Online Appendix to "Ratings Quality over the Business Cycle"

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Abstract

In this appendix we provide proofs to PROPOSITIONS 5, 8, 9, 11, and 12 stated in the main text of *Ratings Quality over the Business Cycle*. We also provide an analysis of how a social planner would choose to invest in ratings quality.

1 Proofs

Proof of Proposition 5
Proof. The effect on the wage w_B^*: First, consider the first-order condition that characterizes w_B^*:

$$-\pi_B(1-\lambda_B)\frac{\partial z_B}{\partial w} - 1 + \delta p_B(1-\lambda_B)\frac{\partial z_B}{\partial w}((1-\tau_B)V_B^* + \tau_B V_R^* - \bar{V}_B) = 0. \quad (1)$$

Taking the total derivative with respect to ω, we obtain:

$$\frac{\partial w_B^*}{\partial \omega} = \frac{\delta p_B(1-\lambda_B)\frac{\partial z_B}{\partial w}((1-\tau_B)\frac{\partial V_B^*}{\partial \omega} + \tau_B\frac{\partial V_R^*}{\partial \omega} - \frac{\bar{V}_B}{\omega})}{\frac{\partial^2 z_B}{\partial w^2}\{\pi_B(1-\lambda_B) - \delta p_B(1-\lambda_B)((1-\tau_B)V_B^* + \tau_B V_R^* - \bar{V}_B)\}}. \quad (2)$$

The denominator will be positive assuming that there is an interior solution for w_B^*. The sign, therefore, depends on the expression:

$$(1-\tau_B)\frac{\partial V_B^*}{\partial \omega} + \tau_B\frac{\partial V_R^*}{\partial \omega} - \frac{\bar{V}_B}{\omega}. \quad (3)$$

We need to find how the fraction of naive investors affects the equilibrium continuation values. As in the proof of Proposition 1, define for $s \in \{R, B\}$:

$$\begin{aligned} G_s \quad : \quad &= -V_s + \pi_s(\lambda_s + (1-\lambda_s)(1-z_s)) - w_s \\ &+\delta\sigma_s((1-\tau_s)V_s + \tau_s V_{-s}) + \delta(1-\sigma_s)\bar{V}_s \end{aligned} \quad (4)$$

We define σ_s as in the text and again note that $\sigma_s \in (0,1)$. We rewrite the value functions as $G_B = G_R = 0$.

We apply the implicit function theorem. This implies that:

$$\begin{aligned} \frac{dV_R^*}{d\omega} &= -\det\begin{bmatrix} \frac{\partial G_B}{\partial \omega} & \frac{\partial G_B}{\partial V_B^*} \\ \frac{\partial G_R}{\partial \omega} & \frac{\partial G_R}{\partial V_B^*} \end{bmatrix} / \det\begin{bmatrix} \frac{\partial G_B}{\partial V_R^*} & \frac{\partial G_B}{\partial V_B^*} \\ \frac{\partial G_R}{\partial V_R^*} & \frac{\partial G_R}{\partial V_B^*} \end{bmatrix} \quad (5) \\ &= -\frac{[\delta(1-\sigma_B)\frac{\bar{V}_B}{\omega}][\delta\sigma_R\tau_R] - [-1+\delta\sigma_B(1-\tau_B)][\delta(1-\sigma_R)\frac{\bar{V}_R}{\omega}]}{\delta^2\tau_B\tau_R\sigma_R\sigma_B - (1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R(1-\tau_R))} \\ &= \frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega} + \delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R) + \delta(1-\delta\sigma_B)\sigma_R\tau_R}, \end{aligned}$$

which is positive (Using Lemma 4, we know that the denominator is negative). Furthermore:

$$\begin{aligned} \frac{dV_B^*}{d\omega} &= -\det\begin{bmatrix} \frac{\partial G_B}{\partial V_R^*} & \frac{\partial G_B}{\partial \omega} \\ \frac{\partial G_R}{\partial V_R^*} & \frac{\partial G_R}{\partial \omega} \end{bmatrix} / \det\begin{bmatrix} \frac{\partial G_B}{\partial V_R^*} & \frac{\partial G_B}{\partial V_B^*} \\ \frac{\partial G_R}{\partial V_R^*} & \frac{\partial G_R}{\partial V_B^*} \end{bmatrix} \quad (6) \\ &= -\frac{\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega} + \delta(1-\sigma_B)\frac{\bar{V}_B}{\omega}(1-\delta\sigma_R(1-\tau_R))}{\delta^2\tau_B\tau_R\sigma_R\sigma_B - (1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R(1-\tau_R))} \\ &= \frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega} + \delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B) + \delta\sigma_B\tau_B(1-\delta\sigma_R)}, \end{aligned}$$

which is also positive.

