Benchmark interest rates when the government is risky – Online Appendix[*]

Patrick Augustin,[†] Mikhail Chernov[‡] Lukas Schmid[§] and Dongho Song[¶]

Abstract

Since the Global Financial Crisis, interest rate swap rates, which represent future uncollateralized interbank borrowing, have fallen below maturity-matched Treasury rates. This is surprising, because U.S. Treasuries, which are deemed expensive because of superior liquidity and safety, should produce yields that are lower than those of swap rates. We show, by no-arbitrage, that sovereign default risk explains negative swap spreads even without frictions such as balance sheet constraints, convenience yield, and hedging demand. We support this explanation with an equilibrium model that jointly accounts for macroeconomic fundamentals and the term structures of interest and U.S. credit default swap rates.

JEL classification codes: C1, E43, E44, G12, H60

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[†]Corresponding author. Desautels Faculty of Management, McGill University, and Canadian Derivatives Institute; 1001 Sherbrooke Street West, Montréal, QC H3A 1G5, Canada; patrick.augustin@mcgill.ca.

[‡]Anderson School of Management, UCLA, NBER, and CEPR; 101 Westwood Plaza, Los Angeles, CA 90095, USA; mikhail.chernov@anderson.ucla.edu.

[§]Fuqua School of Business, Duke University, and CEPR; 100 Fuqua Drive, Durham, NC, 27708, USA; lukas.schmid@duke.edu.

[¶]Carey Business School, Johns Hopkins University; 100 International Drive, Baltimore, MD 21202, USA; dongho.song@jhu.edu.

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A. Derivation of the real pricing kernel

Since utility is defined by a constant elasticity of substitution recursion and the certainty equivalent is homogeneous of degree one, we can scale utility and take logs:

$$\log\left(U_t/C_t\right) \equiv u_t = \rho^{-1}\log\left[(1-\beta)+\beta\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right)^{\rho}\right].$$

Taking a first-order Taylor approximation of u_t around the point $E\left[\log\mu_t\right]=\log\mu$, we obtain the log-linearized form

$$\begin{aligned}
u_t &\approx \rho^{-1}\log\left[(1-\beta)+\beta e^{\rho\log\mu}\right]+\rho^{-1}\frac{\beta e^{\rho\log\mu}\rho}{\left[(1-\beta)+\beta e^{\rho\log\mu}\right]}\left[\log\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right)-\log\mu\right]\\
&\approx b_0+b_1\log\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right),
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \beta e^{\rho\log\mu}\left((1-\beta)+\beta e^{\rho\log\mu}\right)^{-1}\\
b_0 &= \rho^{-1}\log\left[(1-\beta)+\beta e^{\rho\log\mu}\right]-b_1\log\mu.
\end{aligned}$$

The state vector $x_t=(\Delta c_t,\pi_t,d_t,g_t,w_{c,t},w_{\pi,t},w_{y,t},w_{g,t},v_{1,t},v_{2,t})$ describes the economy, where $\Delta c_t=\log(C_{t+1}/C_t)$ is log consumption growth, π_t is inflation, d_t is log output growth, g_t is the government expenditure to output ratio, $w_t=\left[w_{c,t},w_{\pi,t},w_{d,t},w_{g,t}\right]^{\top}$ is the vector of moving average components, and $v_t=\left[v_{1,t},v_{2,t}\right]^{\top}$ is a vector of common stochastic variance processes.

Guess that the log scaled utility is affine in the state vector x_t

$$\begin{aligned}
u_t &= \log u+P_x^{\top}x_t=\log u+P_y^{\top}y_t+p_v^{\top}v_t\\
&= \log u+p_c\Delta c_t+p_{\pi}\pi_t+p_d d_t+p_g g_t+p_{v1}v_{1,t}+p_{v2}v_{2,t},
\end{aligned}$$

which implies that $P_x=\left[P_y^{\top},p_v^{\top}\right]^{\top}=\left[p_c,p_{\pi},p_d,p_g,0,0,0,0,p_{v1},p_{v2}\right]^{\top}$.

Next compute $\log\left(e^{\Delta c_{t+1}+u_{t+1}}\right)$ and $\log\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right)$, plug terms into the log-linarized scaled utility u_t and verify. Given the initial guess, this results in a system of seven equations, which can be solved using the method of undetermined coefficients for the constant and the loadings of the log scaled utility on x_t. For the derivations, define the coordinate vectors e_i $(i=1,2,\ldots 8)$ and e_{v_i} $(i=1,2)$ with all elements equal to zero except element i, which is equal to one.

Step 1: Compute $\log\left(e^{\Delta c_{t+1}+u_{t+1}}\right)$*:*

$$\begin{aligned}
\log\left(e^{\Delta c_{t+1}+u_{t+1}}\right) &= \Delta c_{t+1}+u_{t+1}=e_1^{\top}y_{t+1}+\log u+P_y^{\top}y_{t+1}+p_v^{\top}v_{t+1}\\
&= \log u+\left(P_y+e_1\right)^{\top}\mu_y+\left(P_y+e_1\right)^{\top}\Phi_y y_t+\left(P_y+e_1\right)^{\top}\Phi_{yv}v_t\\
&+ \left(P_y+e_1\right)^{\top}\Sigma_y V_{y,t}^{1/2}\varepsilon_{y,t+1}+p_v^{\top}v_{t+1}.
\end{aligned}$$

Step 2: Compute $\log\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right)$*:*

$$\begin{aligned}
\log\mu_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right) &= \log\left[E_t\left(e^{\Delta c_{t+1}+u_{t+1}}\right)^{\alpha}\right]^{1/\alpha}=\alpha^{-1}\log\left[E_t\left(e^{\alpha\left(\Delta c_{t+1}+u_{t+1}\right)}\right)\right]\\
&= \log u+\left(P_y+e_1\right)^{\top}\mu_y+\left(P_y+e_1\right)^{\top}\Phi_y y_t+\left(P_y+e_1\right)^{\top}\Phi_{yv}v_t\\
&+ \frac{\alpha}{2}\left(P_y+e_1\right)^{\top}\Omega_{y,t}\left(P_y+e_1\right)+\sum_{j=1}^{2}\frac{-v_{v_j}}{\alpha}\log\left(1-\alpha p_{v_j}c_{v_j}\right)+\frac{p_{v_j}\phi_{v_j}}{1-\alpha p_{v_j}c_{v_j}}v_{j,t},
\end{aligned}$$

where $\Omega_{y,t} = \Sigma_y V_{y,t} \Sigma_y^\top$.

Step 3: Plug into u_t and verify:

$$\begin{aligned}
u_t &\approx b_0 + b_1 \log \mu_t \left(e^{\Delta c_{t+1} + u_{t+1}} \right) \\
&= b_0 + b_1 \left[\log u + (P_y + e_1)^\top \mu_y - \sum_{j=1}^{2} \frac{v_{v_j}}{\alpha} \log \left(1 - \alpha p_{v_j} c_{v_j}\right) \right] + b_1 \left[(P_y + e_1)^\top \Phi_y y_t \right] \\
&+ b_1 \left[(P_y + e_1)^\top \Phi_{yv} v_t + \frac{\alpha}{2} (P_y + e_1)^\top \Omega_{y,t} (P_y + e_1) + \sum_{j=1}^{2} \frac{p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} v_{j,t} \right].
\end{aligned}$$

Given the initial guess, this results in a system of seven equations:

$$\begin{aligned}
\log u &= b_0 + b_1 \Big[\log u + (P_y + e_1)^\top \mu_y + \frac{\alpha}{2} (P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{A} \right) \right] \Sigma_y^\top (P_y + e_1) \\
&\quad - \sum_{j=1}^{2} \frac{v_{v_j}}{\alpha} \log \left(1 - \alpha p_{v_j} c_{v_j}\right) \Big] \\
p_c &= b_1 (P_y + e_1)^\top \Phi_y e_1 \\
p_\pi &= u_1 (P_y + e_1)^\top \Phi_y e_2 \\
p_d &= b_1 (P_y + e_1)^\top \Phi_y e_3 \\
p_g &= b_1 (P_y + e_1)^\top \Phi_y e_4 \\
p_{v_1} &= b_1 \left[(P_y + e_1)^\top \Phi_{yv} e_{v_1} + \frac{\alpha}{2} (P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_1 \right) \right] \Sigma_y^\top (P_y + e_1) + \frac{p_{v_1} \phi_{v_1}}{1 - \alpha p_{v_1} c_{v_1}} \right] \\
p_{v_2} &= b_1 \left[(P_y + e_1)^\top \Phi_{yv} e_{v_2} + \frac{\alpha}{2} (P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_2 \right) \right] \Sigma_y^\top (P_y + e_1) + \frac{p_{v_2} \phi_{v_2}}{1 - \alpha p_{v_2} c_{v_2}} \right],
\end{aligned}$$

where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\mathbf{I}$ is the identity matrix, and $\mathbf{1}$ is a column vector of ones, and where we have defined the column vectors $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}_2$ as follows:

$$\begin{aligned}
\mathbf{A} &= [a_c, a_\pi, a_d, a_g, 0, 0, 0, 0]^\top \\
\mathbf{B}_1 &= [b_{cv_1}, b_{\pi v_1}, b_{dv_1}, b_{gv_1}, 0, 0, 0, 0]^\top \\
\mathbf{B}_2 &= [b_{cv_2}, b_{\pi v_2}, b_{dv_2}, b_{gv_2}, 0, 0, 0, 0]^\top .
\end{aligned} \tag{A.1}$$

Since, the equations for p_c, p_π, p_d, and p_g are linear, their solutions are given by

$$\begin{aligned}
p_c &= -e_1^\top \left(b_1^2 \left[\Phi_z - \mathbf{I} \right]^{-1} \phi_c \right) \\
p_\pi &= -e_2^\top \left(b_1^2 \left[\Phi_z - \mathbf{I} \right]^{-1} \phi_c \right) \\
p_y &= -e_3^\top \left(b_1^2 \left[\Phi_z - \mathbf{I} \right]^{-1} \phi_c \right) \\
p_g &= -e_4^\top \left(b_1^2 \left[\Phi_z - \mathbf{I} \right]^{-1} \phi_c \right).
\end{aligned}$$

The equations for p_v are quadratic and have two roots. We choose the root such that $\lim c_j p_{v_j} = 0$ as

$c_j \to 0$. We have that for $j = 1, 2$:

$$\underbrace{\left[b_1\left(\phi_{v_j} - \alpha c_{v_j}\left[(P_y + e_1)^\top \Phi_{yv} e_{v_j} + \frac{\alpha}{2}(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \Sigma_y^\top (P_y + e_1)\right]\right) - 1\right]}_{B_j^*} p_{v_j}$$

$$+ \underbrace{\alpha c_{v_j}}_{A_j^*} p_{v_j}^2 + \underbrace{b_1\left[(P_y + e_1)^\top \Phi_{yv} e_{v_j} + \frac{\alpha}{2}(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \Sigma_y^\top (P_y + e_1)\right]}_{C_j^*} = 0,$$

where the roots to the quadratic equation are determined by:

$$p_{v_j} = \frac{-B_j^* + / - \sqrt{\left(B_j^*\right)^2 - 4A_j^* C_j^*}}{2A_j^*}.$$

Finally, we have that

$$\begin{aligned} \log u &= (1 - b_1)^{-1}\left[b_0 + b_1\left((P_y + e_1)^\top \mu_y + \frac{\alpha}{2}(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{A}\right)\right] \Sigma_y^\top (P_y + e_1)\right.\right. \\ &\quad \left.\left. - \sum_{j=1}^{2} \frac{v_{v_j}}{\alpha} \log\left(1 - \alpha p_{v_j} c_{v_j}\right)\right)\right]. \end{aligned}$$

Plugging terms into the expression for the marginal rate of substitution, we obtain the final solution to the real pricing kernel

$$\begin{aligned} \widehat{m}_{t,t+1} &= \bar{m} + (\rho - 1)\, e_1^\top \Phi_y y_t + (\rho - 1)\, e_1^\top \Phi_{yv} v_t \\ &- \sum_{j=1}^{2}\left[\frac{(\alpha - \rho)\, p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} + \frac{\alpha}{2}(\alpha - \rho)(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \Sigma_y^\top (P_y + e_1)\right] v_{j,t} \\ &+ \left[(\rho - 1)\, e_1 + (\alpha - \rho)(P_y + e_1)\right]^\top \Sigma_y V_{y,t}^{1/2} \varepsilon_{y,t+1} + (\alpha - \rho)\, p_v^\top v_{t+1}, \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} \bar{m} &= \log \beta + (\rho - 1)\, e_1^\top \mu_y + (\alpha - \rho) \sum_{j=1}^{2} \frac{v_{v_j}}{\alpha} \log\left(1 - \alpha p_{v_j} c_{v_j}\right) \\ &- \frac{\alpha}{2}(\alpha - \rho)(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{A}\right)\right] \Sigma_y^\top (P_y + e_1). \end{aligned} \tag{A.3}$$

The numerical solution to the mean log certainty equivalent $E[\log \mu_t] = \log \mu$ depends on the approximated constants from the log-linearization of the scaled log utility b_0 and b_1, which themselves depend on the mean log certainty equivalent $\log \mu$. Model consistency thus requires to solve a fixed-point equation for the mean log certainty equivalent. More specifically, using a convergence criterion of $10e^{-12}$, we solve for the fixed-point equation $\log \mu = f(\log \mu)$.

