Appendix to Diagnostic Bubbles

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Appendix A: Proofs.

Lemma 1. At time t, each trader has access to t signals of precision $\frac{1}{\sigma_\epsilon^2}$, as well as a prior with precision $\frac{1}{\sigma_V^2}$ around 0. By standard results in normal posterior updating, the trader's rational posterior is a normal distribution, with the following mean and variance.

$$\mathbb{E}_{i,t}(V) = \frac{t/\sigma_\epsilon^2}{t/\sigma_\epsilon^2 + 1/\sigma_V^2}\,\overline{s_{\iota,t}} = \pi_t \bar{s}_{it}$$

$$\sigma_t^2(V) = \frac{1}{\frac{t}{\sigma_\epsilon^2} + \frac{1}{\sigma_V^2}} = (1-\pi_t)\sigma_V^2$$

The diagnostic distribution then reads (up to normalization constants):

$$\exp\left\{-\frac{1}{2\sigma_t^2(V)}\left[\left(V - \mathbb{E}_{i,t}(V)\right)^2(1+\theta) - \theta\left(V - \mathbb{E}_{i,t-1}(V)\right)^2\right]\right\}$$

The quadratic and linear terms in V are given by (the constant terms are absorbed by normalization):

$$\exp\left\{-\frac{1}{2\sigma_t^2(V)}\left[V^2 - 2V\left(\mathbb{E}_{i,t}(V)(1+\theta) - \theta\mathbb{E}_{i,t-1}(V)\right)\right]\right\}$$

It follows that the diagnostic distribution is also a normal $\mathcal{N}\left(\mathbb{E}_{i,t}^\theta(V), Var_t^\theta(V)\right)$ with mean:

$$\mathbb{E}_{i,t}^\theta(V) = \mathbb{E}_{i,t}(V) + \theta\left[\mathbb{E}_{i,t}(V) - \mathbb{E}_{i,t-1}(V)\right]$$

and variance:

$$\left(\sigma_t^\theta\right)^2(V) = \sigma_t^2(V)$$

from which the result follows.

Proposition 1. For simplicity, denote $\alpha = \frac{\sigma_\epsilon^2}{\sigma_V^2}$. The consensus is then equal to $\frac{t}{t+\alpha}(1+\theta)V$ for $t \leq k$. Thus, the consensus is increasing from $t = 1$ to k. At $t = 1$, the price is below the fundamental iff $\frac{1}{1+\alpha}(1+\theta) < 1$ if $\theta < \alpha$. At $t = k$, the consensus overvalues the asset, namely $\frac{k}{k+\alpha}(1+\theta) > 1$ if $\theta > \alpha/k$. Next, let us show that $\mathbb{E}_{k+1}^\theta(V) < \mathbb{E}_k^\theta(V)$. It suffices to show:

$$\frac{(1+\theta)k}{k+\alpha} > \frac{(1+\theta)(k+1)}{k+1+\alpha} - \frac{\theta}{1+\alpha}$$
$$\Leftrightarrow \theta > \frac{\alpha}{k} \cdot \frac{\alpha+1}{k+2\alpha+1}$$

Whenever the consensus overvalues the asset at time t, namely $\theta > \alpha/k$, the above condition is satisfied, so the price declines at $t = k+1$. What happens from there on? Taking the derivative of the consensus with respect to t, one obtains:

$$\frac{d\mathbb{E}_t^\theta(V)}{dt} \propto \frac{(1+\theta)}{(t+\alpha)^2} - \frac{\theta}{(t-k+\alpha)^2}$$

The above expression is negative if and only if:

$$\frac{d\mathbb{E}_t^\theta(V)}{dt} < 0 \Leftrightarrow \left(\frac{t-k+\alpha}{t+\alpha}\right)^2 < \frac{\theta}{1+\theta} \Leftrightarrow t < \left(1 - \sqrt{\frac{\theta}{1+\theta}}\right)^{-1} \cdot k - \alpha$$

As a result, after $t^* \equiv max\left[k+1, \left(1-\sqrt{\frac{\theta}{1+\theta}}\right)^{-1} \cdot k - \alpha\right]$, the consensus increases monotonically.