Substituting Expressions 5 and 6 into Expression (3) yields:

$$\begin{aligned}&(1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\ &+\tau_B\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-\frac{\bar{V}_B}{\omega}.\end{aligned} \tag{7}$$

The wage w_B^* and $\frac{\bar{V}_B}{\omega}\geq\frac{\bar{V}_R}{\omega}$: First, suppose that $\frac{\bar{V}_B}{\omega}\geq\frac{\bar{V}_R}{\omega}$; then Equation (7)<0 is implied by:

$$\begin{aligned}&(1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\ &+\tau_B\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-\frac{\bar{V}_B}{\omega} \\ \leq\;&(1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\ &+\tau_B\frac{\delta(1-\delta\sigma_B(1-\tau_B))(1-\sigma_R)\frac{\bar{V}_B}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-\frac{\bar{V}_B}{\omega} \\ =\;&\frac{\bar{V}_B}{\omega}[(1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)+\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\ &+\tau_B\frac{\delta(1-\delta\sigma_B(1-\tau_B))(1-\sigma_R)+\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-1] \\ =\;&\frac{\bar{V}_B}{\omega}[(1-\tau_B)\left(\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)+\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-1\right) \\ &+\tau_B\left(\frac{\delta(1-\delta\sigma_B(1-\tau_B))(1-\sigma_R)+\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-1\right)] \\ =\;&-\frac{\bar{V}_B}{\omega}(1-\delta)[(1-\tau_B)\frac{1-\delta\sigma_R(1-\tau_R)+\delta\sigma_B\tau_B}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\ &+\tau_B\frac{1-\delta\sigma_B(1-\tau_B)+\delta\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}] \\ <\;&0,\end{aligned}$$

where the final inequality comes from the fact that the expression in the brackets is positive.

The wage w_B^* and $\frac{\bar{V}_R}{\omega}>\frac{\bar{V}_B}{\omega}$: Next, suppose that $\frac{\bar{V}_R}{\omega}>\frac{\bar{V}_B}{\omega}$; then, we examine Equation (7) again:

$$
\begin{aligned}
& (1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \qquad (8)\\
& +\tau_B\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-\frac{\bar{V}_B}{\omega}\\
< \; & (1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_R}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}\\
& +\tau_B\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-\frac{\bar{V}_B}{\omega}\\
= \; & \frac{\bar{V}_R}{\omega}[(1-\tau_B)\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)+\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}\\
& +\tau_B\frac{\delta(1-\delta\sigma_B(1-\tau_B))(1-\sigma_R)+\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-1+1-\frac{\bar{V}_B}{\omega}(\frac{\bar{V}_R}{\omega})^{-1}]\\
= \; & \frac{\bar{V}_R}{\omega}[-(1-\delta)(1-\tau_B)\frac{1-\delta\sigma_R+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}\\
& -\tau_B(1-\delta)\frac{1-\delta\sigma_B+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}+1-\frac{\bar{V}_B}{\omega}(\frac{\bar{V}_R}{\omega})^{-1}].
\end{aligned}
$$

Using the definition of $\bar{V}_B$ and $\bar{V}_R$ from the text, we can write:

$$1-\frac{\bar{V}_B}{\omega}(\frac{\bar{V}_R}{\omega})^{-1}=1-\frac{(1-\tau_B-\delta(1-\tau_B-\tau_R))\pi_B+\tau_B\pi_R}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}=-\frac{(1-\tau_B-\tau_R)(1-\delta)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}.$$

It follows that (7) is negative if the final bracketed expression in Equation (8) is negative. Factoring out $-(1-\delta)$, we then want the following to be positive:

$$
\begin{aligned}
& (1-\tau_B)\frac{1-\delta\sigma_R+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \qquad (9)\\
& +\tau_B\frac{1-\delta\sigma_B+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}+\frac{(1-\tau_B-\tau_R)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}.
\end{aligned}
$$

