1-\tau)}{\hat{r}+\lambda_d+\lambda_e} + [\lambda_e\theta + \frac{\lambda_e}{\lambda_e+\lambda_d}\iota\zeta]\right)}{\hat{r}+\lambda_d+\lambda_e-\mu}\end{bmatrix}.$$

Finally, because R can become negative, we need a different solution for the off-equilibrium region where $\delta \leq R/\bar{x}$ and $R < 0$. Luckily, the only general solution satisfying the boundary conditions is 0, so in this region

$$\begin{bmatrix} D(\delta, R, e) \\ E(\delta, R, d) \end{bmatrix} = \begin{bmatrix} c_1\delta + c_2 R \\ c_3\delta + c_4 R \end{bmatrix}$$

for the same constants $c_1 - c_4$. We thus have 6 unknowns, and we require $E(\delta, R, e)$ and $E(\delta, R, d)$ to value match (VM) and smooth paste (SP) at $R/\delta = \bar{x}$ (already calculated) and at $R/\delta = 0$. Recall in the exercise region, $E(\delta, R, e) = \theta\delta - R - D(\delta, R, e)$, where $\theta\delta - R$ is obviously smooth, so imposing VM and SP for $D(\delta, R, e)$ at $R/\delta = 0$ is sufficient and necessary for VM and SP of $E(\delta, R, e)$ at $R/\delta = 0$. These conditions are easiest to impose by switching back to $x = R/\delta$:[9]

$$0 = \begin{bmatrix} K_3\xi(0,\gamma) + K_4\xi(0,\kappa) + K_5\xi(0,\nu) + K_6\xi(0,\phi) \\ -K_3\xi(0,\gamma) - K_4\xi(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi(0,\nu) + K_6\xi(0,\phi)] \end{bmatrix}$$

$$0 = \begin{bmatrix} K_3\xi'(0,\gamma) + K_4\xi'(0,\kappa) + K_5\xi'(0,\nu) + K_6\xi'(0,\phi) \\ -K_3\xi'(0,\gamma) - K_4\xi'(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi'(0,\nu) + K_6\xi'(0,\phi)]. \end{bmatrix}$$

We verify two of these are redundant,[10] and thus these are actually equivalent to two equations:

$$\begin{bmatrix} K_3\xi(0,\gamma) + K_4\xi(0,\kappa) + K_5\xi(0,\nu) + K_6\xi(0,\phi) \\ -K_3\xi(0,\gamma) - K_4\xi(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi(0,\nu) + K_6\xi(0,\phi)] \end{bmatrix}$$
$$= \begin{bmatrix} K_3\xi'(0,\gamma) + K_4\xi'(0,\kappa) + K_5\xi'(0,\nu) + K_6\xi'(0,\phi) \\ -K_3\xi'(0,\gamma) - K_4\xi'(0,\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi'(0,\nu) + K_6\xi'(0,\phi)] \end{bmatrix}$$

In addition to these two equations, we require the four equations corresponding to VM and SP at $\bar{x}$:

[9] Note we arrive at these equalities by first subtracting $c_1\delta + c_2 R$ or $c_3\delta + c_4 R$ from both sides.

[10] Specifically, since ξ converges as $x \to 0$ ($z \to -\infty$), its derivative must converge to zero. This can be shown directly with the asymptotic properties of the Confluent Hypergeometric function.

