Internet Appendix for "The Information Sensitivity of Debt in Good and Bad Times."

Emanuele Brancati Marco Macchiavelli

Bayesian learning and forecast errors

Here we show that under Bayesian learning, current expectations are affected by past forecast errors; in the following derivation, we closely follow Bullard and Suda (2008). Suppose that the true fundamental, θ, follows an AR(1) process:

$$\theta_t = a + b\theta_{t-1} + u_t, \tag{1}$$

where a and b are unknown parameters, and $u_t \sim N(0, \nu^2)$. A Bayesian learner has priors on the parameters of Eq. (1): $\phi_0' = (a_0\ b_0) \sim N(\mu_0, \Omega_0)$. In her mind, the conditional distribution of θ_t given all the information known in the period before is

$$\theta_t \mid \Theta_{t-1}, \phi_{t-1} \sim N(a_{t-1} + b_{t-1}\theta_{t-1}, \nu^2), \tag{2}$$

where Θ_t is the history of θ_s up to period t. By Bayes' rule,

$$f(\phi \mid \Theta_t) \propto f(\Theta_t \mid \phi) f(\phi) \propto f(\theta_t \mid \phi, \Theta_{t-1}) f(\theta_{t-1} \mid \phi, \Theta_{t-2}) \ . \ . \ . \ f(\theta_1 \mid \phi) f(\phi). \tag{3}$$

Define $z_t = (1\ \theta_{t-1})'$ and Z_t being the history of z_s up to period t. Then, $f(\phi \mid \Theta_t) = N(\mu_t, \Omega_t)$, where $\mu_t = \Omega_t \left(\Omega_0^{-1}\phi_0 + \nu^{-2}(Z_t'\Theta_t)\right)$ and $\Omega_t = \left(\Omega_0^{-1} + \nu^{-2}(Z_t'Z_t)\right)^{-1}$. In recursive form, $\Omega_t^{-1} = \Omega_{t-1}^{-1} + \nu^{-2} z_t z_t'$ and $\mu_t = \mu_{t-1} + \Omega_t \nu^{-2} z_t(\theta_t - z_t'\mu_{t-1})$. From Eq. 1, it follows that

$$E_t\theta_{t+1} = z_{t+1}'\mu_t = z_{t+1}'\mu_{t-1} + z_{t+1}'\Omega_t\nu^{-2} z_t(\theta_t - z_t'\mu_{t-1}), \tag{4}$$

where $\theta_t - z_t'\mu_{t-1}$ is the forecast error in the last period. We can also write Eq. 4 as a weighted sum of all the past forecast errors:

$$E_t\theta_{t+1} = z_{t+1}' \sum_{j=0}^{\infty} \Omega_{t-j}\nu^{-2} z_{t-j}(\theta_{t-j} - z_{t-j}'\mu_{t-j-1}). \tag{5}$$

Therefore, today's forecast $E_t\theta_{t+1}$ is a weighted sum of past forecast errors. We would obtain essentially the same expression for the case of recursive learning (Evans and Honkapohja, 2001). Finally, we take a linear approximation of Eq. 5 around the unbiased stochastic steady state[1] to obtain

$$dE_t\theta_{t+1} \approx \sum_{j=0}^{\infty} \bar{c}_{-j} dfe_{t-j} + \sum_{j=0}^{\infty} dc_{t-j}\bar{fe}, \tag{6}$$

where $fe_{t-j} \equiv (\theta_{t-j} - z_{t-j}'\mu_{t-j-1})$, $c_{t-j} \equiv z_{t+1}'\Omega_{t-j}\nu^{-2}z_{t-j}$ and the upper bar denotes a variable at the steady state. Since on average forecast errors are zero, *i.e.* $\bar{fe} = 0$, Eq. 6 simplifies to

$$dE_t\theta_{t+1} \approx \sum_{j=0}^{\infty} \bar{c}_{-j} dfe_{t-j}, \tag{7}$$

which is linear in the forecast errors.

Unknown variance of the error term

In what follows we show that, when the variance of the error term is also unknown, the expected variance can be written recursively; in other words, past expectations of the variance are correlated with its current expectations. Once we assume that past expectations of the error term variance do not directly affect CDS spreads, we have that the previous-period expectation of the error term variance is a valid instrument for its current expectation.

Going back to the previous setup, instead of assuming that $u_t \sim N(0, \nu^2)$, where ν is known, we now suppose that the prior of ν^{-2} follows a Gamma distribution, $\nu^{-2} \sim \Gamma(N, \tau)$; according to the priors, the expected value and the variance of ν^{-2} are N and $2N/\tau^2$ respectively. Hamilton (1994) provides two useful results:[2] first, the bayesian estimate of the coefficient vector is identical to the estimate obtained for the case of known variance of the error term; second, the

[1] By unbiased we mean that forecast errors are on average zero and the notion of a stochastic steady state is required for the sequence of variance-covariance matrices $\{\Omega_{t-j}\}$ not to be degenerate at the steady state, which would have been the case at a non-stochastic steady state.

[2] See Hamilton (1994), Proposition 12.3 on page 356.

time t expected variance of the error term is

$$\begin{aligned} E(\nu^2 \mid Z_t) &= \tau_t^*/N_t^* \\ &\text{where} \\ N_t^* &= N + t \\ \tau_t^* &= \tau + U_t'U_t + (\beta_t - \mu_0)'\Omega_0^{-1}(Z_t'Z_t + \Omega_0^{-1})^{-1}Z_t'Z_t(\beta_t - \mu_0), \end{aligned} \tag{8}$$

for $U_t = [u_1, u_2, ..., u_t]'$ and $\beta_t = (Z_t'Z_t)^{-1}Z_t\theta_t$, the OLS estimator of the AR(1) coefficients a and b. Following Hamilton (1994) on page 357, if we further assume diffuse or uninformative priors, which is represented by $N = \tau = 0$ and $\Omega_0 = \mathbf{0}$, we obtain that the expected variance of the error term can be written recursively in an additive fashion:

$$\begin{aligned} E(\nu^2 \mid Z_t) &= \tfrac{1}{t}U_t'U_t = \tfrac{1}{t}\textstyle\sum_{i=1}^{t} u_i^2 \\ &= \tfrac{t-1}{t}E(\nu^2 \mid Z_{t-1}) + \tfrac{1}{t}u_t^2. \end{aligned} \tag{9}$$

References

Bullard, J., Suda, J., 2008. The stability of macroeconomic systems with bayesian learners. Federal Reserve Bank of St. Louis Working Paper Series 2008-043.

Evans, G., Honkapohja, S., Learning and Expectations in Macroeconomics, Princeton University Press, 2001.

Hamilton, J.D., Time Series Analysis, Vol. 2, Princeton university press, 1994.