Online Appendix

Labor leverage, coordination failures, and aggregate risk

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Production at t=0 and at t=1

In the main model, monopolists choose their operating leverage at $t = 0$, before uncertainty about L is realized at $t = 1$. This captures the idea that firms' technological and organizational choices are long lasting and therefore, that firms cannot respond to a productivity shock by swiftly changing their production technologies. In this section, we extend the model to show that our results remain unchanged when monopolists also produce at $t = 0$ and can condition their choice of operating leverage to the current ($t = 0$) productivity level, provided that they cannot rapidly adjust their operating leverage to accommodate realizations of future ($t = 1$) productivity shocks. Formally, this requires the following changes to the main model. We assume that production also takes place $t = 0$. That is, a productivity shock L_0 is realized at $t = 0$. Then, knowing L_0, each monopolists makes an operating leverage decision, a production decision and the game moves on to $t = 1$. At $t = 1$, a new shock L_1 is realized and monopolists make a second production decision. Importantly, a monopolist cannot adjust its operating leverage between the two production decisions at $t = 0$ and $t = 1$. The productivity shocks $\{L_0, L_1\}$ are uniformly distributed over $[0, \overline{L}]$ and independent. Similar to the main model, the consumer's utility is given by

$$U = \exp\left[\int_0^1 \ln x_1(q)dq\right] + \theta \exp\left[\int_0^1 \ln x_2(q)dq\right], \tag{0A.1}$$

where $x_t(q)$ is the consumption at t defined over a unit interval of goods indexed by q and $\theta \in (0, 1]$ is a time discount factor. Everything else remains the same as in the main model:

(i) Each good q is produced by a sector, and each sector consists of two types of firms. A competitive fringe of firms with a constant returns to scale technology in which one unit of output requires one unit of labour, and a monopolist with access to an increasing returns to scale technology; (ii) In each period t, monopolist q incurs a marginal cost of $\alpha - s_q$ and a fixed cost of $F(s_q)$ to produce, where $F(s_q)$ is increasing, convex, and tends to infinity as $s_q \to \alpha$; and (iii) Production by monopolists requires a monitoring effort β per unit of output.

This extension poses a minor technical challenge. In the baseline model, the equilibrium of the production game is pinned down as the limit of a dispersed information game in which monopolists imperfectly observe L after their choices of operating leverage but before their production decisions. (See Section 3.2.) However, the point of the current extension is to let monopolists observe L_0 before deciding their operating leverages at $t = 0$, which precludes having dispersed information about L_0 when monopolists make their production decisions at $t = 0$. To circumvent this problem while keeping the global games treatment of the production game identical to the baseline model, we assume that all monopolists perfectly observe L_t at the beginning of each period t. However, in each period, the realization of the shock is altered to be $L_t + \mathcal{L}_t$, where $\mathcal{L}_t$ is normally distributed with mean 0 and variance σ^2. Furthermore, in each period, monopolist q also observes a private noisy signal of $\mathcal{L}_t$, $l_{q,t} = \mathcal{L}_t + \xi_{q,t}$, where $\xi_{q,t}$ is normally distributed with mean 0 and variance σ_ξ^2. Importantly, at $t = 0$, monopolist q observes $l_{q,0}$ after its operating leverage decision but before its production decision. This design has the appealing property that we can take the limit to 0 of both σ_ξ^2 and σ^2 in a way that (i) maintains equilibrium uniqueness; (ii) leaves L_t as the only shock at time t (since $\sigma^2 \to 0$, $\mathcal{L}_t$ is 0 almost surely); and (iii) has monopolists producing at t if and only $L_t \geq L^T(s)$, where $L^T(s)$ is the same threshold as in the baseline model. Fig. OA1 illustrates the timing of events.