To conclude, observe that $[\pi_t + \theta(\pi_t - \pi_{t-k})]V \mapsto V$, because $\pi_t \mapsto 1$ as $t \mapsto \infty$.

Consider the other parametric cases. If $\theta > \alpha$ the consensus is immediately overvalued, and follows the same boom bust path as in the leading case. If $\theta < \alpha/k$ the consensus is below fundamental at $t = k$. From there on, it either monotonically increases toward the fundamental value, or it drops a bit but then increases toward the fundamental value. In either case, though, the consensus stays below the fundamental.

Proposition 2. We start by assuming a linear price formula of the form:

$$p_t = a_{2t}\mathbb{E}(V|P_t) + +b_t\left(V - \frac{c_t}{b_t}S_t\right)$$

for $t \leq k$. Denote $s_t^p = \frac{1}{b_t}(p_t - a_{t2}E[V|P_t]) = V - \frac{c_t}{b_t}S_t$ as the public signal obtained about V from the prices. Furthermore, let ζ_t be the precision of the public distribution, i.e. $\zeta_t = \frac{1}{\sigma_{p,t}^2}$, and use the shorthand $E_t^p = E[V|P_t]$. Using standard results of normal posteriors, we obtain:

$$E_t^p = \frac{1}{\sigma_S^2}\sum_{r=1}^{t-1} s_r^p \left(\frac{b_r}{c_r}\right)^2 \frac{1}{\zeta_t}$$

$$\zeta_t = \frac{1}{\sigma_V^2} + \frac{1}{\sigma_S^2}\sum_{r=1}^{t-1}\left(\frac{b_r}{c_r}\right)^2$$

Denoting $\overline{E_t}$ as the average rational fundamental beliefs, it follows that:

$$\bar{E}_t = \int \left[\frac{1}{\sigma_V^2} + \frac{1}{\sigma_S^2}\sum_{r=1}^{t-1} s_r^p \left(\frac{b_r}{c_r}\right)^2 \quad\right]\frac{1}{\zeta_t}\, di = \frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2}+\zeta_t} V + \frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2}+\zeta_t} E_t^p$$

Then, as $t \leq k$, we have $\overline{E_t}^\theta = (1+\theta)\, \bar{E}_t$, and hence our equilibrium condition

$$p_t = \overline{E_t}^\theta - \gamma\sigma_t^2(V)S_t$$

translates to:

$$p_t = (1+\theta)\left(\frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2}+\zeta_t} V + \frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2}+\zeta_t} E_t^p\right) - \gamma\frac{S_t}{\frac{t}{\sigma_\epsilon^2}+\zeta_t}$$

Matching coefficients, we obtain:

$$a_{2t} = (1+\theta)\frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2}+\zeta_t}$$

$$b_t = (1+\theta)\frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2}+\zeta_t}$$

$$c_t = \gamma\left(\frac{t}{\sigma_\epsilon^2}+\zeta_t\right)^{-1}$$

In particular, note $\frac{b_t}{c_t} = \frac{1+\theta}{\gamma}\cdot\frac{t}{\sigma_\epsilon^2}$, and plugging this into our expression for ζ_t, one obtains:

$$\zeta_t = \frac{1}{\sigma_V^2} + \frac{(1+\theta)^2}{\gamma^2\sigma_S^2}\sum_{r=1}^{t-1}\frac{r^2}{\sigma_\epsilon^4} = \frac{1}{\sigma_V^2} + \frac{(1+\theta)^2}{\gamma^2\sigma_S^2\sigma_\epsilon^4}\frac{(t-1)t(2t-1)}{6}$$