B. Valuation

B.1. Term structure of real interest rates

The price of an n-period real zero-coupon bond must satisfy the Euler equation $\widehat{P}_t^n = E_t\left[\widehat{M}_{t,t+n}\right]$. To derive closed-form solutions for the term structure of real interest rates, we conjecture that log zero-coupon bond prices $\widehat{p}_t$ are affine in the state vector x_t

$$\widehat{p}_t^n = \log \widehat{P}_t^n = -\widehat{A}_n - \widehat{B}_{y,n}^\top y_t - \widehat{B}_{v,n}^\top v_t,$$

where the coefficients of the vectors $\widehat{B}_{Y,n}$ and $\widehat{B}_{v,n}$ measure the sensitivity of real bond prices to the risk factors and where n refers to the maturity of the bond. Since the real pricing kernel is an affine function of the state vector, log bond prices are fully characterized by the cumulant-generating function of X_t. The law of iterated expectations implies that $\widehat{P}_t^n$ satisfies the recursion

$$\widehat{P}_t^n = E_t\left[\widehat{M}_{t,t+1}\widehat{P}_{t+1}^{n-1}\right].$$

It can be shown that for all n, the scalar $\widehat{A}_n$ and the components of the column vectors $\widehat{B}_{y_j,n}$ for $j = 1, 2, \ldots, 8$ and $\widehat{B}_{v_j,n}$ for $j = 1, 2$, are given by

$$\begin{aligned}
\widehat{A}_n &= \widehat{A}_{n-1} - \bar{m} + \widehat{B}_{y,n-1}^\top \mu_y + \sum_{j=1}^{2} v_{v_j} \log\left(1 - \left[(\alpha-\rho)\, p_{v_j} - \widehat{B}_{vj,n-1}\right] c_{v_j}\right) \\
&- \frac{1}{2}\left[(\rho-1)\, e_1 + (\alpha-\rho)\,(P_y + e_1) - \widehat{B}_{y,n-1}\right]^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{A}\right)\right] \Sigma_y^\top \\
&\times \left[(\rho-1)\, e_1 + (\alpha-\rho)\,(P_y + e_1) - \widehat{B}_{y,n-1}\right] \\
\widehat{B}_{y_j,n} &= \left[\widehat{B}_{y,n-1} - (\rho-1)\, e_1\right]^\top \Phi_y e_j \\
\widehat{B}_{v_j,n} &= \left[\widehat{B}_{y,n-1} - (\rho-1)\, e_1\right]^\top \Phi_{yv} e_{v_j} \\
&+ \frac{(\alpha-\rho)\, p_{v_j}\phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} + \frac{\alpha}{2}(\alpha-\rho)\,(P_y + e_1)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \Sigma_y^\top (P_y + e_1) \\
&- \frac{1}{2}\left[(\rho-1)\, e_1 + (\alpha-\rho)\,(P_y + e_1) - \widehat{B}_{y,n-1}\right]^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \\
&\times \Sigma_y^\top \left[(\rho-1)\, e_1 + (\alpha-\rho)\,(P_y + e_1) - \widehat{B}_{y,n-1}\right] - \frac{\left[(\alpha-\rho)\, p_{v_j} - \widehat{B}_{v_j,n-1}\right]\phi_{v_j}}{1 - \left[(\alpha-\rho)\, p_{v_j} - \widehat{B}_{v_j,n-1}\right] c_{v_j}},
\end{aligned}$$

with initial conditions $\widehat{A}_0 = 0$, $\widehat{B}_{y,0} = \mathbf{0}$, and $\widehat{B}_{v,0} = \mathbf{0}$, and where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a column vector of ones, e_i $(i = 1, 2, \ldots 8)$ and e_{v_i} $(i = 1, 2)$ are coordinate vectors with all elements equal to zero except element $i = 1$, the column vectors $\mathbf{A}$, and $\mathbf{B}_j$ for $j = 1, 2$ are defined in Equation (A.1), and $\bar{m}$ is defined in Equation (A.3).

It follows naturally that the term structure of real interest rates is given by:

$$\widehat{y}_t^n = n^{-1}\left(\widehat{A}_n + \widehat{B}_{y,n}^\top y_t + \widehat{B}_{v,n}^\top v_t\right).$$

B.2. Term structure of nominal interest rates

The price of an n-period nominal zero-coupon bond must satisfy the Euler equation $P_t^n = E_t\left[M_{t,t+n}\right]$, where $M_{t,t+1}$ defines the nominal stochastic discount factor defined in logs as

$$m_{t,t+1} = \widehat{m}_{t,t+1} - \pi_{t+1} = \widehat{m}_{t,t+1} - e_2^\top y_{t+1},$$

with the real pricing kernel $\widehat{m}_{t,t+1}$ defined in Equation (A.2). To derive closed-form solutions for the term structure of nominal interest rates, we conjecture that log zero-coupon bond prices p_t are affine in the state vector x_t

$$p_t^n = \log P_t^n = -A_n - B_{y,n}^\top y_t - B_{v,n}^\top v_t,$$

where the coefficients of the vectors $B_{y,n}$ and $B_{v,n}$ measure the sensitivity of nominal bond prices to the risk factors and where n refers to the maturity of the bond. Since the nominal pricing kernel is an affine function of the state vector, log bond prices are fully characterized by the cumulant-generating function of x_t. The law of iterated expectations implies that P_t^n satisfies the recursion

$$P_t^n = E_t\left[M_{t,t+1}P_{t+1}^{n-1}\right].$$

It can be shown that for all n, the scalar A_n and the components of the column vectors $B_{y_j,n}$ for $j = 1, 2, \dots, 8$ and $B_{v_j,n}$ for $j = 1, 2$, are given by

$$\begin{aligned}
A_n &= A_{n-1} - \bar{m} + \left[e_2 + B_{y,n-1}\right]^\top \mu_y + \sum_{j=1}^{2} v_{v_j} \log\left(1 - \left[(\alpha-\rho)\, p_{v_j} - B_{vj,n-1}\right] c_{v_j}\right) \\
&- \frac{1}{2}\left[(\rho-1)\, e_1 + (\alpha-\rho)\left(P_y + e_1\right) - e_2 - B_{y,n-1}\right]^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{A}\right)\right] \\
&\times \Sigma_y^\top \left[(\rho-1)\, e_1 + (\alpha-\rho)\left(P_y + e_1\right) - e_2 - B_{y,n-1}\right] \\
B_{y_j,n} &= \left[B_{y,n-1} + e_2 - (\rho-1)\, e_1\right]^\top \Phi_y e_j \\
B_{v_j,n} &= \left[B_{y,n-1} + e_2 - (\rho-1)\, e_1\right]^\top \Phi_{yv} e_{v_j} \\
&+ \frac{(\alpha-\rho)\, p_{v_j}\phi_{v_j}}{1-\alpha p_{v_j} c_{v_j}} + \frac{\alpha}{2}(\alpha-\rho)\left(P_y + e_1\right)^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \Sigma_y^\top \left(P_y + e_1\right) \\
&- \frac{1}{2}\left[(\rho-1)\, e_1 + (\alpha-\rho)\left(P_y + e_1\right) - e_2 - B_{y,n-1}\right]^\top \Sigma_y \left[\mathbf{I} \odot \left(\mathbf{1}^\top \otimes \mathbf{B}_j\right)\right] \\
&\times \Sigma_y^\top \left[(\rho-1)\, e_1 + (\alpha-\rho)\left(P_y + e_1\right) - e_2 - B_{y,n-1}\right] - \frac{\left[(\alpha-\rho)\, p_{v_j} - B_{v_j,n-1}\right]\phi_{v_j}}{1-\left[(\alpha-\rho)\, p_{v_j} - B_{v_j,n-1}\right] c_{v_j}},
\end{aligned}$$

with initial conditions $A_0 = 0$, $B_{y,0} = \mathbf{0}$, and $B_{v,0} = \mathbf{0}$, and where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a column vector of ones, e_i $(i = 1, 2, \dots 8)$ and e_{v_i} $(i = 1, 2)$ are coordinate vectors with all elements equal to zero except element $i = 1$, the column vectors $\mathbf{A}$ and $\mathbf{B}_j$ for $j = 1, 2$ are defined in Equation (A.1), and $\bar{m}$ is defined in Equation (A.3).

It follows naturally that the term structure of nominal interest rates is given by:

$$y_t^n = n^{-1}\left(A_n + B_{y,n}^\top y_t + B_{v,n}^\top v_t\right).$$

B.3. Term structure of risky treasury yields

U.S. default risk is driven by a default intensity h_t defined as

$$h_t = h + h_c \Delta c_t + h_d d_t + h_g g_t + h_{v_1} v_{1,t} + h_{v_2} v_{2,t} = h + h_z^\top z_t + h_v^\top v_t, \tag{B.1}$$

such that $h_z = [h_c, h_\pi, h_d, h_g]^\top$ and $h_v = [h_{v_1}, h_{v_2}]^\top$. We adopt the convention that $h_y = [h_z, 0, 0, 0, 0]^\top$, and $h_{\widetilde{y}} = [h_y, 0, 0, 0]^\top$. We connect the default intensity to $\mathcal{H}_t$, the conditional default probability of a given reference entity at day t via $\mathcal{H}_t \equiv Prob\,(\tau = t \mid \tau \geq t; \mathcal{F}_t) = 1 - e^{-h_t}$, where $\mathcal{F}_t$ denotes all the available information available at time t, with the exception of credit events. This implies that the probability of survival (no credit event) until time t is:

$$\mathcal{S}_t \equiv Prob\,(\tau > t \mid \mathcal{F}_t) = \mathcal{S}_0 \prod_{j=1}^{t}\left(1 - \mathcal{H}_j\right), \quad t \geq 1. \tag{B.2}$$

To price risky zero coupon Treasury bonds, we take into account the convenience yield $s_{1,t}$ with dynamics defined in Equation (4), and loss given default L. Using the law of iterated expectations, it is possible to show that risky bond prices follow the recursion

$$\begin{aligned}
\widetilde{P}_t^n &= E_t\left(M_{t,t+1}e^{s_{1,t+1}}\left[\mathcal{I}(\tau>t+1)+(1-L)\cdot\mathcal{I}(t<\tau\leq t+1)\right]\cdot\widetilde{P}_{t+1}^{n-1}\right)\\
&= E_t\left(M_{t,t+1}e^{s_{1,t+1}}\left[1-L\mathcal{H}_{t+1}\right]\cdot\widetilde{P}_{t+1}^{n-1}\right)\\
&\approx E_t e^{\sum_{j=1}^{n} m_{t+j-1,t+j}-L\cdot h_{t+j}+s_{1,t+j}},
\end{aligned}$$

where we follow Duffie and Singleton (1999) by applying a first order Taylor approximation of $\log(1-L\mathcal{H}_t)$ around 0 such that $\log(1-L\mathcal{H}_t)\approx -L\cdot h_t$. Since all elements of the bond pricing equation are affine functions of the extended state vector, log bond prices are fully characterized by the cumulant-generating function of $\widetilde{x}_t$. The law of iterated expectations implies that $\widetilde{P}_t^n$ satisfies the recursion

$$\widetilde{P}_t^n = E_t\left[M_{t,t+1}e^{-L\cdot h_{t+1}+s_{1,t+1}}\widetilde{P}_{t+1}^{n-1}\right].$$