$$\begin{bmatrix} \theta - \bar{x} - [K_3\xi(\bar{x},\gamma) + K_4\xi(\bar{x},\kappa) + K_5\xi(\bar{x},\nu) + K_6\xi(\bar{x},\phi)] \\ -K_3\xi(\bar{x},\gamma) - K_4\xi(\bar{x},\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi(\bar{x},\nu) + K_6\xi(\bar{x},\phi)] \end{bmatrix}$$

$$= - \begin{bmatrix} -c_1 - c_2\bar{x} \\ c_3 + c_4\bar{x} \end{bmatrix} + \begin{bmatrix} K_1\xi(\bar{x},\gamma) + K_2\xi(\bar{x},\nu) \\ K_1\xi(\bar{x},\gamma) - \frac{\lambda_d}{\lambda_e}K_2\xi(\bar{x},\nu) \end{bmatrix}$$

$$\begin{bmatrix} -1 - [K_3\xi'(\bar{x},\gamma) + K_4\xi'(\bar{x},\kappa) + K_5\xi'(\bar{x},\nu) + K_6\xi'(\bar{x},\phi)] \\ -K_3\xi'(\bar{x},\gamma) - K_4\xi'(\bar{x},\kappa) + \frac{\lambda_d}{\lambda_e}[K_5\xi'(\bar{x},\nu) + K_6\xi'(\bar{x},\phi)] \end{bmatrix}$$

$$= - \begin{bmatrix} -c_2 \\ c_4 \end{bmatrix} + \begin{bmatrix} K_1\xi'(\bar{x},\gamma) + K_2\xi'(\bar{x},\nu) \\ K_1\xi'(\bar{x},\gamma) - \frac{\lambda_d}{\lambda_e}K_2\xi'(\bar{x},\nu) \end{bmatrix}.$$

This is a linear system which is easily solved for $K_1 - K_6$, once one notes that

$$\begin{aligned} \xi'(x,\gamma) &= \gamma x^{\gamma-1} M(-\gamma, -2(\gamma-1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1-\tau)}{\sigma^2 x}) \\ &\quad + x^{\gamma} \frac{-\gamma}{-2(\gamma-1) + \frac{2\mu}{\sigma^2}} \frac{2h(1-\tau)}{\sigma^2 x^2} \\ &\quad \times M(-\gamma+1, -2(\gamma-1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1-\tau)}{\sigma^2 x}). \end{aligned}$$

D Period 1 decision to liquidate or enter Chapter 11

First, we provide a proof of Proposition 4, which gives the solution to the problem of optimally entering Chapter 11.

D.1 Proving Proposition 4

The conditions of Proposition 4 imply the following variational inequality holds:

$$\max(\mathcal{E}(\delta) - B - E^B(\delta), -rE^B(\delta) + \mathcal{D}E^B(\delta) + (1-\tau)(\delta - C_0)) = 0,$$

where

$$\mathcal{D}f(\delta) = f'(\delta)\mu\delta + f''(\delta)\frac{\sigma^2}{2}\delta^2. \tag{15}$$

Since the candidate E^B is smooth, applying Ito's lemma to $e^{-rt}E^B(\delta_t)$ delivers

$$e^{-rt}E^B(\delta_t) = E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + \mathcal{D}E^B(\delta_s)]ds + M_t$$

for a local martingale M_t with $M_0 = 0$. By the variational inequality,

$$\begin{aligned} & E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + \mathcal{D}E^B(\delta_s)]ds + M_t \\ & \leq E^B(\delta_0) + \int_0^t e^{-rs}[-(1-\tau)(\delta_s - C_0)]ds + M_t, \end{aligned}$$

implying

$$e^{-rt}E^B(\delta_t) + \int_0^t e^{-rs}[(1-\tau)(\delta_s - C_0)]ds \leq E^B(\delta_0) + M_t.$$

Let τ_n be the sequence of localizing stopping times for M_t, let T be an arbitrary stopping time, and let $Q_n = T \wedge \tau_n \wedge n$. Then we can apply optional sampling to write

$$\begin{aligned} & \mathbb{E}^{\delta_0}[e^{-rQ_n}E^B(\delta_{Q_n}) + \int_0^{Q_n} e^{-rs}[(1-\tau)(\delta_s - C_0)]ds] \\ & \leq E^B(\delta_0) + \mathbb{E}^{\delta_0}[M_{Q_n}] \\ & = E^B(\delta_0) + M_0 \\ & = E^B(\delta_0). \end{aligned}$$