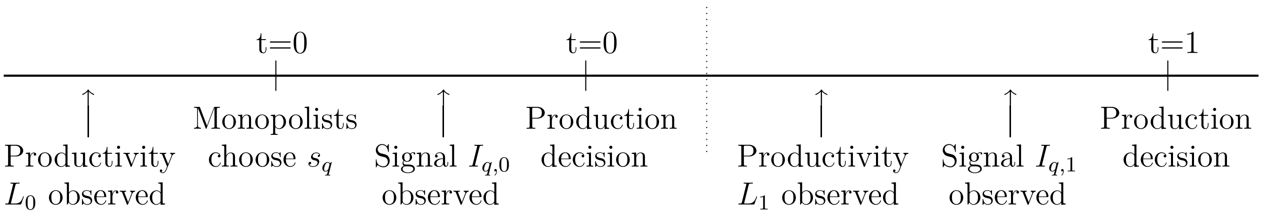

Fig. OA1. Timing of events.

Proposition OA.1 below formally states the result that the global games treatment that we consider in this section delivers the same prediction for the production game as the prediction of the baseline model. (For the proofs of Propositions OA.1 and OA.2 below, see at the end of this appendix.)

Proposition OA.1 *Suppose all monopolists choose operating leverage s at $t = 0$, then if $\sigma \to 0$ and $\frac{\sigma_\xi}{\sigma^2} \to 0$, monopolists operate in period t if and only if $L_t \geq L^T(s)$, where $L^T(s)$ is defined as in Eq. (13).*

Next, we build on Proposition OA.1 to analyze the equilibrium operating leverage at $t = 0$. Suppose monopolist q chooses s_q while all other monopolists choose $s^* \in (0, \alpha)$. If $s_q = s^*$ and $L = L^T(s^*)$, a monopolist q makes a strictly positive profit at $t = 1$ if all other monopolist operate and makes a strictly negative profit if no other monopolist operates. This implies that we can define an interval $[s^* - \tau, s^* + \tau]$ for some $\tau > 0$ such that if $s_q \in [s^* - \tau, s^* + \tau]$ monopolist q produces if and only if $L \geq L^T(s^*)$. Then, s^* being an equilibrium requires that

$$s^* \in \underset{s_q \in [s^*-\tau, s^*+\tau]}{\operatorname{argmax}} \left\{ \begin{array}{c} \mathbf{1}_{\{L_0 \geq L^T(s^*)\}} \left[(1-\alpha-\beta+s_q)\, y(s^*, L_0) - F(s_q) \right] + \\ +\theta \int_{L^T(s^*)}^{\overline{L}} (1-\alpha-\beta+s_q)\, y(s^*, L_1) - F(s_q)\; dL_1 \end{array} \right\}, \tag{0A.2}$$

where $\mathbf{1}_{\{L_0 \geq L^T(s^*)\}}$ is an indicator function that takes a value of one when $L_0 \geq L^T(s^*)$. From Eq. (0A.2), the local necessary first order condition for an interior solution writes

$$\mathbf{1}_{\{L_0 \geq L^T(s^*)\}} \left[y\left(s^*, L_0\right) - F'\left(s^*\right) \right] + \theta \Pr\left(L_1 \geq L^T(s^*) \right) \left[E\left[y | L_1 \geq L^T(s^*) \right] - F'\left(s^*\right) \right] = 0. \tag{0A.3}$$

The next proposition shows that around this local equilibrium condition, there is excessive operating leverage. The logic is identical to the one previously discussed for Proposition 3 and Proposition 10: at the equilibrium leverage s^*, a marginal decrease in the leverage of all monopolists lead them to switch from inactivity to production just below $L^T(s^*)$ (i.e., $\frac{\partial L^T(s^*)}{\partial s} > 0$) which improves welfare.