To derive closed-form solutions for the term structure of risky Treasury rates, we conjecture that log prices of risky zero-coupon bonds $\widetilde{p}_t$ are affine in the extended state vector $\widetilde{x}_t=[y_t^\top,s^\top,v_t^\top]^\top=[\widetilde{y}_t^\top,v_t^\top]^\top$:

$$\widetilde{p}_t^n=\log\widetilde{P}_t^n=-\widetilde{A}_n-\widetilde{B}_{\widetilde{y},n}^\top\widetilde{y}_t-\widetilde{B}_{v,n}^\top v_t.$$

where the coefficients of the vectors $\widetilde{B}_{\widetilde{y},n}$ and $\widetilde{B}_{v,n}$ measure the sensitivity of risky bond prices to the risk factors and where n refers to the maturity of the bond. It can be shown that for all n, the scalar $\widetilde{A}_n$ and the components of the column vectors $\widetilde{B}_{\widetilde{y}_j,n}$ for $j=1,2,\ldots,11$ and $\widetilde{B}_{v_j,n}$ for $j=1,2$, are given by

$$\begin{aligned}
\widetilde{A}_n &= \widetilde{A}_{n-1}+L\cdot h-\bar{m}+\left[\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right]^\top\widetilde{\mu}_y\\
&+ \sum_{j=1}^{2}v_{v_j}\log\Big(1-\Big[(\alpha-\rho)\,p_{v_j}-L\cdot h_{v_j}-\widetilde{B}_{v_j,n-1}\\
&\quad+\left(\widetilde{B}_{s_1,n-1}+1\right)v_{s_1v_j}+\widetilde{B}_{s_2,n-1}v_{s_2v_j}+\widetilde{B}_{s_3,n-1}v_{s_3v_j}\Big]c_{v_j}\Big)\\
&- \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1+(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)-\left(\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right)\right]^\top\widetilde{\Sigma}_y\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{A}}\right)\right]\\
&\times \widetilde{\Sigma}_y^\top\left[(\rho-1)\,\widetilde{e}_1+(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)-\left(\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right)\right]\\
&- \sum_{j=1}^{2}\left(\left(\widetilde{B}_{s_1,n-1}+1\right)v_{s_1v_j}+\widetilde{B}_{s_2,n-1}v_{s_2v_j}+\widetilde{B}_{s_3,n-1}v_{s_3v_j}\right)v_{v_j}c_{v_j}\\
\widetilde{B}_{\widetilde{y}_j,n} &= \left[\left(\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right)-(\rho-1)\,\widetilde{e}_1\right]^\top\widetilde{\Phi}_y\widetilde{e}_j\\
&+ \frac{(\alpha-\rho)\,p_{v_j}\phi_{v_j}}{1-\alpha p_{v_j}c_{v_j}}+\frac{\alpha}{2}(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)^\top\widetilde{\Sigma}_y\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{B}}_j\right)\right]\widetilde{\Sigma}_y^\top(P_{\widetilde{y}}+\widetilde{e}_1)\\
&- \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1+(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)-\left(\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right)\right]^\top\widetilde{\Sigma}_y\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{B}}_j\right)\right]\\
&\times \widetilde{\Sigma_y}^\top\left[(\rho-1)\,\widetilde{e}_1+(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)-\left(\widetilde{e}_2+L\cdot h_{\widetilde{y}}-\widetilde{e}_9+\widetilde{B}_{\widetilde{y},n-1}\right)\right]\\
&- \frac{\left[(\alpha-\rho)\,p_{v_j}-L\cdot h_{v_j}-\widetilde{B}_{v_j,n-1}+\left(\widetilde{B}_{s_1,n-1}+1\right)v_{s_1v_j}+\widetilde{B}_{s_2,n-1}v_{s_2v_j}+\widetilde{B}_{s_3,n-1}v_{s_3v_j}\right]\phi_{v_j}}{1-\left[(\alpha-\rho)\,p_{v_j}-L\cdot h_{v_j}-\widetilde{B}_{v_j,n-1}+\left(\widetilde{B}_{s_1,n-1}+1\right)v_{s_1v_j}+\widetilde{B}_{s_2,n-1}v_{s_2v_j}+\widetilde{B}_{s_3,n-1}v_{s_3v_j}\right]c_{v_j}}\\
&+ \left(\left(\widetilde{B}_{s_1,n-1}+1\right)v_{s_1v_j}+\widetilde{B}_{s_2,n-1}v_{s_2v_j}+\widetilde{B}_{s_3,n-1}v_{s_3v_j}\right)\phi_{v_j},
\end{aligned}$$

with initial conditions $\widetilde{A}_0=0$, $\widetilde{B}_{\widetilde{y},0}=\mathbf{0}$, and $\widetilde{B}_{v,0}=\mathbf{0}$, and where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\widetilde{\mathbf{I}}$ is the identity matrix, $\widetilde{\mathbf{1}}$ is a column vector of ones, $\widetilde{e}_i$ $(i=1,2,\ldots 11)$ and e_{v_i} $(i=1,2)$ are coordinate vectors with all elements equal to zero except element $i=1$, $\bar{m}$ is defined in Equation (A.3), and the column vectors $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_j$ for $j = 1, 2$ are given by

$$\begin{aligned} \widetilde{\mathbf{A}} &= [a_c, a_\pi, a_d, a_g, 0, 0, 0, 0, 1, 1, 1]^\top \\ \widetilde{\mathbf{B}}_j &= \left[b_{cv_j}, b_{\pi v_j}, b_{dv_j}, b_{gv_j}, 0, 0, 0, 0, 0, 0, 0\right]^\top . \end{aligned} \tag{B.3}$$

It follows naturally that the term structure of risky interest rates is given by:

$$\widetilde{y}_t^n = n^{-1}\left(\widetilde{A}_n + \widetilde{B}_{\widetilde{y},n}^\top \widetilde{y}_t + \widetilde{B}_{v,n}^\top v_t\right).$$

B.4. Term structure of LIBOR rates

We work with hypothetical zero-coupon LIBOR bonds L_t^n discounted at the continuously compounded yield ℓ_t^n (defined at the monthly frequency), such that $L_t^n = \exp(-\ell_t^n \cdot n)$, where $n \leq 12$ corresponds to LIBOR rate maturities of up to 12 months. To price LIBOR bonds, we take into account the convenience yield $s_{1,t}$ and bank risk $s_{2,t}$, with dynamics defined in Equation (4), and loss given default L. The LIBOR rate is defined as $\ell_t = \widetilde{y}_t^1 + s_{1,t} + s_{2,t}$. Using the law of iterated expectations, it is possible to show that LIBOR bond prices follow the recursion

$$L_t^n \approx E_t e^{\sum_{j=1}^n m_{t+j-1,t+j} - L \cdot h_{t+j} - s_{2,t+j}}.$$

Following the logic developed for risky Treasury bonds in appendix B.3, it is straightforward to show that the log price of a risky n-period zero coupon LIBOR bond is affine in the extended state space $\widetilde{x}_t$:

$$\log L_t^n = -\bar{A}_n - \bar{B}_{\widetilde{y},n}^\top \widetilde{y}_t - \bar{B}_{v,n}^\top v_t. \tag{B.4}$$

where the constant $\bar{A}_n$ and the coefficients of the column vectors $\bar{B}_{\widetilde{y},n}$ and $\bar{B}_{v,n}$ measure the sensitivity of LIBOR bond prices to the risk factors and where n refers to the maturity of the bond. It can be shown that for all n, the scalar $\bar{A}_n$ and the components of the column vectors $\bar{B}_{\widetilde{y}_j,n}$ for $j = 1, 2, \ldots, 11$ and $\bar{B}_{v_j,n}$ for $j = 1, 2$, are given by

$$\begin{aligned}
\bar{A}_n &= \bar{A}_{n-1} + L \cdot h - \bar{m} + \left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right)^\top \widetilde{\mu}_y \\
&+ \sum_{j=1}^{2} v_{v_j} \log\big(1 - \left[(\alpha - \rho) p_{v_j} - L \cdot h_{v_j} - \bar{B}_{v_j,n-1}\right. \\
&\quad \left. + \bar{B}_{s_1,n-1} v_{s_1 v_j} + \left(\bar{B}_{s_2,n-1} - 1\right) v_{s_2 v_j} + \bar{B}_{s_3,n-1} v_{s_3 v_j}\right] c_{v_j}\big) \\
&- \frac{1}{2}\left[(\rho - 1)\widetilde{e}_1 + (\alpha - \rho)(P_{\widetilde{y}} + \widetilde{e}_1) - \left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right)\right]^\top \widetilde{\Sigma}_y \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{A}}\right)\right] \\
&\times \widetilde{\Sigma}_y^\top \left[(\rho - 1)\widetilde{e}_1 + (\alpha - \rho)(P_{\widetilde{y}} + \widetilde{e}_1) - \left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right)\right] \\
&+ \sum_{j=1}^{2} \bar{B}_{s_1,n-1} v_{s_1 v_j} + \left(\left(\bar{B}_{s_2,n-1} - 1\right) v_{s_2 v_j} + \bar{B}_{s_3,n-1} v_{s_3 v_j}\right) v_{v_j} c_{v_j} \\
\bar{B}_{\widetilde{y}_j,n} &= \left[\left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right) - (\rho - 1)\widetilde{e}_1\right]^\top \widetilde{\Phi}_y \widetilde{e}_j \\
&+ \frac{(\alpha - \rho) p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} + \frac{\alpha}{2}(\alpha - \rho)(P_{\widetilde{y}} + \widetilde{e}_1)^\top \widetilde{\Sigma}_y \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right] \widetilde{\Sigma}_y^\top (P_{\widetilde{y}} + \widetilde{e}_1) \\
&- \frac{1}{2}\left[(\rho - 1)\widetilde{e}_1 + (\alpha - \rho)(P_{\widetilde{y}} + \widetilde{e}_1) - \left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right)\right]^\top \widetilde{\Sigma}_y \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right] \\
&\times \widetilde{\Sigma_y}^\top \left[(\rho - 1)\widetilde{e}_1 + (\alpha - \rho)(P_{\widetilde{y}} + \widetilde{e}_1) - \left(\widetilde{e}_2 + L \cdot h_{\widetilde{y}} + \widetilde{e}_{10} + \bar{B}_{\widetilde{y},n-1}\right)\right] \\
&- \frac{\left[(\alpha - \rho) p_{v_j} - L \cdot h_{v_j} - \bar{B}_{v_j,n-1} + \bar{B}_{s_1,n-1} v_{s_1 v_j} + \left(\bar{B}_{s_2,n-1} - 1\right) v_{s_2 v_j} + \bar{B}_{s_3,n-1} v_{s_3 v_j}\right] \phi_{v_j}}{1 - \left[(\alpha - \rho) p_{v_j} - L \cdot h_{v_j} - \bar{B}_{v_1,n-1} + \bar{B}_{s_1,n-1} v_{s_1 v_j} + \left(\bar{B}_{s_2,n-1} - 1\right) v_{s_2 v_j} + \bar{B}_{s_3,n-1} v_{s_3 v_j}\right] c_{v_j}} \\
&+ \left(\bar{B}_{s_1,n-1} v_{s_1 v_j} + \left(\bar{B}_{s_2,n-1} - 1\right) v_{s_2 v_j} + \bar{B}_{s_3,n-1} v_{s_3 v_j}\right) \phi_{v_j}
\end{aligned}$$

with initial conditions $\bar{A}_0 = 0$, $\bar{B}_{\widetilde{y},0} = \mathbf{0}$, and $\bar{B}_{v,0} = \mathbf{0}$, and where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\widetilde{\mathbf{I}}$ is the identity matrix, $\widetilde{\mathbf{1}}$ is a column vector of ones, $\widetilde{e}_i$ $(i = 1, 2, \ldots 11)$ and e_{v_i} $(i = 1, 2)$ are coordinate vectors with all elements equal to zero except element $i = 1$, $\bar{m}$ is defined in Equation (A.3), and the column vectors $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_j$ are defined in Equation (B.3). It follows naturally that the term structure of risky LIBOR rates is given by:

$$\ell_t^n \quad = \quad n^{-1}\left(\bar{A}_n + \bar{B}_{\widetilde{y},n}^\top \widetilde{y}_t + \bar{B}_{v,n}^\top v_t\right).$$