Since $\delta^\psi \to 0$ as $\delta \to \infty$, there exist constants k^0, k^1 such that $E^B(\delta) \leq k^0 + k^1\delta$. Also, it is clear that

$$|\int_0^t e^{-rs}[(1-\tau)(\delta_s - C_0)]ds| \leq (1-\tau)\int_0^\infty e^{-rs}\delta_s ds + \frac{C_0}{r},$$

where the rightside is integrable. Thus we can apply the lemma of Appendix A to use dominated convergence, and clearly $Q_n \to T$ as $n \to \infty$, so

$$\mathbb{E}^{\delta_0}[e^{-rT}E^B(\delta_T) + \int_0^T e^{-rs}[(1-\tau)(\delta_s - C_0)]ds]$$
$$\leq E^B(\delta_0).$$

By the variational inequality, this implies

$$\mathbb{E}^{\delta_0}[e^{-rT}(\mathcal{E}(\delta_T) - B) + \int_0^T e^{-rs}[(1-\tau)(\delta_s - C_0)]ds]$$
$$\leq E^B(\delta_0).$$

Applying the same argument for the optimal T_B, and using the fact that $\delta \geq \delta_B$ implies $-rE^B(\delta) + \mathcal{D}E^B(\delta) + (1-\tau)(\delta - C_0) = 0$ by the construction of E^B, we get

$$\mathbb{E}^{\delta_0}[e^{-rT_B}(\mathcal{E}(\delta_{T_B}) - B) + \int_0^{T_B} e^{-rs}[(1-\tau)(\delta_s - C_0)]ds] = E^B(\delta_0),$$

completing the proof.

D.2 Liquidation vs Chapter 11 Renegotiation

First, recall that asymptotically $\mathcal{E}(\delta) = E(\delta, R_0, e)$ approaches $(\theta - c_1)\delta - (1 - c_2)R_0$. Thus a sufficient condition for Assumption 1 is that $(\theta - c_1) < (1-\tau)/(r-\mu)$.

In period 1, the firm can decide to liquidate and receive 0 or enter Chapter 11 and receive $\mathcal{E}(\delta) - B = E(\delta, R_0, e) - B$. Specifically, they solve

$$E_0(\delta) = \sup_{T_B, T_L \in F^\delta} \mathbb{E}^\delta[\int_0^{T_B \wedge T_L} e^{-rt}(1-\tau)(\delta_t - C_0)dt + \mathbf{1}(T_B < T_L)e^{-rT_B}[\mathcal{E}(\delta_{T_B}) - B]]. \qquad (16)$$

Note that choosing stopping times T_B, T_L is equivalent to choosing $T = T_B \wedge T_L$ and whether to liquidate or enter Chapter 11 at time T. The latter decision is of course trivial since the firm will always choose the larger of $E(\delta_T, R_0, e) - B$ and 0. Thus (16) can be rewritten equivalently as

$$E_0(\delta) = \sup_{T \in F^\delta} \mathbb{E}^\delta[\int_0^T e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT} g(\delta_T)], \tag{17}$$

where $g(\delta) \equiv \max(0, \mathcal{E}(\delta) - B)$. Further, we can define the Ito process $G_t \equiv \int_0^t e^{-rs}(1-\tau)(\delta_s - C_0)ds$ and

$$\hat{g}(G, \delta, t) \equiv G + C_0/r + e^{-rt} g(\delta) \geq 0$$

to write

$$\hat{E}_0(\delta, G, t) = \sup_{T \in F^{\delta, G, t}} \mathbb{E}^{(\delta, G, t)}[\hat{g}(G_T, \delta_T, T)], \tag{18}$$

which exists by Øksendal (2003) Theorem 10.1.9. It is clear from inspection that $E_0(\delta) = \hat{E}_0(\delta, 0, 0) - C_0/r$. We can thus define, for any fixed C_0, the exercise region $S(C_0) = \{\delta : E_0(\delta) = g(\delta)\}$.