Proposition OA.2 *Consider a generic increasing and convex function $F(.)$ any symmetric interior equilibrium s^* at $t = 1$ satisfies condition in Eq. (0A.3), and a collective marginal decrease in operating leverage below s^* increases welfare.*

Proofs

Proof of Proposition OA.1

Suppose all monopolists choose s at $t = 0$ and monopolist q with signal $l_{q,t}$ produces if and only if $l_{q,t} > l^*$ at time t, for $t = 1, 2$. Consider a monopolist who observes the signal l^*. Its posterior of $\mathcal{L}_t$ is normally distributed with mean $\frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^*$ and variance $\frac{\sigma^2\sigma_\xi^2}{\sigma^2+\sigma_\xi^2}$. Finally, for a given realization of $\mathcal{L}_t$, the mass of monopolists that produce at time t is

$$n(\mathcal{L}_t) \equiv 1 - \Phi\left(\frac{l^* - \mathcal{L}_t}{\sigma_\xi}\right) \tag{0A.4}$$

where $\Phi(.)$ is the cumulative distribution function of the normal distribution with mean 0 and variance 1. Since a monopolist with signal l^* must be indifferent between producing and not producing:

$$\int_{-\infty}^{+\infty} \left\{(1-\alpha-\beta+s)\frac{\mathcal{L}_t - n(\mathcal{L}_t)F}{1-n(\mathcal{L}_t)(1-\alpha+s)} - F\right\} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2\sigma_\xi^2}}\,\varphi\left(\frac{\mathcal{L}_t - \frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^*}{\sqrt{\frac{\sigma^2\sigma_\xi^2}{\sigma^2+\sigma_\xi^2}}}\right) d\mathcal{L}_t = 0. \tag{0A.5}$$

where $\varphi(.)$ is the probability density function of the normal distribution with mean 0 and variance 1. Consider the following change of variable:

$$x_t = \frac{\mathcal{L}_t - l^*}{\sigma_\xi} + \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^* \Leftrightarrow \mathcal{L}_t = x_t\sigma_\xi + \frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^* \tag{0A.6}$$

then Eq. (0A.5) becomes

$$(1-\alpha-\beta+s)\int_{-\infty}^{+\infty}\frac{x_t\sigma_\xi+\frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^*-\Phi(x_t-\frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)F}{1-\Phi(x_t-\frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)(1-\alpha+s)}\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right)dx_t=F. \tag{0A.7}$$

where is x_t is normally distributed with mean 0 and variance $\frac{\sigma^2}{\sigma^2+\sigma_\xi^2}$.

Note

$$\int_0^{+\infty}\frac{x_t}{1-\Phi(x_t-\frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)(1-\alpha+s)}\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right)dx_t \tag{0A.8}$$

$$\leq\int_0^{+\infty}\frac{x_t}{\alpha-s}\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right)dx_t. \tag{0A.9}$$

Let X be normally distributed with mean 0 and variance 1. The RHS of Eq. (0A.9) tends to $\frac{2\mathbb{E}(X|X>0)}{\alpha-s}$ when $\sigma_\xi\to 0$. It follows that

$$\lim_{\sigma_\xi\to 0}\sigma_\xi\int_0^{+\infty}\frac{x_t}{\alpha-s}\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right)dx_t=0 \tag{0A.10}$$

and therefore, for any $b>0$, there exists $\hat{\sigma}_+$ such that, for any l^* and any $\sigma_\xi<\hat{\sigma}_+$,

$$0<\int_0^{+\infty}\frac{x_t\sigma_\xi}{1-\Phi(x_t-\frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)(1-\alpha+s)}\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right)dx_t<b \tag{0A.11}$$

Using

$$\int_{-\infty}^{0} \frac{x_t}{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^*)(1-\alpha+s)} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x_t\right) dx_t \tag{0A.12}$$

$$\geq \int_{0}^{+\infty} \frac{x_t}{\alpha - s} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x\right) dx_t \tag{0A.13}$$

One similarly shows that, for any $b > 0$, there exists $\hat{\sigma}_-$ such that, for any l^* and any $\sigma_\xi < \hat{\sigma}_-$,

$$-b < \int_{-\infty}^{0} \frac{x_t \sigma_\xi}{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^*)(1-\alpha+s)} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x_t\right) dx_t < 0. \tag{0A.14}$$