Consider now the average price $\overline{p_t}$, obtained by setting the supply shocks to their average, $S_t = 0$. Plugging in E_t^p and then ζ_t we find:

$$\overline{p_t} = (1+\theta)\left(\frac{\frac{t}{\sigma_\epsilon^2} + \zeta_t - \frac{1}{\sigma_V^2}}{\frac{t}{\sigma_\epsilon^2} + \zeta_t}\right) V = \frac{(1+\theta)\left(\frac{t}{\sigma_\epsilon^2} + \frac{(1+\theta)^2}{\gamma^2\sigma_S^2\sigma_\epsilon^4}\frac{(t-1)t(2t-1)}{6}\right)}{\left(\frac{1}{\sigma_V^2} + \frac{t}{\sigma_\epsilon^2} + \frac{(1+\theta)^2}{\gamma^2\sigma_S^2\sigma_\epsilon^4}\frac{(t-1)t(2t-1)}{6}\right)} V.$$

Proposition 3. Under learning from prices, the price at time t is $p_t = \mathbb{E}_t^\theta(V|p_{t-1},\dots,p_1;\vec{s}_t,\dots,\vec{s}_1)$. From Proposition 2, the average price path is:

$$p_t = (1+\theta)\left[\frac{\frac{t}{\sigma_\epsilon^2} + \left(\frac{1+\theta}{\sigma_S\sigma_\epsilon^2}\right)^2\frac{t(t-1)(2t-1)}{6}}{\frac{t}{\sigma_\epsilon^2} + \frac{1}{\sigma_V^2} + \left(\frac{1+\theta}{\sigma_S\sigma_\epsilon^2}\right)^2\frac{t(t-1)(2t-1)}{6}}\right] V$$

To explore convexity, rewrite price as:

$$p_t = (1+\theta)V\frac{f}{f+c}$$

where $f = k_1 t + k_2 t(t-1)(2t-1)$, $c = \frac{1}{\sigma_V^2}$, $k_1 = \frac{t}{\sigma_\epsilon^2}$ and $k_2 = \frac{1}{6}\left(\frac{1+\theta}{\sigma_S\sigma_\epsilon^2}\right)^2$. Note that, up to a constant, we have:

$$\partial_t^2 p_t = \partial_t \frac{f'}{(f+c)^2} = (f+c)^{-2}\left[f'' - \frac{2(f')^2}{(f+c)}\right]$$

which has the same sign as $f'' - \frac{2(f')^2}{(f+c)}$. This is positive when:

$$c > \frac{2(f')^2}{f''} - f$$

Convexity requires c to be large, that is σ_V^2 to be small. For example, convexity at $t = 1$ requires:

$$c > \frac{k_1^2 - k_1 k_2 + k_2^2}{3k_2}$$

Rewrite the condition above as:

$$f''c > 2(f')^2 - ff''$$

that is

$$6k_2(2t-1)c > 2\big(k_1 + k_2(6t^2 - 6t + 1)\big)^2 - 6k_2(2t-1)\big(k_1 t + k_2(2t^3 - 3t^2 + t)\big)$$

As we are looking near $t = 1$, let us set $s = t - 1$. Then, the inequality simplifies to:

$$6k_2(2s+1)c > 48k_2^2 s^4 + 96k_2^2 s^3 + 66k_2 s^2 + 12k_1k_2 s^2 + 18k_2^2 s + 6k_1k_2 s + 2k_1^2 + 2k_2^2 - 2k_1k_2$$

In particular, the right-hand side is a quartic with positive coefficients (and in particular it is convex in s), whereas the left hand side is a linear function. Hence, if the left-hand side lies above the right hand side at $t = 1$, the two will cross at $t = t^* > 1$, and never cross again. Hence, for the average price path to be convex at $t \in [1, t^*]$ and concave afterwards, it is necessary and sufficient for the above inequality to hold at $s = 0$, which is given by:

$$c > \frac{k_1^2 - k_1k_2 + k_2^2}{3k_2} \rightarrow \sigma_V^2 < \left(\frac{k_1^2 - k_1k_2 + k_2^2}{3k_2}\right)^{-1} = \sigma_V^{*2}$$

as desired.