Proof of Proposition 5: For any fixed C, let $\delta_B(C), \delta_L(C)$ denote the corresponding optimal thresholds from Proposition 4 and Section 2.2, respectively. By Assumption 1, there exists $\overline{\delta}(C)$ such that

$$\frac{1-\tau}{r-\mu}\delta' - \frac{(1-\tau)C}{r} > g(\delta')$$

for all $\delta' > \overline{\delta}(C)$. Then it cannot be that $E_0(\delta') = g(\delta')$ for $\delta' > \overline{\delta}(C)$, or else deviating to $T = \infty$ would produce a reward greater than the value function, a contradiction. Thus $\overline{\delta}(C) = \sup\{\delta : E_0(\delta) = g(\delta)\}$ is finite. Suppose that $\mathcal{E}(\overline{\delta}(C)) > B$. Then, again by Øksendal (2003) Theorem 10.1.9, if we define $\overline{T} \equiv \inf\{t : \delta_t \leq \overline{\delta}(C)\}$, for $\delta > \overline{\delta}(C)$ we have

$$\begin{aligned} E_0(\delta) &= \mathbb{E}^\delta[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}} g(\delta_{\overline{T}})] \\ &= \mathbb{E}^\delta[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}}(\mathcal{E}(\overline{\delta}(C)) - B)]. \end{aligned}$$

Since $\delta_B(C)$ maximizes this by Proposition 4, it must be that $\overline{\delta}(C) = \delta_B(C)$.[11] Finally, suppose that $\mathcal{E}(\overline{\delta}(C)) \leq B$. Then for $\delta > \overline{\delta}(C)$ we have

$$
\begin{aligned}
E_0(\delta) &= \mathbb{E}^{\delta}[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-r\overline{T}} g(\delta_{\overline{T}})] \\
&= \mathbb{E}^{\delta}[\int_0^{\overline{T}} e^{-rt}(1-\tau)(\delta_t - C_0)dt],
\end{aligned}
$$

and again since $\delta_L(C)$ maximizes this, it must be that $\overline{\delta}(C) = \delta_L(C)$.

Given the existence of $\bar{\delta}(C)$, we may apply a standard formula for the first hitting time of a geometric Brownian motion to write the value function explicitly for $\delta > \bar{\delta}(C)$:

$$
E_0(\delta) = (1-\tau)[\frac{\delta}{r-\mu} - \frac{C}{r}] + (\frac{\delta}{\bar{\delta}(C)})^{\psi}[g(\bar{\delta}(C)) - (1-\tau)[\frac{\bar{\delta}(C)}{r-\mu} - \frac{C}{r}]].
$$

It will be helpful to define

$$
I(\delta, x, C) \equiv (1-\tau)[\frac{\delta}{r-\mu} - \frac{C}{r}] + (\frac{\delta}{x})^{\psi}[g(x) - (1-\tau)[\frac{x}{r-\mu} - \frac{C}{r}]]
$$

and note that, for fixed C, $E_0(\delta) = \sup_{x \leq \delta} I(\delta, x, C)$.

Finally, before we prove Proposition 6, the following lemma is useful.

Lemma D.1 $\mathcal{E}(\delta)$ *diverges to infinity as* $\delta \to \infty$.

