

Online Appendix to: Connecting Two Markets: An Equilibrium Framework for Shorts, Longs, and Stock Loans

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Abstract

This appendix discusses the effect of shocks to the supply of or demand for shares on prices taking into account the joint price setting in both the equity lending market and the stock market. Proofs are left until the end of the discussion.

A. Appendix – The Effect of Supply or Demand Shocks on Prices

This appendix concerns the effect of shocks to the supply of, or demand for, shares on prices. A sale of stock, short or otherwise, can be equivalently defined as a reduction in demand for, or increase in supply of, the stock. Similarly, a purchase, to own or to cover a short position, can be equivalently defined as an increase in demand for or decrease in supply of the stock. For the sake of clarity, we adopt the convention that all purchases and sales affect demand changes, and we define supply as the number of shares outstanding.

If there is an exogenous reduction in demand for a stock, for example through a planned sale of founder's shares, then somebody must absorb the new shares on the market. When the equity lending constraint is slack, the amount that the price must decline is determined by the demand shock and the elasticity of demand, but we show that when the constraint binds, the price response is typically greater: shocks to demand, in either direction, usually cause larger price changes when a stock is hard to borrow than when it is not.

As demand increases, the price should increase. When a stock is not on special this is trivially so, since in this case implicitly differentiating the equity ownership equilibrium condition,

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$D_L(p, p_s) + D_S(p, p_s) + \delta = N$, and rearranging yields:

$$\frac{dp^*}{d\delta} = -\frac{1}{\frac{\partial}{\partial p} D_L + \frac{\partial}{\partial p} D_S}. \quad (\text{A.1})$$

Because $\frac{\partial}{\partial p} D_L$ and $\frac{\partial}{\partial p} D_S$ are negative, $\frac{dp^*}{d\delta}$ is positive. When a stock is on special, there are two relevant equilibrium conditions. Given an increase in demand from longs of δ , our equilibrium conditions are:¹

$$[D_L + \delta] + D_S = N \quad (\text{A.2})$$

and

$$r(p_s^*) \times [D_L + \delta] = -D_S. \quad (\text{A.3})$$

Combining, we get

$$D_L = \frac{N}{1 - r(p_s^*)} - \delta. \quad (\text{A.4})$$

It is more difficult to show that $\frac{dp^*}{d\delta}$ is positive in this case than when the stock is not on special, but it is possible. We leave the mathematics to the Appendix and state the result here.

THEOREM 1: *The equilibrium price of a stock p^* and price to borrow p_s^* are weakly increasing in the size of a demand shock.*

A positive demand shock requires an increase in price to reduce demand and restore equilibrium: higher demand leads to a higher price. The higher price attracts more short sellers, thus increasing demand for shares in the equity lending market. This increases the price to borrow, p_s , restoring equilibrium in the equity lending market. Of course, the price in each market affects demand and supply in the other, so this description ignores a recursion, but it should make clear the basic mechanism causing both prices to rise.

The price to buy and price to borrow are increasing in the size of the demand shock when the stock is on special. When the stock is not on special the price is clearly increasing in the demand shock as well, though the price to borrow remains at zero so long as the demand shock

¹Note that we assume here that once the shares are purchased they are lent out at the same rate as other owners would lend their shares: $r(p_s^*)$. If the shares were not lent out, our equilibrium condition would be $D_L = \frac{N - \delta}{1 - r(p_s^*)}$.

is small. We want to compare the magnitude of the shock in the two situations. This difference is expressed in the following theorem:

THEOREM 2: *When the number of shares available to potential borrowers does not respond much to the price in the equity lending market (i.e., when $\frac{dr}{dp_s^*}$ is close to zero), price responses to demand shocks are higher when a stock is on special than not.*

So when demand increases, price increases, causing shorts to increase demand for shares in the equity lending market. This drives up the price to borrow shares, limiting the number of shorts in the stock market, thus allowing a higher price. The reverse argument follows for negative demand shocks.

Proof of Theorem 1. The result for the case when the stock is not on special can be found in the text. When the stock is on special, we apply Topkis' Theorem. Define the negative squared error in equation A.4 as the objective, let δ be a parameter, and let p_s and p be choice variables:

$$\Pi = - \left(\int_{i^*(p,p_s)}^{\infty} D(p, p_s, i) dF(i) + \delta - \frac{N}{1 - r(p_s)} \right)^2$$

To show that p_s^* and p^* are increasing in δ , we must show that Π is supermodular in all three terms. Because Π is well behaved (everything is twice continuously differentiable) it is sufficient to take three mixed partials and show they are positive.

$$\frac{d\Pi}{d\delta} = -2 \left(\int_{i^*(p,p_s)}^{\infty} D(p, p_s, i) dF(i) + \delta - \frac{N}{1 - r(p_s)} \right)$$

so

$$\frac{d^2\Pi}{d\delta dp} = -2 \frac{d}{dp} \int_{i^*(p,p_s)}^{\infty} D(p, p_s, i) dF(i)$$