Combining Eq. (0A.11) and Eq. (0A.14), there exists $\hat{\sigma}$ such that, for any function l^* and any $\sigma_\xi < \hat{\sigma}$,

$$-b < \int_{-\infty}^{+\infty} \frac{x_t \sigma_\xi}{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^*)(1-\alpha+s)} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x_t\right) dx_t < b \tag{0A.15}$$

Suppose $\frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^* > F$, then if σ_ξ is sufficiently small,

$$\int_{-\infty}^{+\infty} \frac{x_t \sigma_\xi + \frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^* - \Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^*) F}{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2} l^*)(1-\alpha+s)} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x_t\right) dx_t \tag{0A.16}$$

$$> \int_{-\infty}^{+\infty} \left(\frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^* - F\right) \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} \varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}} x_t\right) dx_t - b \tag{0A.17}$$

$$= \frac{\sigma^2}{\sigma^2+\sigma_\xi^2} l^* - F - b \tag{0A.18}$$

It follows that, if $l^* > \max\left\{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}F,\ \frac{\sigma^2+\sigma_\xi^2}{\sigma^2}(F+b+\frac{F}{1-\alpha-\beta+s})\right\}$, the LHS of Eq. (0A.7) is strictly greater than F.

Conversely, suppose $l^* < \frac{\sigma^2+\sigma_\xi^2}{\sigma^2}F$, then if σ_ξ is sufficiently small,

$$\int_{-\infty}^{+\infty} \frac{x_t\sigma_\xi + \frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^* - \Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)F}{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)(1-\alpha+s)} \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right) dx_t \tag{0A.19}$$

$$< \int_{-\infty}^{+\infty} \left(\frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^* - F\right) \sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}\varphi\left(\sqrt{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}}x_t\right) dx_t + b \tag{0A.20}$$

$$= \frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^* - F + b \tag{0A.21}$$

It follows that if $l^* < \min\left\{\frac{\sigma^2+\sigma_\xi^2}{\sigma^2}F,\ \frac{\sigma^2+\sigma_\xi^2}{\sigma^2}(F-b+\frac{F}{1-\alpha-\beta+s})\right\}$, then the LHS of Eq. (0A.7) is strictly smaller than F.

Taken together these two inequalities imply that Eq. (0A.7) has at least one solution, and that there exists an interval $[\underline{l}, \bar{l}]$ such that for any σ_ξ small enough, all solutions to Eq. (0A.7) belong to $[\underline{l}, \bar{l}]$.

Consider next

$$\frac{\overbrace{\frac{\sigma^2}{\sigma^2+\sigma_\xi^2}l^* - \Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)F}^{\equiv N}}{\underbrace{1-\Phi(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*)(1-\alpha+s)}_{\equiv D}} \tag{0A.22}$$

The derivative of Eq. (0A.22) with respect to l^* has the sign of

$$\left[1 + \frac{\sigma_\xi}{\sigma^2}\varphi\left(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*\right)F\right] D - (1-\alpha+s)\frac{\sigma_\xi}{\sigma^2}\varphi\left(x_t - \frac{\sigma_\xi}{\sigma^2+\sigma_\xi^2}l^*\right)N. \tag{0A.23}$$

Note that, for any l^*, $D > \alpha - s$. Furthermore, $\varphi(.)$ is bounded and so is N for any $l^* \in [\underline{l}, \bar{l}]$. It follows that Eq. (0A.23) is strictly positive if $\frac{\sigma_\xi}{\sigma^2}$ is small enough. This together with Eq. (0A.16) and Eq. (0A.19) imply that the LHS of Eq. (0A.7) is strictly increasing on $l^* \in [\underline{l}, \bar{l}]$ if $\frac{\sigma_\xi}{\sigma^2}$ is small enough. Therefore, since (i) Eq. (0A.7) has at least one solution; (ii) there are no solutions to Eq. (0A.7) outside $[\underline{l}, \bar{l}]$ if σ_ξ small enough; and (iii) the LHS of Eq. (0A.7) is strictly increasing on $l^* \in [\underline{l}, \bar{l}]$ if $\frac{\sigma_\xi}{\sigma^2}$ is small, then, there is a unique solution to Eq. (0A.7) if $\frac{\sigma_\xi}{\sigma^2}$ small enough.