Proof of Lemma: Using the notation of Appendix C, we first prove $c_2 > -1$. Multiply the expression for c_2 by $\hat{r}(\hat{r} + \lambda_e + \lambda_d)$:

[11] Specifically, we have $E^B \leq E_0$, so if δ_B were different from $\overline{\delta}(C)$ then E^B could be improved to E_0 by deviating to $\overline{T}$.

$$
\begin{aligned}
&- \hat{r}\frac{\lambda_d}{\lambda_d+\lambda_e}\iota - \lambda_d\iota - \lambda_e\hat{r} - \hat{r}\frac{\lambda_e}{\lambda_d+\lambda_e}\iota \\
&= -\hat{r}\iota - \lambda_d\iota - \lambda_e\hat{r} \\
&> -\hat{r}\iota - \hat{r}r - \lambda_d\iota - \lambda_d r - \lambda_e\hat{r} \\
&= -\hat{r}(\hat{r} + \lambda_d + \lambda_e).
\end{aligned}
$$

Since $c_2 > -1$, it must be that $c_1 < \theta$. If not, then asymptotically as $\delta \to \infty$, the debt value function approaches $c_1\delta + c_2 R > \theta\delta - R$, the reorganized firm value. This is a contradiction, since it implies equity accepts a negative offer. Therefore, in the limit as $\delta \to \infty$, we have $\mathcal{E}(\delta) = E(\delta, R_0, e) = (\theta - c_1)\delta - (1 - c_2)R_0$ starts to increase in δ and thus also goes to infinity.

Proof of Proposition 6: First, we note that if $\hat{C} > C$, then $S(C) \subset S(\hat{C})$.[12] As a result, $\bar{\delta}(C)$ must be weakly increasing in C.

Next, we show that $\bar{\delta}$ diverges to infinity. Suppose by contradiction this weren't the case: there exists K such that $\bar{\delta}(C) \leq K$ for all large enough C. For any $\epsilon > 0$, we can pick $\delta_0 > K$ arbitrarily high so that $(\delta_0/K)^\psi < \epsilon$. For arbitrary C, the value function as above must be

$$
E_0(\delta_0) = (1-\tau)[\frac{\delta_0}{r-\mu} - \frac{C}{r}] + (\frac{\delta_0}{\bar{\delta}(C)})^\psi[g(\bar{\delta}(C)) - (1-\tau)[\frac{\bar{\delta}(C)}{r-\mu} - \frac{C}{r}]].
$$

By continuity, $g(\delta) - (1-\tau)\delta/(r-\mu)$ attains a maximum H on the compact set $[0, K]$, so

$$
\begin{aligned}
E_0(\delta_0) &\leq (1-\tau)[\frac{\delta_0}{r-\mu} - \frac{C}{r}] + (\frac{\delta_0}{\bar{\delta}(C)})^\psi[H + \frac{C(1-\tau)}{r}] \\
&\leq (1-\tau)[\frac{\delta_0}{r-\mu} - \frac{C}{r}] + \epsilon[\max(H,0) + \frac{C(1-\tau)}{r}].
\end{aligned}
$$

Now, send C to infinity, keeping δ_0 constant and taking $\epsilon < 1$. This clearly becomes negative, a contradiction. Then $\bar{\delta}(C)$ increases to infinity as $C \to \infty$, and by the previous lemma $\mathcal{E}(\delta)$ increases

[12] The value function corresponding to $\hat{C}$ is clearly weakly smaller than that corresponding to C. If there were a point $y \in S(C)$ such that $y \notin S(\hat{C})$, then by definition of $S(C), S(\hat{C})$ the value function corresponding to $\hat{C}$ would be strictly larger at y (since the payoff is independent of $C, \hat{C}$) which is a contradiction.

to infinity as $\delta \to \infty$, so there must exist some $\bar{C}$ such that $\mathcal{E}(\bar{\delta}(C)) > B$ whenever $C > \bar{C}$. Since $\mathcal{E}(0) = 0$, if $\bar{\delta}(C)$ is continuous in C then we can take $\bar{C}$ to satisfy $\mathcal{E}(\bar{\delta}(\bar{C})) = B$.