B.5. Term structure of IRS rates

To price IRS rates, we take into account the convenience yield $s_{1,t}$, bank risk $s_{2,t}$, and cost of collateral $s_{3,t}$, with dynamics defined in Equation (4). The formula for an n-period IRS rate is given by

$$IRS_t^n = \left(\sum_{j=1}^{n/\Delta}\left(\widetilde{\Psi}_{j,t}^\ell - \Psi_{j,t}^\ell\right)\right)\left(\sum_{j=1}^{n/\Delta}\Psi_{j,t}^\ell\right)^{-1},$$

where the one-month LIBOR rate is defined as $\ell_t = \widetilde{y}_t^1 + s_{1,t} + s_{2,t}$, Δ defines the time interval between two successive coupon periods, and where the expressions for $\widetilde{\Psi}^\ell$ and Ψ^ℓ are defined as

$$\widetilde{\Psi}_{n,t}^\ell = E_t\left[e^{m_{t,t+n\Delta}+s_{3,t+n\Delta}}e^{\Delta\cdot\ell_{t+(n-1)\Delta}^3}\right] \quad \text{and} \quad \Psi_{n,t}^\ell = E_t\left[e^{m_{t,t+n}+s_{3,t+n}}\right].$$

To derive closed-form solutions for the term structure of IRS rates, we conjecture that the expressions for $\widetilde{\Psi}^\ell$ and Ψ^ℓ are exponentially affine in the extended state vector $\widetilde{x}_t = [y_t^\top, s^\top, v_t^\top]^\top = [\widetilde{y}_t^\top, v_t^\top]^\top$:

$$\widetilde{\Psi}_{n,t}^\ell = e^{\widetilde{A}_n^\ell + \widetilde{B}_{\widetilde{y},n}^{\ell\top}\widetilde{y}_t + \widetilde{B}_{v,n}^{\ell\top}v_t} \quad \text{and} \quad \Psi_{n,t}^\ell = e^{A_n^\ell + B_{\widetilde{y},n}^{\ell\top}\widetilde{y}_t + B_{v,n}^{\ell\top}v_t}.$$

It can be shown that for all n, the scalars $\widetilde{A}_n^\ell$ and A_n^ℓ, and the components of the column vectors $\widetilde{B}_{\widetilde{y}_j,n}^\ell$ and $B_{\widetilde{y}_j,n}^\ell$ for $j = 1, 2, \ldots, 11$, and $\widetilde{B}_{v_j,n}^\ell$ $B_{v_j,n}^\ell$ for $j = 1, 2$, follow the same recursion and are given by

$$\begin{aligned}
A_n^\ell \quad &= \quad A_{n-1}^\ell + \bar{m} - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]^\top \widetilde{\mu}_{\widetilde{y}} - \sum_{j=1}^2 v_{v_j}\log\left(1 - \left[(\alpha-\rho)\,p_{v_j} + B_{v_j,n-1}^\ell\right.\right. \\
&\qquad \left.\left. + B_{s_1,n-1}^\ell v_{s_1 v_j} + B_{s_2,n-1}^\ell v_{s_2 v_j} + \left(B_{s_3,n-1}^\ell + 1\right)v_{s_3 v_j}\right]c_{v_j}\right) \\
&+ \quad \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}} + \widetilde{e}_1) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{A}}\right)\right] \\
&\times \quad \widetilde{\Sigma}_{\widetilde{y}}^\top\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}} + \widetilde{e}_1) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right] \\
&- \quad \sum_{j=1}^2\left(B_{s_1,n-1}^\ell v_{s_1 v_j} + B_{s_2,n-1}^\ell v_{s_2 v_j} + \left(B_{s_3,n-1}^\ell + 1\right)v_{s_3 v_j}\right)v_{v_j}c_{v_j} \\
B_{\widetilde{y}_j,n}^\ell \quad &= \quad \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}}\widetilde{e}_j \\
B_{v_j,n}^\ell \quad &= \quad \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}v}e_{v_j} \\
&- \quad \frac{(\alpha-\rho)\,p_{v_j}\phi_{v_j}}{1 - \alpha p_{v_j}c_{v_j}} - \frac{\alpha}{2}(\alpha-\rho)\,(P_{\widetilde{y}} + \widetilde{e}_1)^\top \widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right]\widetilde{\Sigma}_{\widetilde{y}}^\top(P_{\widetilde{y}} + \widetilde{e}_1) \\
&+ \quad \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}} + \widetilde{e}_1) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right] \\
&\times \quad \widetilde{\Sigma}_{\widetilde{y}}^\top\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}} + \widetilde{e}_1) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^\ell - \widetilde{e}_{11}\right]\right]
\end{aligned}$$

$$+ \quad \frac{\left[(\alpha-\rho)\, p_{v_j} + B^{\ell}_{v_j,n-1} + B^{\ell}_{s_1,n-1} v_{s_1 v_j} + B^{\ell}_{s_2,n-1} v_{s_2 v_j} + \left(B^{\ell}_{s_3,n-1}+1\right) v_{s_3 v_j}\right] \phi_{v_j}}{1-\left[(\alpha-\rho)\, p_{v_j} + B^{\ell}_{v_j,n-1} + B^{\ell}_{s_1,n-1} v_{s_1 v_j} + B^{\ell}_{s_2,n-1} v_{s_2 v_j} + \left(B^{\ell}_{s_3,n-1}+1\right) v_{s_3 v_j}\right] c_{v_j}}$$
$$- \quad \left(B^{\ell}_{s_1,n-1} v_{s_1 v_j} + B^{\ell}_{s_2,n-1} v_{s_2 v_j} + \left(B^{\ell}_{s_3,n-1}+1\right) v_{s_3 v_j}\right) \phi_{v_j},$$

where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\widetilde{\mathbf{I}}$ is the identity matrix, $\widetilde{\mathbf{1}}$ is a column vector of ones, $\widetilde{e}_i$ $(i = 1, 2, \ldots 11)$ and e_{v_i} $(i = 1, 2)$ are coordinate vectors with all elements equal to zero except element $i = 1$, $\bar{m}$ is defined in Equation (A.3), and the column vectors $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_j$ are defined in Equation (B.3).

While the expressions $\widetilde{\Psi}^{\ell}$ and Ψ^{ℓ} have the same recursions, they have different starting conditions. For $\Psi^{\ell}_{n,t}$, the recursion starts at 0, with initial condition given by $A^{\ell}_0 = 0$, and all elements of $B^{\ell}_{\widetilde{y},0} = 0$ and $B^{\ell}_{v,0} = 0$, except for $B^{\ell}_{s_3,0} = 1$. For $\widetilde{\Psi}^{\ell}_{n,t}$, the recursion starts at $n = \Delta$ (i.e., $\Delta = 3$ for quarterly coupon payments), with starting condition given by:

$$\widetilde{\Psi}^{\ell}_{\Delta,t} \quad = \quad e^{\Delta \cdot \ell^3_t} E_t\left[e^{m_{t,t+\Delta}+s_{3,t+\Delta}}\right].$$

Since $\log E_t\left[e^{m_{t,t+\Delta}+s_{3,t+\Delta}}\right] = A^{\ell}_{\Delta} + B^{\ell\top}_{\widetilde{y},\Delta}\widetilde{y}_t + B^{\ell\top}_{v,\Delta} v_t$, and $\ell^3_t = \frac{1}{3}\left(\bar{A}_3 + \bar{B}^{\top}_{\widetilde{y},3}\widetilde{y}_t + \bar{B}^{\top}_{v,3} v_t\right)$, the initial condition for $\widetilde{\Psi}^{\ell}_{n,t}$ is given by:

$$\widetilde{\Psi}^{\ell}_{\Delta,t} \quad = \quad e^{\frac{\Delta}{3}\bar{A}_3 + \frac{\Delta}{3}\bar{B}^{\top}_{\widetilde{y},3}\widetilde{y}_t + \frac{\Delta}{3}\bar{B}^{\top}_{v,3} v_t + A^{\ell}_{\Delta} + B^{\ell\top}_{\widetilde{y},\Delta}\widetilde{y}_t + B^{\ell\top}_{v,\Delta} v_t} = e^{A^{\ell_{init}}_{\Delta} + \left(B^{\ell_{init}}_{\widetilde{y},\Delta}\right)^{\top}\widetilde{y}_t + \left(B^{\ell_{init}}_{v,\Delta}\right)^{\top} v_t},$$

where the constant $A^{\ell_{init}}_{\Delta}$ and the elements of the column vectors $B^{\ell_{init}}_{\widetilde{y}_j,\Delta}$ for $j = 1, 2, \ldots, 11$ and $B^{\ell_{init}}_{v_j,\Delta}$ for $j = 1, 2$ are given by:

$$\begin{aligned} A^{\ell_{init}}_{\Delta} &= \frac{\Delta}{3}\bar{A}_3 + A^{\ell}_{\Delta} \\ B^{\ell_{init}}_{\widetilde{y}_j,\Delta} &= \frac{\Delta}{3}\bar{B}_{\widetilde{y}_j,3} + B^{\ell}_{\widetilde{y}_j,\Delta} \\ B^{\ell_{init}}_{v_j,\Delta} &= \frac{\Delta}{3}\bar{B}_{v_j,3} + B^{\ell}_{v_j,\Delta}. \end{aligned}$$

B.6. Term structure of OIS rates

To price OIS rates, we take into account the convenience yield $s_{1,t}$ and cost of collateral $s_{3,t}$, with dynamics defined in Equation (4). The formula for an n-period OIS rate is given by

$$OIS^n_t = \left(\sum_{j=1}^{n/\Delta}\left(\widetilde{\Psi}^{o}_{j,t} - \Psi^{o}_{j,t}\right)\right)\left(\sum_{j=1}^{n/\Delta}\Psi^{o}_{j,t}\right)^{-1},$$

where the one-month OIS rate is defined as $o_t = \widetilde{y}^1_t + s_{1,t}$, Δ defines the time interval between two successive coupon periods, and where the expressions for $\widetilde{\Psi}^{o}$ and Ψ^{o} are defined as

$$\widetilde{\Psi}^{o}_{n,t} = E_t\left[e^{m_{t,t+n}+s_{3,t+n}}\exp\left(\sum_{j=1}^{\Delta} o_{t+n\Delta-j}\right)\right] \quad \text{and} \quad \Psi^{o}_{n,t} = E_t\left[e^{m_{t,t+n}+s_{3,t+n}}\right].$$

To derive closed-form solutions for the term structure of OIS rates, we conjecture that the expressions for $\widetilde{\Psi}^{o}$ and Ψ^{o} are exponentially affine in the extended state vector $\widetilde{x}_t = [y^{\top}_t, s^{\top}, v^{\top}_t]^{\top} = [\widetilde{y}^{\top}_t, v^{\top}_t]^{\top}$:

$$\widetilde{\Psi}^{o}_{n,t} = e^{\widetilde{A}^{o}_n + \widetilde{B}^{o\top}_{\widetilde{y},n}\widetilde{y}_t + \widetilde{B}^{o\top}_{v,n} v_t} \quad \text{and} \quad \Psi^{o}_{n,t} = e^{A^{\ell}_n + B^{o\top}_{\widetilde{y},n}\widetilde{y}_t + B^{o\top}_{v,n} v_t}.$$