Assume from now on that $\frac{\sigma_\xi}{\sigma^2}$ is small enough. We show next that $\lim_{\sigma_\xi \to 0} \sigma_\xi l^* = 0$. Suppose it was not true, then either $l^* \to +\infty$ and from Eq. (0A.16), the LHS of Eq. (0A.7) tends to $+\infty$, which implies Eq. (0A.7) cannot hold; or $l^* \to -\infty$ and from Eq. (0A.19), the LHS of Eq. (0A.7) tends to $-\infty$, which implies Eq. (0A.7) cannot hold.

Suppose now that $\sigma_\xi \to 0$. Then Eq. (0A.7) has a unique solution defined by

$$(1 - \alpha - \beta + s) \int_{-\infty}^{+\infty} \frac{l^* - \Phi(x_t)F}{1 - \Phi(x_t)(1 - \alpha + s)} \varphi(x_t)\, dx_t = F. \tag{0A.24}$$

Finally, consider the change of variable $z = \Phi(x_t)$, Eq. (0A.24) becomes

$$(1 - \alpha - \beta + s) \int_0^1 \frac{l^* - zF}{1 - z(1 - \alpha + s)} dz = F \tag{0A.25}$$

From Eq. (0A.25), the proof of Proposition 1 and defining $L^T(s)$ as the limit of l^* when $\sigma_\xi \to 0$:

$$L^T(s) = \frac{F}{1 - \alpha + s} + \frac{\beta F}{(1 - \alpha - \beta + s) \ln\left(\frac{1}{\alpha - s}\right)}. \tag{0A.26}$$

Q.E.D.

Proof of Proposition OA.2

Part I: Local Necessary Equilibrium Condition.

Note first, that the proof of Proposition OA.1 does not make use of the specific functional form for $F(s_q)$ and therefore, the operating threshold $L^T(s)$ in Proposition OA.1 holds for a generic function $F(s_q)$. Suppose monopolist q chooses s_q while all other monopolists choose $s^* \in (0, \alpha)$. If $s_q = s^*$ and $L = L^T(s^*)$, monopolist q makes a strictly positive profit at $t = 1$ if all other monopolist operate and makes a strictly negative profit if no other monopolist operates. This implies that we can define an interval $[s^* - \tau, s^* + \tau]$ for some $\tau > 0$ such that if $s_q \in [s^* - \tau, s^* + \tau]$, monopolist q produces if and only if $L \geq L^T(s^*)$. Then, s^* being an equilibrium requires that

$$s^* \in \underset{s_q \in [s^*-\tau, s^*+\tau]}{\operatorname{argmax}} \mathbf{1}_{\{L_0 \geq L^T(s^*)\}} \left((1 - \alpha - \beta + s_q)\, y(s^*, L_0) - F(s_q)\right) + \\ + \theta \int_{L^T(s^*)}^{\overline{L}} (1 - \alpha - \beta + s_q)\, y(s^*, L_1) - F(s_q)\; dL_1, \tag{0A.27}$$

where $\mathbf{1}_{\{L_0 \geq L^T(s^*)\}}$ is an indicator function that takes the value of one when $L_0 \geq L^T(s^*)$. If $L_0 \geq L^T(s^*)$, the local necessary condition for an interior solution to the monopolist optimization problem in Eq. (0A.27) writes:

$$y\left(s^*, L_0\right) - F'\left(s^*\right) + \theta \Pr\left(L_1 \geq L^T(s^*)\right) \left[E\left[y | L_1 \geq L^T(s^*)\right] - F'\left(s^*\right)\right] = 0. \tag{0A.28}$$

If $L_0 < L^T(s^*)$, the local necessary condition for an interior solution to the monopolist optimization problem in Eq. (0A.27) writes:

$$E\left[y|L_1 \geq L^T(s^*)\right] - F'\left(s^*\right) = 0. \tag{0A.29}$$

Part II: Excessive leverage.