Consider $\frac{(1-\tau_B-\tau_R)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}$. First, note that $\frac{d}{d\tau_B}(\frac{(1-\tau_B-\tau_R)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B})=-(\pi_B-\pi_R)\frac{\pi_R+\tau_R(\pi_B-\pi_R)}{(\pi_R-\delta\pi_R+\tau_R\pi_B-\tau_R\pi_R+\delta\tau_B\pi_R+\delta\tau_R\pi_R)^2}<0$ and so

$$\begin{aligned}
\frac{(1-\tau_B-\tau_R)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B} &\geq \frac{(1-\tau_B-\tau_R)(\pi_B-\pi_R)}{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}|_{\tau_{B=1}} \\
&= \frac{-\tau_R(\pi_B-\pi_R)}{(1-\tau_R(1-\delta))\pi_R+\tau_R\pi_B} \\
&= \frac{-\tau_R(\pi_B-\pi_R)}{(1+\tau_R\delta)\pi_R+\tau_R(\pi_B-\pi_R)} \\
&= \frac{-1}{\frac{(1+\tau_R\delta)\pi_R}{\tau_R(\pi_B-\pi_R)}+1} \\
&> -1.
\end{aligned}$$

Thus, (9) is positive given that:

$$\begin{aligned}
0 \quad < \quad & (1-\tau_B)\frac{1-\delta\sigma_R+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\
& +\tau_B\frac{1-\delta\sigma_B+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-1 \\
= \quad & (1-\tau_B)\left(\frac{1-\delta\sigma_R+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-1\right) \\
& +\tau_B\left(\frac{1-\delta\sigma_B+\delta\sigma_B\tau_B+\delta\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}-1\right) \\
= \quad & \delta\sigma_B(1-\tau_B)\frac{1-\delta\sigma_R+\delta\tau_B\sigma_R+\delta\sigma_R\tau_R}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\
& +\delta\sigma_R\tau_B\frac{1-\delta\sigma_B+\delta\sigma_B\tau_B+\delta\sigma_B\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R},
\end{aligned}$$

which is true.

The effect on the wage w_R^*:

Analogously to above, $\frac{\partial w_R^*}{\partial\omega}$ has the same sign as:

$$(1-\tau_R)\frac{\partial V_R^*}{\partial\omega}+\tau_R\frac{\partial V_B^*}{\partial\omega}-\frac{\bar{V}_R}{\omega}, \tag{10}$$

which can be rewritten as:

$$\begin{aligned}
&(1-\tau_R)\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}+\delta^2(1-\sigma_B)\sigma_R\tau_R\frac{\bar{V}_B}{\omega}}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \qquad (11)\\
&+\tau_R\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)\frac{\bar{V}_B}{\omega}+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-\frac{\bar{V}_R}{\omega}.
\end{aligned}$$

If $\frac{\bar{V}_R}{\omega}\geq\frac{\bar{V}_B}{\omega}$, it is easy to see that factoring the whole expression by $\frac{\bar{V}_R}{\omega}$ makes this negative.

Now, suppose that $\frac{\bar{V}_R}{\omega}<\frac{\bar{V}_B}{\omega}$. Factoring Equation (11) by $\frac{\bar{V}_B}{\omega}$ gives us:

$$\frac{\bar{V}_B}{\omega}\{(1-\tau_R)\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}+\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \tag{12}$$

$$+\tau_R\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)+\delta^2\sigma_B\tau_B(1-\sigma_R)\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}\}.$$

As $\frac{\bar{V}_B}{\omega}>0$, we will focus on the expression in curly brackets. Re-arranging that expression gives us:

$$\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}[(1-\tau_R)\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \tag{13}$$

$$+\tau_R\frac{\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-1]$$

$$+(1-\tau_R)\frac{\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}$$

$$+\tau_R\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}.$$

We would like to demonstrate that Expression (13) is negative. We will do this by placing an upper bound on it. The second and third lines of Expression (13) are positive. It is easy to show that the bracketed expression in the first line is negative. The term $\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}$, of course, is positive. Note that $\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}=\frac{(1-\tau_R-\delta(1-\tau_B-\tau_R))\pi_R+\tau_R\pi_B}{(1-\tau_B-\delta(1-\tau_B-\tau_R))\pi_B+\tau_B\pi_R}$. This expression is increasing in π_R. We will evaluate $\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}$ at $\pi_R=0$, which makes the first line of Expression (13) smaller. This gives us $\frac{\bar{V}_R}{\omega}(\frac{\bar{V}_B}{\omega})^{-1}=\frac{\tau_R}{(1-\tau_B-\delta(1-\tau_B-\tau_R))}$. Therefore, Expression (13) is smaller than:

$$
\begin{aligned}
& \frac{\tau_R}{(1-\tau_B-\delta(1-\tau_B-\tau_R))}[(1-\tau_R)\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \\
& +\tau_R\frac{\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-1] \\
& +(1-\tau_R)\frac{\delta^2(1-\sigma_B)\sigma_R\tau_R}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \\
& +\tau_R\frac{\delta(1-\delta\sigma_R(1-\tau_R))(1-\sigma_B)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)} \\
= \;& \frac{\tau_R}{(1-\tau_B-\delta(1-\tau_B-\tau_R))}[(1-\tau_R)\frac{\delta[1-\delta\sigma_B(1-\tau_B)](1-\sigma_R)}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R} \\
& +\tau_R\frac{\delta^2\sigma_B\tau_B(1-\sigma_R)}{(1-\delta\sigma_R(1-\tau_R))(1-\delta\sigma_B)+\delta\sigma_B\tau_B(1-\delta\sigma_R)}-1] \\
& +\frac{(1-\tau_B-\delta(1-\tau_B-\tau_R))}{\delta\sigma_B+\delta\sigma_R-\delta^2\sigma_B\sigma_R-\delta\sigma_B\tau_B-\delta\sigma_R\tau_R+\delta^2\sigma_B\tau_B\sigma_R+\delta^2\sigma_B\sigma_R\tau_R-1}\frac{\delta(\sigma_B-1)\tau_R}{(1-\tau_B-\delta(1-\tau_B-\tau_R))} \\
= \;& (\delta-1)\frac{1-\delta(1-\tau_B-\tau_R)}{(1-\delta\sigma_B(1-\tau_B))(1-\delta\sigma_R)+\delta(1-\delta\sigma_B)\sigma_R\tau_R}\frac{\tau_R}{(1-\tau_B-\delta(1-\tau_B-\tau_R))}.
\end{aligned} \tag{14}
$$

This final expression is clearly negative. As Expression (13) has a negative upper bound, it is negative. This implies that $\frac{\partial w_R^*}{\partial \omega}$ is negative. ■

Proof of Proposition 8

Proof. In equilibrium, the wage $w_{i,s}^*$ for $s \in \{R, B\}$ is optimally chosen and so satisfies the first-order condition:

$$
\begin{aligned}
0 \;=\;& -1+\frac{\partial z_{i,s}}{\partial w}(1-\lambda_s)\{-\pi_{D,s}+\delta p_s[\rho(-1+2z_{j,s})+(1-\rho)(1-(1-\lambda_s)(1-z_{j,s})p_s)]EV_{D,s}^* \\
& +\delta p_s[\rho(1-z_{j,s})+(1-\rho)(1-\lambda_s)(1-z_{j,s})p_s]EV_{M,s}^*\}.
\end{aligned} \tag{15}
$$

Imposing symmetry, we write the equilibrium wage for this duopoly case as $w_{D,s}^*$, drop the i and j subscripts on the $z_{i,s}$ and $z_{j,s}$ functions, and rewrite the CRA's first-order condition (Equation 15) as:

$$
0=-1+\frac{\partial z_s}{\partial w}(1-\lambda_s)\{-\pi_{D,s}+\delta p_s[\rho(-1+2z_s)+(1-\rho)\sigma_{D,s}^*)]EV_{D,s}^*+\delta p_s[\rho(1-z_s)+(1-\rho)(1-\sigma_{D,s}^*)]EV_{M,s}^*\}. \tag{16}
$$

Define $\bar{A}_s$ as the right hand side of equation 16. Furthermore, define

$$
\bar{Y}_s=(1-\lambda_s)\{-\pi_{D,s}+\delta p_s[\rho(-1+2z_s)+(1-\rho)\sigma_{D,s}^*]EV_{D,s}^*+\delta p_s[\rho(1-z_s)+(1-\rho)(1-\sigma_{D,s}^*)]EV_{M,s}^*\}. \tag{17}
$$