It can be shown that for all n, the scalars $\widetilde{A}_n^o$ and A_n^o, and the components of the column vectors $\widetilde{B}_{\widetilde{y}_j,n}^o$ and $B_{\widetilde{y}_j,n}^o$ for $j = 1, 2, \dots, 11$, and $\widetilde{B}_{v_j,n}^o$ $B_{v_j,n}^o$ for $j = 1, 2$, follow the same recursion and are given by

$$\begin{aligned}
A_n^o &= A_{n-1}^o + \bar{m} - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]^\top \widetilde{\mu}_{\widetilde{y}} - \sum_{j=1}^{2} v_{v_j} \log\Big(1 - \Big[(\alpha-\rho)\, p_{v_j} + B_{v_j,n-1}^o \\
&\quad + B_{s_1,n-1}^o v_{s_1 v_j} + B_{s_2,n-1}^o v_{s_2 v_j} + \left(B_{s_3,n-1}^o + 1\right) v_{s_3 v_j}\Big]\, c_{v_j}\Big) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{A}}\right)\right] \\
&\times \widetilde{\Sigma}_{\widetilde{y}}^\top \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]\right] \\
&- \sum_{j=1}^{2} \left(B_{s_1,n-1}^o v_{s_1 v_j} + B_{s_2,n-1}^o v_{s_2 v_j} + \left(B_{s_3,n-1}^o + 1\right) v_{s_3 v_j}\right) v_{v_j} c_{v_j} \\
B_{\widetilde{y}_j,n}^o &= \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}} \widetilde{e}_j \\
&- \frac{(\alpha-\rho)\, p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} - \frac{\alpha}{2}(\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right)^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \mathbf{B}_j\right)\right] \widetilde{\Sigma}_{\widetilde{y}}^\top \left(P_{\widetilde{y}} + \widetilde{e}_1\right) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \mathbf{B}_j\right)\right] \\
&\times \widetilde{\Sigma}_{\widetilde{y}}^\top \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B_{\widetilde{y},n-1}^o - \widetilde{e}_{11}\right]\right] \\
&+ \frac{\left[(\alpha-\rho)\, p_{v_j} + B_{v_j,n-1}^o + B_{s_1,n-1}^o v_{s_1 v_j} + B_{s_2,n-1}^o v_{s_2 v_j} + \left(B_{s_3,n-1}^o + 1\right) v_{s_3 v_j}\right] \phi_{v_j}}{1 - \left[(\alpha-\rho)\, p_{v_j} + B_{v_j,n-1}^o + B_{s_1,n-1}^o v_{s_1 v_j} + B_{s_2,n-1}^o v_{s_2 v_j} + \left(B_{s_3,n-1}^o + 1\right) v_{s_3 v_j}\right] c_{v_j}} \\
&- \left(B_{s_1,n-1}^o v_{s_1 v_j} + B_{s_2,n-1}^o v_{s_2 v_j} + \left(B_{s_3,n-1}^o + 1\right) v_{s_3 v_j}\right) \phi_{v_j}
\end{aligned}$$

where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\widetilde{\mathbf{I}}$ is the identity matrix, $\widetilde{\mathbf{1}}$ is a column vector of ones, $\widetilde{e}_i$ ($i = 1, 2, \dots 11$) and e_{v_i} ($i = 1, 2$) are coordinate vectors with all elements equal to zero except element $i = 1$, $\bar{m}$ is defined in Equation (A.3), and the column vectors $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_j$ are defined in Equation (B.3).

While the expressions $\widetilde{\Psi}^o$ and Ψ^o have the same recursions, they have different starting conditions. For $\Psi_{n,t}^o$, the recursion starts at 0, with initial condition given by $A_0^o = 0$, and all elements of $B_{\widetilde{y},0}^o = 0$ and $B_{v,0}^o = 0$, except for $B_{s_3,0}^o = 1$. For $\widetilde{\Psi}_{n,t}^o$, the recursion starts at $n = \Delta$ (i.e., $\Delta = 3$ for quarterly coupon payments), with starting condition given by:

$$\widetilde{\Psi}_{\Delta,t}^o = E_t\left[e^{m_{t,t+\Delta} + s_{3,t+\Delta} + \sum_{j=1}^{\Delta} o_{t+\Delta-j}}\right] = e^{\widetilde{A}_\Delta^o + \widetilde{B}_{\widetilde{y},\Delta}^{o\top} \widetilde{y}_t + \widetilde{B}_{v,\Delta}^{o\top} v_t},$$

where the expressions for $\widetilde{A}_\Delta^o$, $\widetilde{B}_{\widetilde{y},\Delta}^o$, and $\widetilde{B}_{v,\Delta}^o$ are obtained recursively. Observe that

$$\widetilde{\Psi}_{\Delta,t}^o = e^{o_t} E_t\left[e^{m_{t,t+1} + s_{3,t+1}} e^{o_{t+1}} E_{t+1}\left[e^{m_{t+1,t+2} + s_{3,t+2}} \dots e^{o_{t+\Delta-1}} E_{t+\Delta-1}\left[e^{m_{t+\Delta-1,t+\Delta} + s_{3,t+\Delta}}\right]\right]\right],$$

and define $\widetilde{\Psi}_{n,t}^o$ to be equal to $\widetilde{\Psi}_{n,t}^{o_{init}}$ characterized as

$$\widetilde{\Psi}_{n,t}^{o_{init}} = E_t\left[e^{o_t + m_{t,t+1} + s_{3,t+1}} \widetilde{\Psi}_{n-1,t+1}^{o_{init}}\right],$$

It can be shown that for all $n = 1, 2, 3, \dots, \Delta$

$$\widetilde{\Psi}_{n,t}^{o_{init}} = e^{A_n^{o_{init}} + B_{\widetilde{y},n}^{o_{init}\top} \widetilde{y}_t + B_{v,n}^{o_{init}\top} v_t}, \tag{B.5}$$

where the scalar $A_n^{o_{init}}$ and components of the column vectors $B_{\widetilde{y}_j,n}^{o_{init}}$ for $j = 1, 2, \dots, 11$ (except for $B_{s_1,n}^{o_{init}}$)

and $B^{o_{init}}_{v_j,n}$ are given by:

$$
\begin{aligned}
A^{o_{init}}_n &= \widetilde{A}_1 + A^{o_{init}}_{n-1} + \bar{m} - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]^\top \widetilde{\mu}_{\widetilde{y}} - \sum_{j=1}^{2} v_{v_j} \log\Big(1 - \Big[(\alpha-\rho)\, p_{v_j} + B^{o_{init}}_{v_j,n-1} \\
&\quad + B^{o_{init}}_{s_1,n-1} v_{s_1 v_j} + B^{o_{init}}_{s_2,n-1} v_{s_2 v_j} + \left(B^{o_{init}}_{s_3,n-1} + 1\right) v_{s_3 v_j}\Big] c_{v_j}\Big) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{A}}\right)\right] \\
&\times \widetilde{\Sigma}^\top_{\widetilde{y}} \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right] \\
&- \sum_{j=1}^{2} \left(B^{o_{init}}_{s_1,n-1} v_{s_1 v_j} + B^{o_{init}}_{s_2,n-1} v_{s_2 v_j} + \left(B^{o_{init}}_{s_3,n-1} + 1\right) v_{s_3 v_j}\right) v_{v_j} c_{v_j} \\
B^{o_{init}}_{\widetilde{y}_j,n} &= \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}} \widetilde{e}_1 + \widetilde{B}_{\widetilde{y}_j,1} \\
B^{o_{init}}_{s_1,n} &= \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}} \widetilde{e}_9 + \widetilde{B}_{s_1,1} + 1 \\
B^{o_{init}}_{v_j,n} &= \left[(\rho-1)\,\widetilde{e}_1 - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Phi}_{\widetilde{y}v} e_{v_j} + \widetilde{B}_{v_j,1} \\
&- \frac{(\alpha-\rho)\, p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} - \frac{\alpha}{2}(\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right)^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \mathbf{B}_j\right)\right] \widetilde{\Sigma}^\top_{\widetilde{y}} \left(P_{\widetilde{y}} + \widetilde{e}_1\right) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \mathbf{B}_j\right)\right] \\
&\times \widetilde{\Sigma}^\top_{\widetilde{y}} \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) - \left[\widetilde{e}_2 - B^{o_{init}}_{\widetilde{y},n-1} - \widetilde{e}_{11}\right]\right] \\
&+ \frac{\left[(\alpha-\rho)\, p_{v_j} + B^{o_{init}}_{v_j,j-1} + B^{o_{init}}_{s_1,n-1} v_{s_1 v_j} + B^{o_{init}}_{s_2,n-1} v_{s_2 v_j} + \left(B^{o_{init}}_{s_3,n-1} + 1\right) v_{s_3 v_j}\right] \phi_{v_j}}{1 - \left[(\alpha-\rho)\, p_{v_j} + B^{o_{init}}_{v_j,n-1} + B^{o_{init}}_{s_1,n-1} v_{s_1 v_j} + B^{o_{init}}_{s_2,n-1} v_{s_2 v_j} + \left(B^{o_{init}}_{s_3,n-1} + 1\right) v_{s_3 v_j}\right] c_{v_j}} \\
&- \left(B^{o}_{s_1,n-1} v_{s_1 v_j} + B^{o}_{s_2,n-1} v_{s_2 v_j} + \left(B^{o}_{s_3,n-1} + 1\right) v_{s_3 v_j}\right) \phi_{v_j},
\end{aligned}
$$

with starting condition $\widetilde{\Psi}^{o_{init}}_{1,t} = e^{o_t} E_t\left[e^{m_{t,t+1} + s_{3,t+1}}\right]$. Since $E_t\left[e^{m_{t,t+1} + s_{3,t+1}}\right] = \Psi^o_{1,t} = e^{A^o_1 + B^{o\top}_1 \widetilde{x}_t}$, $o_t = \widetilde{y}^1_t + s_{1,t}$, with $\widetilde{y}^1_t = \widetilde{A}_1 + \widetilde{B}^\top_1 \widetilde{x}_t$, we have that:

$$
\begin{aligned}
\widetilde{\Psi}^{o_{init}}_{1,t} &= e^{o_t} E_t\left[e^{m_{t,t+1} + s_{3,t+1}}\right] e^{\widetilde{A}_1 + A^o_1 + \left(\widetilde{B}_{\widetilde{y},1} + B^o_{\widetilde{y},1} + \widetilde{e}_9\right)^\top \widetilde{y}_t + \left(\widetilde{B}_{v,1} + B^o_{v,1}\right)^\top v_t} \\
&= e^{A^{o_{init}}_1 + \left(B^{o_{init}}_{\widetilde{y},1}\right)^\top \widetilde{x}_t + \left(B^{o_{init}}_{v,1}\right)^\top v_t},
\end{aligned}
$$

where the constant $A^{o_{init}}_1$ and the elements of the column vectors $B^{o_{init}}_{\widetilde{y}_j,1}$ for $j = 1, 2, \ldots, 11$ (except for $B^{o_{init}}_{s_1,1}$) and $B^{o_{init}}_{v_j,1}$ for $j = 1, 2$ are given by:

$$
\begin{aligned}
A^{o_{init}}_1 &= \widetilde{A}_1 + A^o_1 \\
B^{o_{init}}_{\widetilde{y}_j,1} &= \widetilde{B}_{\widetilde{y}_j,1} + B^o_{\widetilde{y}_j,1} \\
B^{o_{init}}_{s_1,1} &= \widetilde{B}_{s_1,1} + B^o_{s_1,1} + 1 \\
B^{o_{init}}_{v_j,1} &= \widetilde{B}_{v_j,1} + B^o_{v_j,1}.
\end{aligned}
$$

B.7. Term structure of CDS premiums

To price CDS premiums, we take into account the cost of collateral $s_{3,t}$, with dynamics defined in Equation (4), the hazard rate defined in Equation (B.1), and the corresponding survival probabilities defined in Equation (B.2). The formula for an n-period CDS premium is given by

$$CDS_t^n = L\cdot\left(\sum_{j=1}^{n}\left(\widetilde{\Psi}_{j,t}^c-\Psi_{j,t}^c\right)\right)\left(\sum_{j=1}^{n/\Delta}\Psi_{j\Delta,t}^c+\sum_{j=1}^{n}\left(\frac{j}{\Delta}-\lfloor\frac{j}{\Delta}\rfloor\right)(\widetilde{\Psi}_{j,t}^c-\Psi_{j,t}^c)\right)^{-1},$$

where the floor function $\lfloor\cdot\rfloor$ rounds to the nearest lower integer, Δ defines the time interval between two successive coupon periods, and where the expressions for $\widetilde{\Psi}^c$ and Ψ^c are defined as

$$\widetilde{\Psi}_{n,t}^c = E_t\left[e^{m_{t,t+n}+s_{3,t+n}}\frac{\mathcal{S}_{t+n-1}}{\mathcal{S}_t}\right] \quad\text{and}\quad \Psi_{n,t}^c = E_t\left[e^{m_{t,t+n}+s_{3,t+n}}\frac{\mathcal{S}_{t+n}}{\mathcal{S}_t}\right].$$