Before analyzing the social planner's optimization problem, we show that the operating threshold, $L^T(s^*)$, increases with operating leverage (i.e., $\frac{\partial L^T(s^*)}{\partial s} > 0$) a result that we will later use in the proof.

Lemma 0A.1 $\frac{\partial L^T(s^*)}{\partial s} > 0.$

Proof of Lemma 0A.1.

Consider first the case in which $L_0 \geq L^T(s^*)$. The local necessary equilibrium condition in Eq. (0A.28) writes

$$y\left(L_0, s^*\right) - F'\left(s^*\right) + \theta \frac{\overline{L} - L^T(s^*)}{\overline{L}} \left[\frac{y(L^T(s^*), s^*) + y(\overline{L}, s^*)}{2} - F'\left(s^*\right)\right] = 0. \tag{0A.30}$$

Since $y(L, s^*)$ is increasing in L, and $L^T(s^*) < \overline{L}$, then it follows that

$$y(L^T(s^*), s^*) < F'(s^*), \tag{0A.31}$$

as otherwise, the left-hand-side in Eq. (0A.30) would be greater than zero.

$L_1^T(s^*)$ is implicitly given by the indifference condition

$$\int_0^1 (1-\alpha-\beta+s^*)\hat{y}(L^T(s^*), s^*, z) - F(s^*)\ dz = 0 \tag{0A.32}$$

where

$$\hat{y}(L, s, z) \equiv \frac{L - zF(s)}{1 - z(1-\alpha+s)}. \tag{0A.33}$$

$\frac{\partial \hat{y}(L,s,z)}{\partial z}(L^T(s^*), s^*, .)$ has the sign of $(1-\alpha+s^*)L^T(s^*) - F(s^*)$ which, from Eq. (13), is strictly positive. It follows that for any $z \in (0,1)$,

$$\hat{y}(L^T(s^*), s^*, z) < \hat{y}(L^T(s^*), s^*, 1) = y(L^T(s^*), s^*) < F'(s^*) \tag{0A.34}$$

where the last inequality just repeats Eq. (0A.31).

Next, for any $z \in (0,1)$, $\frac{\partial \hat{y}(L,s,z)}{\partial s}(L^T(s^*), s^*, .)$ has the sign of $y(L^T(s^*), s^*, z) - F'(s^*)$ which, from Eq. (0A.34), is strictly negative.

Finally, total differentiation of the indifference condition in Eq. (0A.32) with respect to s^* yields

$$\begin{aligned}
&\int_0^1 \hat{y}(L^T(s^*), s^*, z) - F'(s^*)\ dz \\
&+ \int_0^1 (1-\alpha-\beta+s^*)\frac{\partial \hat{y}(L,s,z)}{\partial s}(L^T(s^*), s^*, .)\ dz \\
&+ \int_0^1 (1-\alpha-\beta+s^*)\frac{\partial \hat{y}(L,s,z)}{\partial L}(L^T(s^*), s^*, .)\frac{\partial L^T(s^*)}{\partial s^*}\ dz = 0.
\end{aligned} \tag{0A.35}$$

Eq. (0A.34) implies that the first line of this equation is strictly negative. We have shown

$\frac{\partial \hat{y}(L,s,z)}{\partial s}(L^T(s^*), s^*, .) < 0$ for any $z \in (0,1)$ which implies that the second line is also negative. Finally $\frac{\partial \hat{y}(L,s,z)}{\partial L}(L^T(s^*), s^*, .) > 0$ for any $z \in (0,1)$. Therefore $\frac{\partial L^T(s^*)}{\partial s^*} > 0$.