Then,

$$
\frac{d\bar{A}_s}{dw}=\frac{\partial^2 z_s}{\partial w^2}\bar{Y}_s+\delta(\frac{\partial z_s}{\partial w})^2(1-\lambda_s)p_s\{[\rho+(1-\rho)(1-\lambda_s)p_s](EV_{D,s}^*-EV_{M,s}^*)+\rho EV_{D,s}^*\}<0, \tag{18}
$$

where the inequality follows since $\bar{Y}_s > 0$ (assuming that there is an interior solution), $\frac{\partial^2 z_s}{\partial w^2} < 0$, $\frac{\partial z_s}{\partial w} > 0$ and $[\rho + (1-\rho)(1-\lambda_s)p_s]\left[EV^*_{D,s} - EV^*_{M,s}\right] + \rho EV^*_{D,s} < 0$ by the assumption that we are in the strategic substitutes case.

Note that we can represent the first-order condition as:

$$\frac{1}{\frac{\partial z_s}{\partial w}} = \bar{Y}_s. \tag{19}$$

The left-hand side is increasing in w. It also shifts up if γ is larger.

The right-hand side is decreasing in w. We now examine what happens to the right-hand side when the parameters change. In particular, how can we compare $\bar{Y}_B$ and $\bar{Y}_R$?

When states are independent across time, $\tau_B = 1 - \tau_R$, which implies that $EV^*_{D,B} = EV^*_{D,R}$ and $EV^*_{M,B} = EV^*_{M,R}$.

We consider each of our parameters in turn.

First, consider $\pi_{D,s}$:

$$\frac{d\bar{Y}_s}{d\pi_{D,s}} = -(1-\lambda_s) < 0. \tag{20}$$

Next,

$$\frac{d\bar{Y}_s}{dp_s} = \delta(1-\lambda_s)\{(2(1-\rho)p_s+\rho z_s)EV^*_{D,s}+(1-z_s)[\rho+2(1-\rho)(1-\lambda_s)p_s](EV^*_{M,s}-EV^*_{D,s})\},$$

which is positive, since $[\rho + (1-\rho)(1-\lambda_s)p_s]\left[EV^*_{D,s} - EV^*_{M,s}\right] + \rho EV^*_{D,s} < 0$ implies $EV^*_{M,s} > EV^*_{D,s}$.

Turning next to λ_s,

$$\frac{d\bar{Y}_s}{d\lambda_s} = -\frac{\bar{Y}_s}{1-\lambda_s} + \delta p_s^2(1-z_s)(1-\rho)(1-\lambda_s)(EV^*_{D,s} - EV^*_{M,s}) < 0, \tag{21}$$

once again, since $EV^*_{M,s} > EV^*_{D,s}$.

Finally, we turn to consider γ_s,

$$\frac{d\bar{Y}_s}{d\gamma_s} = \frac{\partial z_s}{\partial \gamma_s}\delta p_s(1-\lambda_s)\{\rho EV^*_{D,s} + (\rho + (1-\rho)(1-\lambda_s)p_s)(EV^*_{D,s} - EV^*_{M,s})\} > 0. \tag{22}$$

Once again, this gives us an ambiguous result for γ_s. ■

Proof of Proposition 9

Proof. Given assumption A1, $EV^*_{M,B} < EV^*_{M,R}$. Assumption A2 implies that $EV^*_{D,B} < EV^*_{D,R}$. Using the proof of the previous proposition, this implies that $\bar{Y}_B < \bar{Y}_R$ when $\pi_{D,B} > \pi_{D,R}$, $p_B < p_R$, and $\lambda_B > \lambda_R$ hold (and setting $\gamma_B = \gamma_R$), which means that $w^*_{D,B} < w^*_{D,R}$. As in the previous proposition, setting $\gamma_B > \gamma_R$ has an ambiguous effect. ■

Proof of Proposition 11

Proof. Consider the first-order condition with symmetry imposed in Equation 16. Define $\bar{A}_s$ as in Proposition 8 and $\bar{Y}_s$ as in Equation (17).

Once again, we can represent the first order condition as:

$$\frac{1}{\frac{\partial z_s}{\partial w}} = \bar{Y}_s \tag{23}$$

The left-hand side is increasing in w. It also shifts up if γ is larger.

The right-hand side is decreasing in w. We now examine what happens to the right-hand side when the parameters change. In particular, how can we compare $\bar{Y}_B$ and $\bar{Y}_R$?