The remainder of the proof shows $\bar{\delta}(C)$ is continuous in C. By the analysis of Appendix C, as $\delta \to \infty$ we have that $\mathcal{E}(\delta)$ converges to an affine function $a\delta + b$ with $0 < a$. By Assumption 1, $a < (1-\tau)/(r-\mu)$. Then there exists a constant d such that $\mathcal{E}(\delta) < a\delta + d$, so

$$\begin{aligned} &\frac{(1-\tau)C}{r} - \frac{(1-\tau)\delta}{r-\mu} + \mathcal{E}(\delta) - B \\ &< \frac{(1-\tau)C}{r} - \delta(\frac{(1-\tau)}{r-\mu} - a) + d. \end{aligned}$$

Then defining

$$\phi(C) \equiv (\frac{(1-\tau)}{r-\mu} - a)^{-1} d + (\frac{(1-\tau)}{r-\mu} - a)^{-1} \frac{(1-\tau)C}{r},$$

we have for any C that $\delta \geq \phi(C)$ implies

$$\frac{(1-\tau)C}{r} - \frac{(1-\tau)\delta}{r-\mu} + \mathcal{E}(\delta) - B < 0.$$

In this case, for $\delta > \phi(C)$, it must be that the value function $E_0(\delta)$ is defined by a lower threshold $\bar{\delta}(C)$ with $\bar{\delta}(C) < \delta$, since exercise for $\delta \geq \phi(C)$ is strictly suboptimal. We are now ready to show, for any $\tilde{C}$, that $\bar{\delta}(C)$ is continuous on $[0, \tilde{C}]$. To see this, fix $\delta > \phi(\tilde{C})$. Pick some arbitrary $C \in [0, \tilde{C}]$ and let $E_0^C(\delta)$ be the associated value function. As above,

$$E_0^C(\delta) = \sup_{x \leq \delta} I(\delta, x, C),$$

and since we just showed the value function has a lower threshold $x \leq \phi(C)$, it must be that this equals

$$= \sup_{x \leq \phi(C)} I(\delta, x, C).$$

This is a parameterized constrained optimization where the objective is continuous, and the correspondence $C \to [0, \phi(C)]$ is clearly continuous and compact valued. Define

$$I^*(\delta, x, C) \equiv \sup_{x \leq \phi(C)} I(\delta, x, C).$$

Applying Berge's Theorem, the correspondence

$$C \rightrightarrows Z(C) \equiv \{x : I^*(\delta, x, C) = I(\delta, x, C)\}$$

mapping C to the set of optimal x corresponding to C is upper hemicontinuous. Since any $x \in Z(C)$ must be in $S(C)$ (the region where $E_0^C(x) = g(x)$), we have $\bar{\delta}(C) \geq \sup Z(C)$, but if $\bar{\delta}(C) > \sup Z(C)$ then the firm is not acting optimally by definition of $Z(C)$, so $\bar{\delta}(C) = \sup Z(C)$. A standard argument shows the supremum of the image of an upper hemicontinuous correspondence is continuous, completing the proof.[13]

E Fixed costs of Chapter 11

This appendix provides more details on the costs of Chapter 11. Both debtors and creditors hire professionals (i.e., accountants, lawyers, investment bankers, financial advisors) who charge nontrivial fees. Fees incurred during bankruptcy are typically reimbursed from the estate (the firm's assets):

"The large bulk of bankruptcy professional fees and expenses are awarded under Bankruptcy Code Section 330(a). Section 330(a) awards are to professionals employed by the DIP... or employed by an official committee... the DIP pays the awards from the estate (LoPucki and Doherty (2011))."

Creditors have further opportunities for fee reimbursement through 11 USC § 503(b) and § 506(b). Weiss (1990) estimates that such fees average 3.1% of firm value, but LoPucki and Doherty (2011) give many reasons why this is an underestimate. In extreme cases like the bankruptcy of Allied Holdings, fees can reach 22% of firm assets (LoPucki and Doherty (2011) Appendix A).