Proposition 4. Consider the price path

$$p_t = (1+\theta)^{T-t+1}\left[\prod_{r=t}^{T} \pi_r\right] V$$

where $\pi_t = \frac{t}{t+\frac{\sigma_\epsilon^2}{\sigma_V^2}}$ and denote $\hat{\pi} \equiv [\prod_{r=1}^{T} \pi_r]^{\frac{1}{T}}$. First note that $p_t - p_{t-1} = p_t[1 - (1+\theta)\pi_{t-1}]$. So price increases at t if $(1+\theta)\pi_{t-1} < 1$ and it decreases otherwise. Because π_t is monotonically increasing, it follows that the price path is either always increasing (if $(1+\theta)\pi_T < 1$), or always decreasing (if $(1+\theta)\pi_1 > 1$) or is first increasing and then decreasing. This holds provided $(1+\theta)\pi_1 < 1 < (1+\theta)\pi_T$, which reads $\theta \in \left(\frac{\sigma_\epsilon^2}{T\sigma_V^2}, \frac{1-\hat{\pi}}{\hat{\pi}}\right)$. In particular, we then have both $p_1 = (1+\theta)^T[\prod_{r=1}^{T} \pi_r]V < V$ and $p_T = (1+\theta)\pi_T V > V$.

Appendix B: Material on Dynamic Trading

1. Recap of He and Wang Dynamic REE

We shall closely follow the notation of He and Wang 1995. Let $Q_{t+1} = P_{t+1} - P_t$ be the price innovation at time $t+1$. Let $\Psi_{t+1}^i = \left[1,\ \mathbb{E}_{i,t+1}(V), \mathbb{E}_{p,t+1}(V), \mathbb{E}_{pt}(V)\right]$ be the relevant state space. (Adding the lagged value of public expectations ensures that P_{t+1} is a linear function of the state variables.) As always, denote γ as the risk aversion parameter. We denote:

$$Q_{t+1} = A_{Q,t+1}\,\Psi_t^i\; + B_{Q,t+1}\epsilon_{t+1}^i$$

$$\Psi_{t+1}^i = A_{\Psi,t+1}\,\Psi_t^i\; + B_{\Psi,t+1}\epsilon_{t+1}^i,$$

where ϵ_{t+1}^i is a vector of shocks (including both the supply shock, the innovation to fundamentals, and signal noise). Setting X_t as the portfolio weight, we have that wealth evolves according to $W_{t+1}^i = W_t^i + X_t^i Q_{t+1}$. The individual optimizes $\mathbb{E}_{it}^{\theta}\left[-e^{-\gamma W_T^i}\right]$, where $W_{final}^i = W_T^i + X_T^i(V - P_T)$.

According to He and Wang 1995, one can apply backwards induction on the proposed value function. They show that the value function takes the following convenient form:

$$J\left(W_{t+1}^i;\ \Psi_{t+1}^i\right) = -\exp\left[-\gamma W_{t+1}^i - \frac{1}{2}\left(\Psi_{t+1}^i\right)' U_{t+1}\Psi_{t+1}^i\right],$$

Standard operations involving multivariate Gaussian integrals can be used to solve the coefficients recursively. By setting:

$$\Xi_{t+1} = \left(\Sigma_{t+1}^{-1} + B'_{\Psi,t+1}U_{t+1}B_{\Psi,t+1}\right)^{-1}$$

$$\rho_{t+1} = \sqrt{|\Xi_{t+1}|/|\Sigma_{t+1}|}$$

$$F_t = \left[B_{Q,t+1}\Xi_{t+1}B'_{Q,t+1}\right]^{-1}\left(A_{Q,t+1} - B_{Q,t+1}\Xi_{t+1}B'_{\Psi,t+1}\,U_{t+1}A_{\Psi,t+1}\right),$$

we can complete the recursion for U_t by computing

$$M_t = F'_t\left(B_{Q,t+1}\Xi_{t+1}B'_{Q,t+1}\right)F_t - \left(B'_{\Psi,t+1}U_{t+1}A_{\Psi,t+1}\right)'\Xi_{t+1}\left(B'_{\Psi,t+1}U_{t+1}A_{\Psi,t+1}\right),$$

and setting $U_t = M_t - 2\log \rho_{t+1} I_{1,1}^{4\times 4}$, where $I_{1,1}^{4\times 4}$ is a matrix that only has the (1, 1) element set to 1. Furthermore, one can easily show that the portfolio choice for individual i is given by:

$$X_t^i = \frac{1}{\gamma} F_t \Psi_t^i.$$

Thus, we need the following information to start computing the equilibrium coefficients:

1. The update matrix $A_{Q,t+1}, A_{\Psi,t+1}, B_{Q,t+1}, B_{\Psi,t+1}$.
2. The final objective matrix U_T (In the final period, Q_{T+1} should be $V - P_T$)

2. Applying dynamic REE to our setting

A. Introducing diagnostic distortions

Recall that our state space for each individual's dynamic programming problem was $\Psi_{t+1}^i = [1\ \mathbb{E}_{i,t+1}(V)\ \mathbb{E}_{p,t+1}(V)\ \mathbb{E}_{pt}(V)]$. A natural way to model diagnostic expectations in this notation is to assume that the individual has distorted expectations of the transition probabilities of the future state space. Under rationality and normal shocks, the distribution of the future states consists of a multivariate normal. One can show then that the diagnostic distribution of the future state $f^\theta(\Psi_{t+1}|F_t)$ consists of a shift of the rational distribution by $1+\theta$ of its current mean, as done in BGMS 2019. In the notation of He and Wang 1995, this is equivalent to distorting the transition matrix $A_{\Psi,t+1}$ by:

$$A_{\Psi,t+1}^{diag} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\theta & 0 & 0 \\ 0 & 0 & 1+\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot A_{\Psi,t+1},$$

while maintaining $B_{\Psi,t+1}$. Note that the entries corresponding to the constant term and E_{pt} are not inflated. This is because the two quantities are measurable at time t, and hence diagnostic expectations does not distort the quantity. Denoting $P_t = \Lambda_t \Psi_t$, we then obtain:

$$A_{Q,t+1}^{diag} = \Lambda_{t+1} A_{\Psi,t+1}^{diag} - \Lambda_t$$

$$B_{Q,t+1} = \Lambda_{t+1} B_{\Psi,t+1}.$$

All that remains for our purposes are specifying U_t, which is the dynamic component of the objective function. For this, we use the U_t^{rat} computed from the above recursive equations assuming $\theta = 0$. Intuitively, the agent trades by using U_t^{rat} as sufficient statistics to account for dynamic trading. This assumption greatly simplifies computation, and can be interpreted as a diagnostic perturbation of the dynamic rational expectations equilibrium: we are preserving the rational dynamic motivation, while introducing a distortion in future state transition probabilities.

B. Market clearing and computing the equilibrium coefficients

As in Section 3, we stipulate the following equilibrium coefficients (a_t, b_t, c_t), where

$$p_t = a_t\, E_{pt} + b_t V - c_t S_t.$$

Setting $\zeta_t = \frac{1}{\sigma_{pt}^2}$ as the accumulated precision of the public price signals by time t, we have the following equations from standard Bayesian computations:

$$\zeta_t = \frac{1}{\sigma_V^2} + \frac{1}{\sigma_S^2}\sum_{r=1}^{t}\left(\frac{b_r}{c_r}\right)^2$$

$$E_{pt} = \zeta_t^{-1}\left(\frac{1}{\sigma_S^2}\sum_{r=1}^{t}\left(\frac{b_r}{c_r}\right)^2\left(V - \frac{c_r}{b_r}S_r\right)\right)$$

$$E_{it}[V] = \frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2} + \zeta_t}\ \bar{s}_t^{\,i} + \frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2} + \zeta_t} E_{pt}.$$

By definition, according to individual i's beliefs, the true value V is distributed according to: $V \sim N\left(E_{i,t}[V],\ \left(\zeta_t + \frac{t}{\sigma_\epsilon^2}\right)^{-1}\right)$. Hence, we can express: $V = E_{i,t}[V] + u,\ u \sim N\left(0,\ \left(\zeta_t + \frac{t}{\sigma_\epsilon^2}\right)^{-1}\right)$. Furthermore, denote $s_t^i = V + \epsilon_t^i$. For simplicity, let us suppress the terms corresponding to the constant term in the state space – they remain 1. After standard algebra, one obtains the following rational transition equation for the private and public expectations of the fundamental value.