Consider now the case in which $L_0 < L^T(s^*)$. The local necessary equilibrium condition in Eq. (0A.29) writes

$$\frac{\overline{L} - L^T(s^*)}{\overline{L}} \left[\frac{y(L^T(s^*), s^*) + y(\overline{L}, s^*)}{2} - F'(s^*) \right] = 0. \tag{0A.36}$$

Since $y(L, s^*)$ is increasing in L, and $L^T(s^*) < \overline{L}$, then it follows that

$$y(L^T(s^*), s^*) < F'(s^*), \tag{0A.37}$$

$L^T(s^*)$ is implicitly given by the indifference condition

$$\int_0^1 (1 - \alpha - \beta + s^*)\hat{y}(L^T(s^*), s^*, z) - F(s^*)\; dz = 0 \tag{0A.38}$$

and identical derivations to the ones in the first case, shows that $\frac{\partial L^T(s^*)}{\partial s^*} > 0$. This completes the proof of Lemma 0A.1. *Q.E.D.*

The social planner's objective function for a given L_0 writes

$$\begin{aligned} W(s, L_0) =& \mathbf{1}_{\{L_0 \geq L^T(s)\}} \left((1 - \alpha - \beta + s)\frac{L_0 - F(s)}{\alpha - s} - F(s) \right) + \\ &+ \theta \int_{\min\{L^T(s), \overline{L}\})}^{\overline{L}} (1 - \alpha - \beta + s)\frac{L_1 - F(s)}{\alpha - s} - F(s)\; dL_1, \end{aligned} \tag{0A.39}$$

and hence, the optimization problem writes

$$s^{Opt} \in \arg\max_{S} W(s, L_0). \tag{0A.40}$$

Since at the optimum $L^T(s) < \overline{L}$, the first derivative writes:

- For $L_0 > L^T(s)$ and for $L_0 = L^T(s)$ when $L^T(s) \to L_0^+$:

$$\begin{aligned}
&\frac{L_0 - F(s)}{\alpha - s} - F'(s) + (1-\alpha-\beta+s)\frac{\partial}{\partial s}\left(\frac{L_0 - F(s)}{\alpha - s}\right) \\
&+ \theta \int_{L^T(s)}^{\overline{L}} \frac{L_1 - F(s)}{\alpha - s} - F'(s)\ dL_1 + \theta(1-\alpha-\beta+s)\int_{L^T(s)}^{\overline{L}} \frac{\partial}{\partial s}\left(\frac{L_1 - F(s)}{\alpha - s}\right)\ dL_1 \\
&- \theta\left[(1-\alpha-\beta+s)\frac{L^T(s) - F(s)}{\alpha - s} - F(s)\right]\frac{\partial L^T(s)}{\partial s}.
\end{aligned} \tag{0A.41}$$

- For $L_0 < L^T(s)$ and for $L_0 = L^T(s)$ when $L^T(s) \to L_0^-$:

$$\begin{aligned}
&\theta \int_{L^T(s)}^{\overline{L}} \frac{L_1 - F(s)}{\alpha - s} - F'(s)\ dL_1 + \theta(1-\alpha-\beta+s)\int_{L^T(s)}^{\overline{L}} \frac{\partial}{\partial s}\left(\frac{L_1 - F(s)}{\alpha - s}\right)\ dL_1 \\
&- \theta\left[(1-\alpha-\beta+s)\frac{L^T(s) - F(s)}{\alpha - s} - F(s)\right]\frac{\partial L^T(s)}{\partial s}.
\end{aligned} \tag{0A.42}$$

Case 1: $L_0 > L^T(s^*)$. We can write Eq. (0A.41) as

$$\begin{aligned}&\frac{1-\beta}{\alpha-s}\left[\frac{L_0-F(s)}{\alpha-s}-F'(s)+\theta\int_{L^T(s)}^{\overline{L}}\frac{L_1-F(s)}{\alpha-s}-F'(s)\;dL_1\right]\\&-\theta\left[(1-\alpha-\beta+s)\frac{L^T(s)-F(s)}{\alpha-s}-F(s)\right]\frac{\partial L^T(s)}{\partial s}.\end{aligned}\tag{0A.43}$$