While these fees are typically awarded and thus subtracted from the total estate to be split between creditors, prepetition fees are an important exception. Firms hire professionals prior to filing for Chapter 11 (prepetition), and the court "does not award prepetition fees" (LoPucki and

[13] For any ϵ, C, by upper hemicontinuity there exists ϵ_1 such that for all y satisfying $|y - C| < \epsilon_1$ and any $z \in Z(C), u \in Z(y)$, we have $|z - u| < \epsilon/3$. Pick $z \in Z(C)$ with $|\sup(Z(C)) - z| < \epsilon/3$. Then, for any y satisfying $|y - C| < \epsilon_1$, choose $u \in Z(y)$ with $|\sup(Z(y)) - u| < \epsilon/3$ so by triangle inequality $|\sup(Z(y)) - \sup(Z(C))| < \epsilon$.

Doherty (2011)). Indeed, while firms are supposed to report prepetition fees and expenses in connection with a future Chapter 11 under 11 USC § 329(a), they frequently fail to report. Within the dataset used for LoPucki and Doherty (2011) (which is generously provided on LoPucki's website), prepetition fees averaged 43% of the subsequent total 11 USC § 330(a) awards in cases where the firm reported both. These fees must be paid by the firm (i.e., equityholders) since they are not awardable.

Fixed costs: Larger firms and firms with longer bankruptcies certainly pay more in professional fees. However, there is substantial empirical evidence suggesting these fees have a fixed cost component. White (1989) assumes entering Chapter 11 entails a fixed cost, citing court fees, lawyers' fees, and the lost time of management. White (2016) surveys the literature on small business bankruptcy and states "that the costs of Chapter 11 reorganization are high and that they have a fixed component that prevents small corporations from using the procedure."

Early bankruptcy studies focused on particular industries, and many found evidence of fixed costs. For example, Warner (1977) examines railroad bankruptcies and finds "this evidence suggests that there are substantial fixed costs associated with the railroad bankruptcy process, and hence economies of scale with respect to bankruptcy costs." He also finds that the length of bankruptcy cannot explain these costs. Guffey and Moore (1991) find that direct bankruptcy costs exhibit substantial economies of scale in a sample of trucking firms.

Later studies examine bankruptcies across many industries and find similar results. For example, Table 1 of LoPucki and Doherty (2004) presents results of a regression of log fees on log assets, log number of days in bankruptcy, the number of professional firms, and a constant. They find significant coefficients of 0.414 and 0.535 on assets and length of bankruptcy, respectively. The constant term, however, is large and positive with a t-statistic nearly three times as large as either assets or bankruptcy length. This suggests that larger and longer bankruptcies entail higher fees, but there is a significant fixed cost faced by all firms. As an extreme example, Farm Fresh Inc., with less than $200 million in assets upon declaring bankruptcy, spent three million in fees on a 44 day bankruptcy (LoPucki and Doherty (2011) Appendix A).

BWZ (2006) examine direct and indirect costs of bankruptcy in a comprehensive sample of small and large corporate bankruptcies in Arizona and New York from 1995 to 2001. In section V.B, they show that the ratio of chapter 11 expenses to pre-bankruptcy assets is drastically higher for small firms than for large firms. For firms with less than \$100,000 in pre-bankruptcy assets, expenses average 31.5% of assets, while for firms with \$100,000 to \$1 million in assets, fees average 10.2% of assets. For firms with more than \$10 million in assets, fees average 1.3% of assets. This stark result is consistent with a large fixed cost component of legal fees. Again, the economies of scale which bankruptcy fees exhibit are not driven by shorter bankruptcies for larger firms. In section IV.A they find the weakly positive relationship between firm size (measured in asset value) and bankruptcy duration is not statistically significant.

Put together, these five studies (Warner (1977); Guffey and Moore (1991); LoPucki and Doherty (2004); LoPucki and Doherty (2011); BWZ (2006)) all document economies of scale in bankruptcy fees that strongly suggest a fixed cost component to bankruptcy fees.

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