The law of iterated expectations implies that $\widetilde{\Psi}_{j,t}^c$ and $\Psi_{j,t}^c$ satisfy the recursions

$$\widetilde{\Psi}_{n,t}^c = E_t\left[e^{m_{t,t+1}+s_{3,t+1}}\left(1-\mathcal{H}_{t+1}\right)\widetilde{\Psi}_{n-1,t+1}^c\right],\ \Psi_{n,t}^c = E_t\left[e^{m_{t,t+1}+s_{3,t+1}}\left(1-\mathcal{H}_{t+1}\right)\Psi_{n-1,t+1}^c\right],$$

starting at $n=1$ for $\widetilde{\Psi}_{n,t}^c$ and at $n=0$ for $\Psi_{n,t}^c$. To evaluate the expressions for $\widetilde{\Psi}^c$ and Ψ^c, we conjecture that they are exponentially affine functions of the extended state vector $\widetilde{x}_t$:

$$\widetilde{\Psi}_{n,t}^c = e^{\widetilde{A}_n^c+\left(\widetilde{B}_{\widetilde{y},n}^c\right)^\top\widetilde{y}_t+\left(\widetilde{B}_{v,n}^c\right)^\top v_t} \quad\text{and}\quad \Psi_{n,t}^c = e^{A_n^c+\left(B_{\widetilde{y},n}^c\right)^\top\widetilde{y}_t+\left(B_{v,n}^c\right)^\top v_t}.$$

It can be shown that for all n, the scalars $\widetilde{A}_n^c$ and A_n^c, and the components of the column vectors $\widetilde{B}_{\widetilde{y}_j,n}^c$ and $B_{\widetilde{y}_j,n}^c$ for $j=1,2,\ldots,11$, and $\widetilde{B}_{v_j,n}^c$ for $j=1,2$, follow the same recursion and are given by

$$\begin{aligned}
\widetilde{A}_n^c &= \widetilde{A}_{n-1}^c - h + \bar{m} - \left[\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right]^\top \widetilde{\mu}_{\widetilde{y}} \\
&- \sum_{j=1}^{2} v_{v_j}\log\Big(1-\Big[(\alpha-\rho)\,p_{v_j} - h_{v_j} + \widetilde{B}_{v_j,n-1}^c \\
&\qquad + \widetilde{B}_{s_1,n-1}^c v_{s_1 v_j} + \widetilde{B}_{s_2,n-1}^c v_{s_2 v_j} + \left(\widetilde{B}_{s_3,n-1}^c+1\right) v_{s_3 v_j}\Big]\, c_{v_j}\Big) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1) - \left(\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right)\right]^\top \widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{A}}\right)\right] \\
&\times \widetilde{\Sigma}_{\widetilde{y}}^\top\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1) - \left(\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right)\right] \\
&- \sum_{j=1}^{2}\left(\widetilde{B}_{s_1,n-1}^c v_{s_1 v_j} + \widetilde{B}_{s_2,n-1}^c v_{s_2 v_j} + \left(\widetilde{B}_{s_3,n-1}^c+1\right) v_{s_3 v_j}\right) v_{v_j} c_{v_j} \\
\widetilde{B}_{\widetilde{y}_j,n}^c &= \left[(\rho-1)\,\widetilde{e}_1 - \left(\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right)\right]^\top \widetilde{\Phi}_{\widetilde{y}}\widetilde{e}_j \\
&- \frac{(\alpha-\rho)\,p_{v_j}\phi_{v_j}}{1-\alpha p_{v_j}c_{v_j}} - \frac{\alpha}{2}(\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1)^\top\widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{B}}_j\right)\right]\widetilde{\Sigma}_{\widetilde{y}}^\top\,(P_{\widetilde{y}}+\widetilde{e}_1) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1) - \left(\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right)\right]^\top \widetilde{\Sigma}_{\widetilde{y}}\left[\widetilde{\mathbf{I}}\odot\left(\widetilde{\mathbf{1}}^\top\otimes\widetilde{\mathbf{B}}_j\right)\right] \\
&\times \widetilde{\Sigma}_{\widetilde{y}}^\top\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\,(P_{\widetilde{y}}+\widetilde{e}_1) - \left(\widetilde{e}_2 + h_{\widetilde{y}} - \widetilde{B}_{\widetilde{y},n-1}^c - \widetilde{e}_{11}\right)\right] \\
&+ \frac{\left[(\alpha-\rho)\,p_{v_j} - h_{v_j} + \widetilde{B}_{v_j,n-1}^c + \widetilde{B}_{s_1,n-1}^c v_{s_1 v_j} + \widetilde{B}_{s_2,n-1}^c v_{s_2 v_j} + \left(\widetilde{B}_{s_3,n-1}^c+1\right) v_{s_3 v_j}\right]\phi_{v_j}}{1-\left[(\alpha-\rho)\,p_{v_j} - h_{v_j} + \widetilde{B}_{v_j,n-1}^c + \widetilde{B}_{s_1,n-1}^c v_{s_1 v_j} + \widetilde{B}_{s_2,n-1}^c v_{s_2 v_j} + \left(\widetilde{B}_{s_3,n-1}^c+1\right) v_{s_3 v_j}\right] c_{v_j}} \\
&- \left(\widetilde{B}_{s_1,n-1}^c v_{s_1 v_j} + \widetilde{B}_{s_2,n-1}^c v_{s_2 v_j} + \left(\widetilde{B}_{s_3,n-1}^c+1\right) v_{s_3 v_j}\right)\phi_{v_j},
\end{aligned}$$

where $\otimes$ defines the Kronecker product, $\odot$ the Hadamar product, $\widetilde{\mathbf{I}}$ is the identity matrix, $\widetilde{\mathbf{1}}$ is a column vector of ones, $\widetilde{e}_i$ $(i=1,2,\ldots 11)$ and e_{v_i} $(i=1,2)$ are coordinate vectors with all elements equal to zero except element $i = 1$, $\bar{m}$ is defined in Equation (A.3), and the column vectors $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}_j$ are defined in Equation (B.3).

Even though the expressions for $\widetilde{\Psi}^c$ and Ψ^c follow the same recursion, they have different initial conditions. The initial condition for Ψ^c is given by $\log \Psi^c_{0,t} = A^c_0 + B^{c\top}_{\widetilde{y},0}\widetilde{y}_t + B^{c\top}_{v,0} v_t$, where the scalar $A^c_0 = 0$ and the elements of the column vectors $B^c_{\widetilde{y}_j,0} = 0$ for $j = 1, 2, \ldots, 11$, and $B^c_{v,0} = 0$ for $j = 1, 2$, except for $B^c_{s_3,0} = 1$. The initial condition for $\widetilde{\Psi}^c$ is given by:

$$\widetilde{\Psi}^c_{1,t} = E_t\left[e^{m_{t,t+1}+s_{3,t+1}}\right] = e^{\widetilde{A}^c_1 + \widetilde{B}^{c\top}_{\widetilde{y},1}\widetilde{y}_t + \widetilde{B}^{c\top}_{v,1} v_t},$$

where the scalar $\widetilde{A}^c_1$ and components of the column vectors $\widetilde{B}^c_{\widetilde{y}_j,1}$ for $j = 1, 2, \ldots, 11$ and $\widetilde{B}^c_{v_j,1}$ for $j = 1, 2$ are given by:

$$\begin{aligned}
\widetilde{A}^c_1 &= \bar{m} + \left[\widetilde{e}_{11} - \widetilde{e}_2\right]^\top \widetilde{\mu}_Y - \sum_{j=1}^{2} v_{v_j} \log\left(1 - \left[(\alpha-\rho)\, p_{v_j} + v_{s_3 v_j}\right] c_{v_j}\right) - \sum_{j=1}^{2} v_{s_3 v_j} v_{v_j} c_{v_j} \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) + \widetilde{e}_{11} - \widetilde{e}_2\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{A}}\right)\right] \\
&\times \widetilde{\Sigma}^\top_{\widetilde{y}} \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) + \widetilde{e}_{11} - \widetilde{e}_2\right] \\
\widetilde{B}^c_{\widetilde{y}_j,1} &= \left[(\rho-1)\,\widetilde{e}_1 + \widetilde{e}_{11} - \widetilde{e}_2\right]^\top \widetilde{\Phi}_{\widetilde{y}} \widetilde{e}_j \\
\widetilde{B}^c_{v_j,1} &= \left[(\rho-1)\,\widetilde{e}_1 + \widetilde{e}_{11} - \widetilde{e}_2\right]^\top \widetilde{\Phi}_{\widetilde{y}v} e_{v_j} \\
&- \frac{(\alpha-\rho)\, p_{v_j} \phi_{v_j}}{1 - \alpha p_{v_j} c_{v_j}} - \frac{\alpha}{2}(\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right)^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right] \widetilde{\Sigma}^\top_{\widetilde{y}} \left(P_{\widetilde{y}} + \widetilde{e}_1\right) \\
&+ \frac{1}{2}\left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) + \widetilde{e}_{11} - e_2\right]^\top \widetilde{\Sigma}_{\widetilde{y}} \left[\widetilde{\mathbf{I}} \odot \left(\widetilde{\mathbf{1}}^\top \otimes \widetilde{\mathbf{B}}_j\right)\right] \\
&\times \widetilde{\Sigma}^\top_{\widetilde{y}} \left[(\rho-1)\,\widetilde{e}_1 + (\alpha-\rho)\left(P_{\widetilde{y}} + \widetilde{e}_1\right) + \widetilde{e}_{11} - \widetilde{e}_2\right] + \frac{\left[(\alpha-\rho)\, p_{v_j} + v_{s_3 v_j}\right] \phi_{v_j}}{1 - \left[(\alpha-\rho)\, p_{v_j} + v_{s_3 v_j}\right] c_{v_j}} - v_{s_3 v_j} \phi_{v_j}.
\end{aligned}$$

C. Estimation

We provide two different state-space representations. The first one, shown in Section C.1, is used when macroeconomic fundamentals are only used in the estimation. The second one, shown in Section C.2, is an extended state-space representation for which we can (potentially) jointly use macroeconomic fundamentals and asset data.

C.1. State-space representation

The underlying state transition dynamics run at a monthly frequency. We first provide the case in which all observables are available at the monthly frequency. The corresponding state-transition dynamics are shown in Section C.1.1 and the measurement equation is presented in Section C.1.2. In the presence of mixed-frequency observables, i.e., some observables are available at the quarterly frequency, we explain how to adjust the state space in Section C.1.3 with an illustrative example. Finally, in Section C.1.4, we explain the availability of data and provide our state-space representation that we estimate.