To a first order approximation,² this equals

$$\frac{d^2\Pi}{d\delta dp} = -2 \int_{i^*(p,p_s)}^{\infty} D_1 dF(i) \geq 0$$

Because D_1 is negative, the integrand is negative, so the mixed partial is positive. Now with

²The approximation is because the lower limit of integration is also affected by p and p_s . However, at $i^*(p, p_s)$ demand is 0, so the change in demand resulting from a shift in this limit is zero to a first order approximation.

respect to p_s

$$\frac{d^2\Pi}{d\delta dp_s} = -2 \left(\frac{d}{dp_s} \int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) - \frac{N}{(1-r(p_s))^2} \frac{dr}{dp_s} \right)$$

Since we assume that instantaneous demand from longs does not depend on the price of borrowing shares,

$$\int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) = \int_{i^*(p,0)}^{\infty} D(p,0,i) dF(i)$$

so we get

$$\frac{d^2\Pi}{d\delta ds} = \frac{2N}{(1-r(p_s))^2} \frac{dr}{ds} \geq 0$$

The final mixed partial is with respect to p and p_s :

$$\begin{aligned} \frac{d\Pi}{dp_s} &= -2 \left(\int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) + \delta - \frac{N}{1-r(p_s)} \right) \\ &\quad \times \frac{d}{dp_s} \left(\int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) + \delta - \frac{N}{1-r(p_s)} \right) \\ &= -2 \left(\int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) + \delta - \frac{N}{1-r(p_s)} \right) \\ &\quad \times \frac{N}{(1-r(p_s))^2} \frac{dr}{dp_s} \end{aligned}$$

so taking the mixed partial:

$$\begin{aligned} \frac{d^2\Pi}{dsdp} &= -2 \left[\frac{d}{dp} \left(\int_{i^*(p,p_s)}^{\infty} D(p,p_s,i) dF(i) + \delta - \frac{N}{1-r(p_s)} \right) \right] \\ &\quad \times \frac{N}{(1-r(p_s))^2} \frac{dr}{dp_s} \\ &= -2 \left(\int_{i^*(p,p_s)}^{\infty} D_1 dF(i) \right) \frac{N}{(1-r(p_s))^2} \frac{dr}{dp_s} > 0 \end{aligned}$$

again, to a first order approximation. Since D_1 is negative and $\frac{dr}{dp_s}$ is positive, the mixed partial is positive. \square

Proof of Theorem 2. We differentiate the equilibrium condition in Equation (A.4) with respect to δ , remembering that p^* and p_s^* depend on δ :

$$\frac{d}{d\delta} D_L = \frac{d}{d\delta} \frac{N}{1-r(p_s^*)} - 1.$$

To a first order approximation,

$$\frac{\partial}{\partial p} D_L \times \frac{dp^*}{d\delta} + \frac{\partial}{\partial p_s} D_L \times \frac{dp_s^*}{d\delta} = \frac{N}{[1 - r(p_s^*)]^2} \frac{dr}{dp_s^*} \frac{dp_s^*}{d\delta} - 1.$$

Because we assume that those who demand positive quantities are uninterested in the share borrowing price, $\frac{\partial}{\partial p_s} D_L = 0$. We can therefore simplify to get

$$\frac{\partial}{\partial p} D_L \times \frac{dp^*}{d\delta} = \frac{N}{[1 - r(p_s^*)]^2} \frac{dr}{dp_s^*} \frac{dp_s^*}{d\delta} - 1,$$

and rearranging yields

$$\frac{dp^*}{d\delta} = \left[-\frac{1}{\frac{\partial}{\partial p} D_L} \right] \times \left[1 - \frac{N}{[1 - r(p_s^*)]^2} \frac{dr}{dp_s^*} \frac{dp_s^*}{d\delta} \right]. \quad (\text{A.5})$$

The denominator in the first term is closer to zero than the denominator in Equation (A.1).

Since this denominator is negative, the first term itself is less negative than in Equation (A.1).

Let $\frac{dr}{dp_s^*}$ tend to zero:

$$\begin{aligned} \lim_{\frac{dr}{dp_s^*} \rightarrow 0} \frac{dp^*}{d\delta} &= \left[-\frac{1}{\frac{\partial}{\partial p} D_L} \right] \times \left[1 - \frac{N}{[1 - r(p_s^*)]^2} \frac{dp_s^*}{d\delta} \left(\lim_{\frac{dr}{dp_s^*} \rightarrow 0} \frac{dr}{dp_s^*} \right) \right] \\ &= -\frac{1}{\frac{\partial}{\partial p} D_L}, \end{aligned}$$

which is strictly greater than the derivative of price with respect to a demand shock when a stock is not on special or naked shorting is allowed. Because the right hand side of equation A.5 is continuous, for $\frac{dr}{dp_s^*}$ close to zero, the inequality remains. \square