When states are independent across time, $\tau_B = 1 - \tau_R$, which implies that $EV^*_{D,B} = EV^*_{D,R}$ and $EV^*_{M,B} = EV^*_{M,R}$.

We consider each of our parameters in turn.

First, consider $\pi_{D,s}$:

$$\frac{d\bar{Y}_s}{d\pi_{D,s}} = -(1-\lambda_s) < 0. \tag{24}$$

Next,

$$\frac{d\bar{Y}_s}{dp_s} = \delta(1-\lambda_s)\{(2(1-\rho)p_s+\rho z_s)EV^*_{D,s}+(1-z_s)[\rho+2(1-\rho)(1-\lambda_s)p_s](EV^*_{M,s}-EV^*_{D,s})\} > 0,$$

which is positive, given Condition 1.

Turning, next to λ_s,

$$\frac{d\bar{Y}_s}{d\lambda_s} = -\frac{\bar{Y}_s}{1-\lambda_s} + \delta p_s^2(1-z_s)(1-\rho)(1-\lambda_s)(EV^*_{D,s} - EV^*_{M,s}) < 0, \tag{25}$$

which is negative, given Condition 1.

Finally, we turn to consider γ_s,

$$\frac{d\bar{Y}_s}{d\gamma_s} = \frac{\partial z_s}{\partial \gamma_s}\delta p_s(1-\lambda_s)\{\rho EV^*_{D,s} + (\rho + (1-\rho)(1-\lambda_s)p_s)(EV^*_{D,s} - EV^*_{M,s})\} < 0. \tag{26}$$

This now gives us an *un*ambiguous result for γ_s. ■

Proof of Proposition 12

Proof. Given assumption A1, $EV^*_{M,B} < EV^*_{M,R}$. Assumption A2 implies that $EV^*_{D,B} < EV^*_{D,R}$. Using the proof of the previous proposition, this implies that $\bar{Y}_B < \bar{Y}_R$ when $\pi_{D,B} > \pi_{D,R}$, $p_B < p_R$, $\lambda_B > \lambda_R$, *and* $\gamma_B > \gamma_R$ hold , which means that $w^*_{D,B} < w^*_{D,R}$. ■

2 Social Planner

In this section, we examine how much a social planner would invest in accuracy. We will take two approaches. First, we will allow the planner to commit to wage levels. Second, we will restrict the social planner to the same constraints the monopolist CRA has, i.e. unobservable investment and the same grim trigger punishment strategy by investors. We define $\bar{U}_s$ (where $s \in \{B, R\}$) as the outside option of the investor, which implies the investment is welfare enhancing if it offers an expected return above $\bar{U}_s$. The following assumption is consistent with the model:

$$1 - p_s < \bar{U}_s < 1 \qquad s \in \{B, R\}$$

This assumption states that the investor gets positive surplus from a good investment and negative surplus from a bad investment.[1]

We consider two possibilities for the planner's behavior.

2.1 Planner setting wages independent of past outcomes

If the planner can commit to a wage, she faces a static problem:

$$\max_{w_s} (\lambda_s(1 - \bar{U}_s) + (1 - \lambda_s)(1 - z_s)(1 - p_s - \bar{U}_s)) - w_s$$

The solution is given by the first order condition:

$$\frac{dz(w_s, \gamma_s)}{dw} = \frac{1}{(1 - \lambda_s)(\bar{U}_s - (1 - p_s))}$$

We order the parameters according to booms and recessions as before: $\lambda_B > \lambda_R$, $p_B < p_R$, and $\gamma_B > \gamma_R$. Note that fees no longer are a part of the analysis, as they represent a transfer between the issuer and CRA and do not change welfare. It seems reasonable to set $\bar{U}_B > \bar{U}_R$, i.e. outside options for investors are larger in a boom than in a recession.

Given the ordering on parameters, it is not clear if the social planner's wage choices are countercyclical. For three of the four parameters, the direction of accuracy is the same as what CRAs would choose (countercyclical), but for somewhat different reasons. When there are more bad issues, the likelihood of having a negative surplus is larger, so the planner wants to minimize the probability of making a mistake. When the probability of default is larger, the planner wants to be more accurate to avoid the negative surplus. When the labor market is tighter, the cost of accuracy for the planner also goes up, making her choose less accuracy. However, a higher outside option for investors in the boom makes it more costly to mislead them. This effect points toward procyclical accuracy.