$$\begin{pmatrix} E_{p,t+1} \\ E_{i,t+1}[V] \end{pmatrix} = \begin{pmatrix} \frac{\zeta_t}{\zeta_{t+1}} & 1-\frac{\zeta_t}{\zeta_{t+1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_{p,t} \\ E_{i,t}[V] \end{pmatrix} + \begin{pmatrix} \frac{\zeta_{t+1}-\zeta_t}{\zeta_{t+1}}\left(u-\frac{c_{t+1}}{b_{t+1}}S_{t+1}\right) \\ \frac{\frac{1}{\sigma_\epsilon^2}+\zeta_{t+1}-\zeta_t}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}}u+\frac{\frac{1}{\sigma_\epsilon^2}}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}}\epsilon_{t+1}^i-\frac{\zeta_{t+1}-\zeta_t}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}}\frac{c_{t+1}}{b_{t+1}}\ S_{t+1} \end{pmatrix}$$

This implies:

$$A_{\Psi,\mathrm{t+1}} = \begin{pmatrix} 1 & 0 & 0 \\ 1-\frac{\zeta_t}{\zeta_{t+1}} & \frac{\zeta_t}{\zeta_{t+1}} & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow A_{\Psi,\mathrm{t+1}}^{diag} = \begin{pmatrix} 1+\theta & 0 & 0 \\ (1+\theta)\left(1-\frac{\zeta_t}{\zeta_{t+1}}\right) & (1+\theta)\frac{\zeta_t}{\zeta_{t+1}} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B_{\Psi,\mathrm{t+1}} = \begin{pmatrix} \frac{\frac{1}{\sigma_\epsilon^2}+\zeta_{t+1}-\zeta_t}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}} & \frac{\frac{1}{\sigma_\epsilon^2}}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}} & -\frac{\zeta_{t+1}-\zeta_t}{\frac{t+1}{\sigma_\epsilon^2}+\zeta_{t+1}}\frac{c_{t+1}}{b_{t+1}} \\ 1-\frac{\zeta_t}{\zeta_{t+1}} & 0 & -\left(1-\frac{\zeta_t}{\zeta_{t+1}}\right)\frac{c_{t+1}}{b_{t+1}} \\ 0 & 0 & 0 \end{pmatrix}$$

with the variance of $\epsilon_t^i = \begin{pmatrix} u \\ \epsilon \\ S \end{pmatrix}$ set to $\begin{pmatrix} \left(\zeta_t+\frac{t}{\sigma_\epsilon^2}\right)^{-1} & 0 & 0 \\ 0 & \sigma_\epsilon^2 & 0 \\ 0 & 0 & \sigma_S^2 \end{pmatrix}$. Furthermore, one obtains:

$$p_{t+1} = \Lambda_{t+1}\Psi_{t+1} = \begin{pmatrix} 0 & a_{t+1}-\frac{b_{t+1}}{1-\frac{\zeta_t}{\zeta_{t+1}}} & -b_{t+1}\frac{\frac{\zeta_t}{\zeta_{t+1}}}{1-\frac{\zeta_t}{\zeta_{t+1}}} \end{pmatrix} \cdot \Psi_{t+1}$$

As mentioned above,

$$A_{\mathrm{Q,t+1}}^{diag} = \Lambda_{t+1}A_{\Psi,\mathrm{t+1}}^{diag} - \Lambda_t$$

$$B_{Q,t+1} = \Lambda_{t+1}B_{\Psi,\mathrm{t+1}}.$$

To compute the equilibrium coefficients, we impose market-clearing: the investor demand should equal the supply at t:

$$\frac{1}{\gamma}F_t^{diag}\int \Psi_t^{\mathrm{i}}\,di = S_t.$$

Note:

$$\int \Psi_t^i di = \begin{pmatrix} V \cdot \frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2} + \zeta_t} + \frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2} + \zeta_t} E_{p,t} \\ E_{p,t} \\ E_{p,t-1} \end{pmatrix} = \begin{pmatrix} \frac{\zeta_t}{\frac{t}{\sigma_\epsilon^2} + \zeta_t} & \frac{\frac{t}{\sigma_\epsilon^2}}{\frac{t}{\sigma_\epsilon^2} + \zeta_t} & 0 \\ 1 & 0 & 0 \\ \frac{\zeta_t}{\zeta_{t-1}} & -\frac{1}{\sigma_S^2}\left(\frac{b_t}{c_t}\right)^2 \frac{1}{\zeta_{t-1}} & \frac{1}{\sigma_S^2}\frac{b_t}{c_t}\frac{1}{\zeta_{t-1}} \end{pmatrix} \begin{pmatrix} E_{p,t} \\ V \\ S_t \end{pmatrix}$$

$$= \Gamma_t(a_t, b_t, c_t) \begin{pmatrix} E_{p,t} \\ V \\ S_t \end{pmatrix}.$$

Hence, one can solve for the equilibrium coefficients (a_t, b_t, c_t) that satisfies the market-clearing conditions:

$$\underbrace{\frac{1}{\gamma} F_t^{diag}(a_t, b_t, c_t)}_{Dynamic\ diagnostic\ demand\ given\ rational\ beliefs} \cdot \underbrace{\Gamma_t(a_t, b_t, c_t) \begin{pmatrix} E_{p,t} \\ V \\ S_t \end{pmatrix}}_{average\ rational\ beliefs} = \underbrace{(0 \quad 0 \quad 1) \begin{pmatrix} E_{p,t} \\ V \\ S_t \end{pmatrix}}_{supply}.$$

Using the formula we have for $F_t^{diag} = \left[B_{Q,t+1}\Xi_{t+1}B'_{Q,t+1}\right]^{-1}\left(A_{Q,t+1}^{diag} - B_{Q,t+1}\Xi_{t+1}B'_{\Psi,t+1}U_{t+1}A_{\Psi,t+1}^{diag}\right)$, one can use the above market clearing equation to numerically solve for (a_t, b_t, c_t) inductively backwards, given the boundary conditions[1]:

$$a_T = \frac{(1+\theta)\zeta_T}{\frac{T}{\sigma_\epsilon^2} + \zeta_T}$$

$$b_T = \frac{\frac{(1+\theta)T}{\sigma_\epsilon^2}}{\frac{T}{\sigma_\epsilon^2} + \zeta_T}$$

$$c_T = \gamma\left(\frac{T}{\sigma_\epsilon^2} + \zeta_T\right)^{-1}$$

[1] It is not entirely trivial to solve for the coefficients numerically from the above equations. First, one can deduce that the equilibrium coefficients must satisfy:

$$\Lambda_t = \Lambda_{t+1} A_{\Psi,t+1}^{diag} - B_{Q,t+1}\Xi_{t+1}B'_{\Psi,t+1}U_{t+1}A_{\Psi,t+1}^{diag} - \gamma \cdot B_{Q,t+1}\Xi_{t+1}B'_{Q,t+1}(0 \quad 0 \quad 1)\Gamma_t^{-1}.$$

As the first coefficient of Λ_t is 0, one can show that this pins down ζ_{t-1} as a univariate zero, from which one can easily compute Γ_t. The concrete coefficients (a_t, b_t, c_t) then follows from $\Lambda_t = \left(0 \quad a_t - \frac{b_t}{1-\frac{\zeta_{t-1}}{\zeta_t}} \quad -b_t\frac{\frac{\zeta_{t-1}}{\zeta_t}}{1-\frac{\zeta_{t-1}}{\zeta_t}}\right)$. The precise numerical procedure can be given upon request.

In summary, the equilibrium coefficients can be computed from the following procedure:

1. Guess the final public precision ζ_T in a given grid.
2. Compute the boundary coefficients (a_T, b_T, c_T).
3. Inductively compute the coefficients (a_t, b_t, c_t) backwards.
4. Verify $\zeta_0 = \zeta_1 - \frac{1}{\sigma_S^2}\left(\frac{b_1}{c_1}\right)^2 = \frac{1}{\sigma_V^2}$.