C.1.1. State transition dynamics

Define $x_t = [z_t^\top, w_t^\top, v_t^\top]^\top$. We describe the joint dynamics of x_t below

$$\underbrace{\begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \\ z_{3,t+1} \\ z_{4,t+1} \\ w_{1,t+1} \\ w_{2,t+1} \\ w_{3,t+1} \\ w_{4,t+1} \\ v_{1,t+1} \\ v_{2,t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \mu_{z1} \\ \mu_{z2} \\ \mu_{z3} \\ \mu_{z4} \\ 0 \\ 0 \\ 0 \\ 0 \\ \nu_{v_1} c_{v_1} \\ \nu_{v_2} c_{v_2} \end{bmatrix}}_{\mu} + \underbrace{\begin{bmatrix} \phi_{z11} & \phi_{z12} & \phi_{z13} & \phi_{z14} & 1 & 0 & 0 & 0 & \phi_{zv11} & \phi_{zv12} \\ \phi_{z21} & \phi_{z22} & \phi_{z23} & \phi_{z24} & 0 & 1 & 0 & 0 & \phi_{zv21} & \phi_{zv22} \\ \phi_{z31} & \phi_{z32} & \phi_{z33} & \phi_{z34} & 0 & 0 & 1 & 0 & \phi_{zv31} & \phi_{zv32} \\ \phi_{z41} & \phi_{z42} & \phi_{z43} & \phi_{z44} & 0 & 0 & 0 & 1 & \phi_{zv41} & \phi_{zv42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{v_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{v_2} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \\ z_{4,t} \\ w_{1,t} \\ w_{2,t} \\ w_{3,t} \\ w_{4,t} \\ v_{1,t} \\ v_{2,t} \end{bmatrix}}_{x_t}$$

$$+ \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{w11} & \theta_{w12} & \theta_{w13} & \theta_{w14} & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{w21} & \theta_{w22} & \theta_{w23} & \theta_{w24} & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{w31} & \theta_{w32} & \theta_{w33} & \theta_{w34} & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{w41} & \theta_{w42} & \theta_{w43} & \theta_{w44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\Sigma_A} \times \underbrace{\begin{bmatrix} \nu_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu_{21} & \nu_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu_{31} & \nu_{32} & \nu_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu_{41} & \nu_{42} & \nu_{43} & \nu_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\Sigma_B}$$

$$\times \left(\underbrace{\begin{bmatrix} V_{1,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{2,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{3,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{4,t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{v_1,t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{v_2,t} \end{bmatrix}}_{V_t} \right)^{1/2} \underbrace{\begin{bmatrix} \varepsilon_{z_1,t+1} \\ \varepsilon_{z_2,t+1} \\ \varepsilon_{z_3,t+1} \\ \varepsilon_{z_4,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \eta_{v_1,t+1} \\ \eta_{v_2,t+1} \end{bmatrix}}_{\eta_{t+1}},$$

for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2\}$,

$$V_{i,t} \quad = \quad a_i + b_{i1} v_{1,t} + b_{i2} v_{2,t}, \quad \omega_{v_j,t} = \nu_{v_j} c_{v_j}^2 + 2 c_{v_j} \phi_{v_j} v_{j,t}$$

with $\varepsilon_{z_i,t+1} \sim \mathcal{N}(0, 1)$ and $\eta_{v_j,t+1}$ being a zero mean unit variance shock. In vector notations, we express the state space by

$$\underbrace{\begin{bmatrix} z_{t+1} \\ w_{t+1} \\ v_{t+1} \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \mu_z \\ \mathbf{0} \\ \nu_v \odot c_v \end{bmatrix}}_{\mu} + \underbrace{\begin{bmatrix} \Phi_{z4\times4} & \mathbf{I}_{4\times4} & \Phi_{zv4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \Phi_{v2\times2} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} z_t \\ w_t \\ v_t \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} \mathbf{I}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \Theta_{w4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{I}_{2\times2} \end{bmatrix}}_{\Sigma_A} \quad \text{(C.1)}$$

$$\times \underbrace{\begin{bmatrix} \nu_{z4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{I}_{2\times2} \end{bmatrix}}_{\Sigma_B} \times \left(\underbrace{\begin{bmatrix} V_{z,t_{4\times4}} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \omega_{v,t_{2\times2}} \end{bmatrix}}_{V_t} \right)^{1/2} \times \underbrace{\begin{bmatrix} \varepsilon_{z,t+1} \\ \mathbf{0} \\ \eta_{v,t+1} \end{bmatrix}}_{\eta_{t+1}}.$$

C.1.2. Measurement equation

For ease of illustration, assume that the observables are available at a monthly frequency. Define $o_t = [\Delta c_t, d_t, g_t, \pi_t]^\top$. Then, the measurement equation becomes

$$o_t = \beta^\top x_t + u_t, \quad u_t \sim N(0, \Sigma_u) \tag{C.2}$$

where $\beta = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0]^\top$ and Σ_u is a measurement error (diagonal) variance-covariance matrix.

C.1.3. Dealing with the mixed-frequency issue

When some observables are available at a quarterly frequency, we need to adjust both measurement and transition equations to deal with the mixed-frequency issue. We provide an example whereby the dimension of z_t, w_t, and v_t are reduced to half for ease of illustration. We assume that the first observable is available at a quarterly while the second observable is available at a monthly frequency. We introduce the superscript q to indicate if the observable is available at the quarterly frequency. Thus, $o_t = [z^q_{1,t}, z_{2,t}]'$. Also for simplicity, we do not allow for measurement errors. There are two cases to consider.

1. If $z^q_{1,t}$ is expressed in growth rates, adjust the measurement loading β and state vector to

$$\beta = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_t = \begin{bmatrix} z_{1,t} \\ z_{1,t-1} \\ z_{1,t-2} \\ z_{1,t-3} \\ z_{1,t-4} \\ z_{2,t} \\ z_{2,t-1} \\ z_{2,t-2} \\ z_{2,t-3} \\ z_{2,t-4} \\ w_{1,t} \\ w_{2,t} \\ v_{1,t} \end{bmatrix}. \tag{C.3}$$

We can relate the mixed-frequency observables to the state vector by

$$o_t = \begin{bmatrix} z^q_{1,t} \\ z_{2,t} \end{bmatrix} = \begin{bmatrix} \frac{z_{1,t}+2z_{1,t-1}+3z_{1,t-2}+2z_{1,t-3}+z_{1,t-4}}{3} \\ z_{2,t} \end{bmatrix} = \beta^\top x_t. \tag{C.4}$$

2. If $z^q_{1,t}$ is expressed in levels, adjust the measurement loading β and state vector to

$$\beta = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_t = \begin{bmatrix} z_{1,t} \\ z_{1,t-1} \\ z_{1,t-2} \\ z_{1,t-3} \\ z_{1,t-4} \\ z_{2,t} \\ z_{2,t-1} \\ z_{2,t-2} \\ z_{2,t-3} \\ z_{2,t-4} \\ w_{1,t} \\ w_{2,t} \\ v_{1,t} \end{bmatrix}. \tag{C.5}$$

We can relate the mixed-frequency observables to the state vector by

$$o_t = \begin{bmatrix} z^q_{1,t} \\ z_{2,t} \end{bmatrix} = \begin{bmatrix} \frac{z_{1,t}+z_{1,t-1}+z_{1,t-2}}{3} \\ z_{2,t} \end{bmatrix} = \beta^\top x_t. \tag{C.6}$$

C.1.4. Implementation

We use quarterly consumption growth (Δc^q_t), output growth (d^q_t), and log government expenditure-to-output ratio (g^q_t), and monthly inflation (π_t) in the estimation. Except for consumption growth data, we are using the highest available frequency. Our choice of using quarterly consumption growth avoids modeling measurement errors in monthly consumption growth (see Schorfheide et al. (2018) for a detailed discussion), which significantly reduces the dimension of the state vector leading to a much more tractable estimation problem. Note that Δc^q_t and d^q_t are expressed in growth rates, but π_t and g^q_t are expressed in levels. Following the idea described in Section C.1.3, we modify the measurement equation loading β and state vector X_t to equate the observables to our state variables

$$o_t = \begin{bmatrix} \Delta c^q_t \\ d^q_t \\ g^q_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \frac{z_{1,t}+2z_{1,t-1}+3z_{1,t-2}+2z_{1,t-3}+z_{1,t-4}}{3} \\ \frac{z_{2,t}+2z_{2,t-1}+3z_{2,t-2}+2z_{2,t-3}+z_{2,t-4}}{3} \\ \frac{z_{3,t}+z_{3,t-1}+z_{3,t-2}}{3} \\ z_{4,t} \end{bmatrix}. \tag{C.7}$$

The most efficient characterization of the state vector is

$$x_t = \begin{bmatrix} z_{1,t}, z_{1,t-1}, z_{1,t-2}, z_{1,t-3}, z_{1,t-4}, z_{2,t}, z_{2,t-1}, z_{2,t-2}, z_{2,t-3}, z_{2,t-4}, \\ z_{3,t}, z_{3,t-1}, z_{3,t-2}, z_{4,t}, w_{1,t}, w_{2,t}, w_{3,t}, w_{4,t}, v_{1,t}, v_{2,t} \end{bmatrix}^\top. \tag{C.8}$$

The coefficient matrices in (C.1) are adjusted accordingly to match the dimension of (C.8). It is easy to deduce the form of β from (C.7) and (C.8).

Because of the conditionally linear structure of our state-space form, we can directly apply the Rao-Blackwellization particle filter as in Schorfheide et al. (2018). The details are omitted for brevity.

C.2. State-space representation: Extended form

We now introduce an extended state-space representation in which we additionally introduce s factors which are crucial elements for asset prices. We allow the s factors to depend on the lagged values of z and ν factors to model inter-dependence.

C.2.1. State transition dynamics

$$
\begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \\ z_{3,t+1} \\ z_{4,t+1} \\ w_{1,t+1} \\ w_{2,t+1} \\ w_{3,t+1} \\ w_{4,t+1} \\ s_{1,t+1} \\ s_{2,t+1} \\ s_{3,t+1} \\ v_{1,t+1} \\ v_{2,t+1} \end{bmatrix}
=
\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu_{s_1} \\ \mu_{s_2} \\ \mu_{s_3} \\ \nu_{v_1} c_{v_1} \\ \nu_{v_2} c_{v_2} \end{bmatrix}
+
\begin{bmatrix}
\phi_{z11} & \phi_{z12} & \phi_{z13} & \phi_{z14} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{zv11} & \phi_{zv12} \\
\phi_{z21} & \phi_{z22} & \phi_{z23} & \phi_{z24} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi_{zv21} & \phi_{zv22} \\
\phi_{z31} & \phi_{z32} & \phi_{z33} & \phi_{z34} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \phi_{zv31} & \phi_{zv32} \\
\phi_{z41} & \phi_{z42} & \phi_{z43} & \phi_{z44} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \phi_{zv41} & \phi_{zv42} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi_{sz11} & \phi_{sz12} & \phi_{sz13} & \phi_{sz14} & 0 & 0 & 0 & 0 & \phi_{s_{11}} & \phi_{s_{12}} & \phi_{s_{13}} & \phi_{sv11} & \phi_{sv12} \\
\phi_{sz21} & \phi_{sz22} & \phi_{sz23} & \phi_{sz24} & 0 & 0 & 0 & 0 & \phi_{s_{21}} & \phi_{s_{22}} & \phi_{s_{23}} & \phi_{sv21} & \phi_{sv22} \\
\phi_{sz31} & \phi_{sz32} & \phi_{sz33} & \phi_{sz34} & 0 & 0 & 0 & 0 & \phi_{s_{31}} & \phi_{s_{32}} & \phi_{s_{33}} & \phi_{sv31} & \phi_{sv32} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{v_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_{v_2}
\end{bmatrix}
\begin{bmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \\ z_{4,t} \\ w_{1,t} \\ w_{2,t} \\ w_{3,t} \\ w_{4,t} \\ s_{1,t} \\ s_{2,t} \\ s_{3,t} \\ v_{1,t} \\ v_{2,t} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_{w11} & \theta_{w12} & \theta_{w13} & \theta_{w14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_{w21} & \theta_{w22} & \theta_{w23} & \theta_{w24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_{w31} & \theta_{w32} & \theta_{w33} & \theta_{w34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta_{w41} & \theta_{w42} & \theta_{w43} & \theta_{w44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\times
\begin{bmatrix}
\nu_{11} & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
\nu_{21} & \nu_{22} & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
\nu_{31} & \nu_{32} & \nu_{33} & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
\nu_{41} & \nu_{42} & \nu_{43} & \nu_{44} & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 0 \\
-(\nu_{11}+\nu_{21}+\nu_{31}+\nu_{41})+\nu_{sz11} & -(\nu_{22}+\nu_{32}+\nu_{42})+\nu_{sz12} & -(\nu_{33}+\nu_{43})+\nu_{sz13} & -\nu_{44}+\nu_{sz14} & \mathbf{0}_{1\times 4} & [\nu_{s_1},0,0] & -1+\nu_{sv11} & -1+\nu_{sv12} \\
-(\nu_{11}+\nu_{21}+\nu_{31}+\nu_{41})+\nu_{sz21} & -(\nu_{22}+\nu_{32}+\nu_{42})+\nu_{sz22} & -(\nu_{33}+\nu_{43})+\nu_{sz23} & -\nu_{44}+\nu_{sz24} & \mathbf{0}_{1\times 4} & [0,\nu_{s_2},0] & -1+\nu_{sv21} & -1+\nu_{sv22} \\
-(\nu_{11}+\nu_{21}+\nu_{31}+\nu_{41})+\nu_{sz31} & -(\nu_{22}+\nu_{32}+\nu_{42})+\nu_{sz32} & -(\nu_{33}+\nu_{43})+\nu_{sz33} & -\nu_{44}+\nu_{sz34} & \mathbf{0}_{1\times 4} & [0,0,\nu_{s_3}] & -1+\nu_{sv31} & -1+\nu_{sv22} \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 1 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 3} & 0 & 1
\end{bmatrix}
$$