It is also worthwhile to note that the level of investment in accuracy is not directly comparable to the main text, as it now depends on outside options.

[1] The fact that we have assumed that $\bar{U}_s$ is state-contingent is consistent with our model. There we do not assume that $\bar{z}$ (the cutoff for which investors will continue to purchase rated investments) is state-contingent, but it could be, as long as we redefine $\bar{z} = \max\{\bar{z}_B, \bar{z}_R\}$, i.e. the constraint is not binding. We mention this in the text in footnote 11.

2.2 Planner subject to similar reputation concerns as CRA

Now we restrict the planner to the same constraints as the monopoly CRA, namely a punishment for mistakes. Consider a value function for the social planner for each state:

$$\begin{aligned} W_B &= \max_{w_B}(\lambda_B(1-\bar{U}_B)+(1-\lambda_B)(1-z_B)(1-p_B-\bar{U}_B))-w_B \\ &\quad +\delta\sigma_B((1-\tau_B)W_B+\tau_B W_R) \\ W_R &= \max_{w_R}(\lambda_R(1-\bar{U}_R)+(1-\lambda_R)(1-z_R)(1-p_R-\bar{U}_R))-w_R \\ &\quad +\delta\sigma_R((1-\tau_R)W_R+\tau_R W_B) \end{aligned} \tag{27}$$

We denote equilibrium values with a star $(^*)$ and define σ_s where $s \in \{B, R\}$ as before. We can show the solution exists and is unique in a manner analogous to Lemma 1, but for brevity do not include full details here.[2]

The first-order conditions for the wages w_B and w_R, respectively:

$$\frac{\partial z}{\partial w}(w_B^*, \gamma_B) = \frac{1}{1-\lambda_B}\frac{1}{\delta p_B((1-\tau_B)W_B^*+\tau_B W_R^*)-(1-p_B-\bar{U}_B)} \tag{28}$$

$$\frac{\partial z}{\partial w}(w_R^*, \gamma_R) = \frac{1}{1-\lambda_R}\frac{1}{\delta p_R((1-\tau_R)W_R^*+\tau_R W_B^*)-(1-p_R-\bar{U}_R)} \tag{29}$$

We will again assume that surplus is higher in a boom:

Assumption A1': The value to a social planner of being in a boom is larger than the value of being in a recession ($W_B^* > W_R^*$)

Define continuation values from the boom and recession states, respectively, as:

$$EW_B^* \ :\ = (1-\tau_B)W_B^* + \tau_B W_R^*, \text{ and} \tag{30}$$

$$EW_R^* \ :\ = (1-\tau_R)W_R^* + \tau_R W_B^*. \tag{31}$$

It follows that $w_B^* \leq w_R^*$ and there is more accuracy in recessions than in booms when:

$$(1-\lambda_B)(\delta p_B EW_B^* - (1-p_B-\bar{U}_B)) \leq (1-\lambda_R)(\delta p_R EW_R^* - (1-p_R-\bar{U}_R)). \tag{32}$$

When $\tau_B = 1 - \tau_R$, each period's state is an iid draw and $EW_B = EW_R$. We obtain the following result; which can be shown in a similar fashion to Proposition 2 of the paper, however, for brevity details are omitted.

Proposition 1 *If states are independent across time ($\tau_B = 1 - \tau_R$), then it is welfare maximizing to have more investment in ratings quality in a recession than in a boom when they differ only in terms of default rates, the fraction*

[2] In the modification, the second order condition for wages is satisfied without any assumptions beyond $\frac{\partial^2 z}{\partial w^2} < 0$.

of good issues, and labor market tightness. It is welfare maximizing to have more investment in a boom than a recession when they differ only in terms of investors' outside options. In general, there is no unambiguous ranking.

If booms and recessions do not arise independently of history, then Proposition 1 cannot be applied directly. Nevertheless, given Assumption 1', we can state the following (and prove it using similar reasoning to Proposition 3 of the paper):

Proposition 2 *If there is negative correlation between states, then it is welfare maximizing to have more investment in ratings quality in a recession than in a boom when they differ only in terms of default rates, the fraction of good issues, and labor market tightness. It is welfare maximizing to have more investment in a boom than a recession when they differ only in terms of investors' outside options. In general, there is no unambiguous ranking.*