Using Eq. (0A.28), Eq. (0A.43) evaluated at s^* writes:

$$-\theta\left[(1-\alpha-\beta+s^*)\frac{L^T(s^*)-F(s^*)}{\alpha-s^*}-F(s^*)\right]\frac{\partial L^T(s^*)}{\partial s^*}\tag{0A.44}$$

Since at the operating threshold (i.e., at $L=L^T(s)$) firms make a strictly positive profit,

$$(1-\alpha-\beta+s)\frac{L^T(s)-F(s)}{\alpha-s}-F(s)>0\tag{0A.45}$$

for any s, and using Lemma 0A.1 (i.e., $\frac{\partial L^T(s^*)}{\partial s}>0$), it follows that the first derivative in Eq. (0A.41) is strictly negative at $s=s^*$, that is, a marginal decrease in s around s^* increases welfare.

Case 2: $L_0<L^T(s^*)$. We can write Eq. (0A.42) as

$$\begin{aligned}&\theta\frac{1-\beta}{\alpha-s}\int_{L^T(s)}^{\overline{L}}\frac{L_1-F(s)}{\alpha-s}-F'(s)\;dL_1\\&-\theta\left[(1-\alpha-\beta+s)\frac{L^T(s)-F(s)}{\alpha-s}-F(s)\right]\frac{\partial L^T(s)}{\partial s}.\end{aligned}\tag{0A.46}$$

Using Eq. (0A.29), Eq. (0A.46) evaluated at s^* writes,

$$-\theta\left[(1-\alpha-\beta+s^*)\frac{L^T(s^*)-F(s^*)}{\alpha-s^*}-F(s^*)\right]\frac{\partial L^T(s^*)}{\partial s^*} \tag{0A.47}$$

for any s, and using Lemma 0A.1 (i.e., $\frac{\partial L^T(s^*)}{\partial s} > 0$), it follows that the first derivative in Eq. (0A.42) is strictly negative at $s = s^*$, that is, a marginal decrease in s around s^* increases welfare.

Case 3: $L_0 = L^T(s^*)$. The social welfare function is not continuos at $L_0 = L^T(s^*)$ as an increase in s beyond s^* leads monopolists not to produce at $t = 0$. Consider first the case in which $L_0 \to L^T(s)^+$. Using Eq. (0A.28), Eq. (0A.41) evaluated at s^* writes as in Eq. (0A.44):

$$-\theta\left[(1-\alpha-\beta+s^*)\frac{L^T(s^*)-F(s^*)}{\alpha-s^*}-F(s^*)\right]\frac{\partial L^T(s^*)}{\partial s} \tag{0A.48}$$

Since at the operating threshold (i.e., at $L = L^T(s)$) firms make a strictly positive profit,

$$(1-\alpha-\beta+s)\frac{L^T(s)-F(s)}{\alpha-s}-F(s) > 0 \tag{0A.49}$$

for any s, and using Lemma 0A.1 (i.e., $\frac{\partial L^T(s^*)}{\partial s} > 0$), it follows the first derivative for $L_0 = L^T(s^*)$ when $L^T(s) \to L^T(s^*)^+$ (that is, when $s \to s^{*-}$ since $\frac{\partial L^T(s^*)}{\partial s} > 0$) is negative, that is, a marginal decrease in s around s^* from below increases welfare.

Consider now the case in which $L_0 \to L^T(s)^-$ (that is, in which $s \to s^{*+}$). Since $L^T(s^*)$ is increasing in s^* (Lemma 0A.1) and since monopolists make strictly positive profits at the threshold, $L = L^T(s^*)$, it follows there is a discrete increase in the welfare function at s^* when s approaches s^* from above and monopolists start producing at $t = 0$, that is,

$$\lim_{s \to s^{*+}} W(s, L^T(s)) < W(s^*, L^T(s^*)). \tag{0A.50}$$

Therefore, an increase in operating leverage beyond s^* decreases welfare in an interval $(s^*, s^* + \delta)$ for some $\delta > 0$. In summary, at $L_0 = L^T(s^*)$, a decrease in s^* increases welfare and an increase in s^* decreases welfare. *Q.E.D.*