$$
\times
\left(
\begin{bmatrix}
V_{1,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & V_{2,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & V_{3,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & V_{4,t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{v_1,t} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{v_2,t}
\end{bmatrix}
\right)^{1/2}
\begin{bmatrix} \varepsilon_{z_1,t+1} \\ \varepsilon_{z_2,t+1} \\ \varepsilon_{z_3,t+1} \\ \varepsilon_{z_4,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ \varepsilon_{s_1,t+1} \\ \varepsilon_{s_2,t+1} \\ \varepsilon_{s_3,t+1} \\ \eta_{v_1,t+1} \\ \eta_{v_2,t+1} \end{bmatrix},
$$

for $i \in \{1,2,3,4\}$, $l \in \{1,2,3\}$ and $j \in \{1,2\}$,

$$V_{i,t} \quad = \quad a_i + b_{i1} v_{1,t} + b_{i2} v_{2,t}, \quad \omega_{v_j,t} = \nu_{v_j} c_{v_j}^2 + 2 c_{v_j} \phi_{v_j} v_{j,t}$$

with $\varepsilon_{z_i,t+1} \sim \mathcal{N}(0,1)$, $\varepsilon_{s_l,t+1} \sim \mathcal{N}(0,1)$, and $\eta_{v_j,t+1}$ being a zero mean unit variance shock.

In vector notations, we express the state space by

$$\underbrace{\begin{bmatrix} z_{t+1} \\ w_{t+1} \\ s_{t+1} \\ v_{t+1} \end{bmatrix}}_{\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix} \mu_z \\ \mathbf{0} \\ \mu_s \\ \nu_v \odot c_v \end{bmatrix}}_{\tilde{\mu}} + \underbrace{\begin{bmatrix} \Phi_{z4\times4} & \mathbf{I}_{4\times4} & \mathbf{0}_{4\times3} & \Phi_{zv4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \Phi_{sz3\times4} & \mathbf{0}_{3\times4} & \Phi_{s3\times3} & \Phi_{sv3\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{0}_{2\times3} & \Phi_{v2\times2} \end{bmatrix}}_{\tilde{\Phi}} \underbrace{\begin{bmatrix} z_t \\ w_t \\ s_t \\ v_t \end{bmatrix}}_{\tilde{x}_t} \tag{C.9}$$

$$+ \underbrace{\begin{bmatrix} \mathbf{I}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \Theta_{w4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \mathbf{J}_{3\times4} & \mathbf{J}_{3\times4} & \mathbf{I}_{3\times3} & \mathbf{J}_{3\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{0}_{2\times3} & \mathbf{I}_{2\times2} \end{bmatrix}}_{\tilde{\Sigma}_A} \times \underbrace{\begin{bmatrix} \nu_{z4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ -\mathbf{J}_{3\times4}\cdot\nu_{z4\times4} + \nu_{sz3\times4} & \mathbf{0}_{3\times4} & \nu_{s3\times3} & -\mathbf{J}_{3\times2} + \nu_{sv3\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{0}_{2\times3} & \mathbf{I}_{2\times2} \end{bmatrix}}_{\tilde{\Sigma}_B}$$

$$\times \left(\underbrace{\begin{bmatrix} V_{z,t_{4\times4}} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \mathbf{0}_{4\times3} & \mathbf{0}_{4\times2} \\ \mathbf{0}_{3\times4} & \mathbf{0}_{3\times4} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times2} \\ \mathbf{0}_{2\times4} & \mathbf{0}_{2\times4} & \mathbf{0}_{2\times3} & \omega_{v,t_{2\times2}} \end{bmatrix}}_{\tilde{V}_t} \right)^{1/2} \times \underbrace{\begin{bmatrix} \varepsilon_{z,t+1} \\ \mathbf{0} \\ \varepsilon_{s,t+1} \\ \eta_{v,t+1} \end{bmatrix}}_{\tilde{\eta}_{t+1}}.$$

C.2.2. Measurement equation

Consider various different maturities of the risky Treasury zero-coupon yields (T), interest rate swap premiums (I), CDS premiums (C), and OIS spreads (O). We introduce new notations to relate the observed rates to our state variables, given our solution coefficients. Define

$$\begin{aligned} y^T_{m,t} &= \Xi^T(A^T_m, B^T_m, \tilde{A}^T_m, \tilde{B}^T_m, \tilde{x}_t) = \Xi^T(\tilde{A}^T_m, \tilde{B}^T_m, \tilde{x}_t) \\ y^C_{m,t} &= \Xi^C(A^C_m, B^C_m, \tilde{A}^C_m, \tilde{B}^C_m, \tilde{x}_t) \\ y^I_{m,t} &= \Xi^I(A^I_m, B^I_m, \tilde{A}^I_m, \tilde{B}^I_m, \tilde{x}_t) \\ y^O_{m,t} &= \Xi^I(A^O_m, B^O_m, \tilde{A}^O_m, \tilde{B}^O_m, \tilde{x}_t) \end{aligned} \tag{C.10}$$

to match the m-maturity rate of the observable to our state variables $\tilde{x}_t$. We provide the derivation of solution coefficients $A^j_m, B^j_m, \tilde{A}^j_m, \tilde{B}^j_m$ and an expression for $\Xi^j(\cdot)$ in B for $j \in \{T, I, C, O\}$. We select the maturities of 1y, 3y, 5y, 7y, 10y, and 15y in the estimation, which are collected in

$$y^j_t = \begin{bmatrix} y^j_{1,t} \\ \vdots \\ y^j_{m,t} \\ \vdots \\ y^j_{15,t} \end{bmatrix} = \begin{bmatrix} \Xi^j(A^j_1, B^j_1, \tilde{A}^j_1, \tilde{B}^j_1, \tilde{x}_t) \\ \vdots \\ \Xi^j(A^j_m, B^j_m, \tilde{A}^j_m, \tilde{B}^j_m, \tilde{x}_t) \\ \vdots \\ \Xi^j(A^j_{15}, B^j_{15}, \tilde{A}^j_{15}, \tilde{B}^j_{15}, \tilde{x}_t) \end{bmatrix}. \tag{C.11}$$

We consider y^T_t, y^C_t, y^I_t in the estimation and use y^O_t as out-of-sample validation. We have defined $s_{1,t}$ and $s_{2,t}$ as observables in the main body of our paper. Define vectors e_{s_1} and e_{s_2} that select $s_{1,t}$ and $s_{2,t}$ from $\tilde{x}_t$, respectively. Put together,

$$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} e^\top_{s_1} \\ e^\top_{s_2} \end{bmatrix} \tilde{x}_t \tag{C.12}$$

disciplines the dynamics of the s factors. In sum, our state-space representation is comprised of state transition equations (C.9) and measurement equations (C.11) for $j \in \{T, C, I\}$ and (C.12). There are two ways in which we can proceed.

1. A joint estimation of macroeconomic observables and prices:
   We augment our measurement equations with (C.7) and adjust the state transition equation (C.9) to deal with mixed-frequency observations as explained in Section C.1.3. While the joint estimation approach can be appealing, it is computationally challenging since we have to increase the dimension of our state vector substantially. More importantly, because the system no longer preserves the conditionally linear structure, e.g., (C.11), we cannot apply the solution proposed by Schorfheide et al. (2018), and thus the non-linear filtering algorithm can be highly inefficient.
2. Two-stage estimation in which macroeconomic observables and prices are separated:
   For this, we treat the filtered estimates of $\hat{z}_t$ and $\hat{w}_t$ from the first stage estimation, which only involves macroeconomic data, as observables for the second stage estimation. Among our state vector $\tilde{x}_t$, we are assuming that $z_t, w_t, s_{1,t}, s_{2,t}$ are observed factors and treating $s_{3,t}, v_{1,t}, v_{2,t}$ as latent factors. We can then partition the state vector into

   $$\tilde{x}_t = (\tilde{x}_t^{o,\top}, \tilde{x}_t^{l,\top})^\top \tag{C.13}$$

   where the superscript o and l indicate "observed" and "latent" respectively. The non-linearing filtering technique only deals with $\tilde{x}_t^l$ in the state transition equation (C.9), since the other variables are observed. In this case, the measurement equations are (C.11) for $j \in \{T, C, I\}$ and (C.12).

C.2.3. Particle filter

We use a particle-filter approximation of the likelihood function and embed this approximation into a fairly standard random walk Metropolis algorithm. In the subsequent exposition, we omit the dependence of all densities on the parameter vector Θ. In slight abuse of notations, we denote all observables with

$$y_t = \left[\ y_t^{C,\top}, y_t^{T,\top}, y_t^{I,\top}, s_{1,t}, s_{2,t}\ \right]^\top \tag{C.14}$$

The particle filter approximates the sequence of distributions $\{p(\tilde{x}_t^l | y_{1:t})\}_{t=1}^T$ by a set of pairs $\left\{\tilde{x}_t^{l,(i)}, \pi_t^{(i)}\right\}_{i=1}^N$, where $\tilde{x}_t^{l,(i)}$ is the ith particle vector, $\pi_t^{(i)}$ is its weight, and N is the number of particles. As a by-product, the filter produces a sequence of likelihood approximations $\hat{p}(y_t | y_{1:t-1})$, $t = 1, \ldots, T$.

- Initialization: We generate the particle values $\tilde{x}_0^{l,(i)}$ from the unconditional distribution. We set $\pi_0^{(i)} = 1/N$ for each i.
- Propagation of particles: We simulate (C.9) forward to generate $\tilde{x}_t^{l,(i)}$ conditional on $\tilde{x}_{t-1}^{l,(i)}$ and observed $\tilde{x}_{t-1}^o$. We use $q(\tilde{x}_t^{l,(i)} | \tilde{x}_{t-1}^{l,(i)}, \tilde{x}_{t-1}^o, y_t)$ to represent the distribution from which we draw $\tilde{x}_t^{l,(i)}$.
- Correction of particle weights: Define the unnormalized particle weights for period t as

  $$\tilde{\pi}_t^{(i)} \quad = \quad \pi_{t-1}^{(i)} \times \frac{p(y_t | \tilde{x}_t^{l,(i)}, \tilde{x}_t^o) p(\tilde{x}_t^{l,(i)} | \tilde{x}_{t-1}^{l,(i)}, \tilde{x}_{t-1}^o)}{q(\tilde{x}_t^{l,(i)} | \tilde{x}_{t-1}^{l,(i)}, \tilde{x}_{t-1}^o, y_t)}.$$

  The term $\pi_{t-1}^{(i)}$ is the initial particle weight and the ratio $\frac{p(y_t | \tilde{x}_t^{l,(i)}, \tilde{x}_t^o) p(\tilde{x}_t^{l,(i)} | \tilde{x}_{t-1}^{l,(i)}, \tilde{x}_{t-1}^o)}{q(\tilde{x}_t^{l,(i)} | \tilde{x}_{t-1}^{l,(i)}, \tilde{x}_{t-1}^o, y_t)}$ is the importance weight of the particle.

  The approximation of the log likelihood function is given by

  $$\log \hat{p}(y_t | y_{1:t-1}) = \log \hat{p}(y_{t-1} | y_{1:t-2}) + \log \left(\sum_{i=1}^N \tilde{\pi}_t^{(i)} \right).$$

- Resampling: Define the normalized weights

$$\pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}}$$

  and generate N draws from the distribution $\{\tilde{x}_t^{l,(i)}, \pi_t^{(i)}\}_{i=1}^N$ using multinomial resampling. In slight abuse of notation, we denote the resampled particles and their weights also by $\tilde{x}_t^{l,(i)}$ and $\pi_t^{(i)}$, where $\pi_t^{(i)} = 1/N$.

References

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