Internet Appendix for “Collateral Constraints and Asset Prices”

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IA.1. Additional results

Proposition IA.1 (Existence of boundaries $\underline{v}$ and $\overline{v}$). There exist constant boundaries $\underline{v}$ and $\overline{v}$ for the state variable v_t process (31) *that solve equations $\Phi_A(\overline{v})$ and $\Phi_A(\underline{v})$.*

Proof of Proposition IA.1. We here show the existence of $\overline{v}$ that solves $\Phi_A(\overline{v}) = 0$, where $\Phi_A(v)$ is given by equation (A26) in the Appendix of the paper. The proof for $\underline{v}$ is analogous.

We note that $\Phi_A(v_t) \geq 0$ because of the constraint $W_{At} \geq 0$. Suppose, $\overline{v}$ does not exist, and hence $\Phi_A(v_t) > 0$ for all v_t. From Eq. (14) for consumption share s we observe that $s(v_t) \to 0$ when $v_t \to +\infty$. For arbitrary $\varepsilon \in (0, l_A)$ choose v_t sufficiently large, so that $s(v_t) - l_A < -\varepsilon$. Let $T(v_t)$ be the stopping time, defined as

$$T(v_t) = \inf\{\tau : s(v_\tau) - l_A \geq -\varepsilon\}. \tag{IA.1}$$

From Eq. (A26) for $\Phi_A(v_t)$ we obtain the following inequality:

$$\begin{aligned}
\Phi_A(v_t)s(v_t)^{1-\gamma_A} &\leq -\varepsilon \mathbb{E}_t^A\left[\sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A}\Delta t\right] \\
&+ \mathbb{E}_t^A\left[\sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A}(s(v_\tau)-\varepsilon)1_{\{s(v_\tau)\geq\varepsilon\}}\Delta t\right] \\
&\leq -\varepsilon(l_A-\varepsilon)^{-\gamma_A}\mathbb{E}_t^A\left[\sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right] \\
&+ \max(1;\varepsilon^{1-\gamma_A})\mathbb{E}_t^A\left[\sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right].
\end{aligned} \tag{IA.2}$$

Next, we show that $T(v_t) \to +\infty$ as $v_t \to +\infty$. Let $\widehat{v}$ be such that $s(\widehat{v}) = l_A - \varepsilon$. Then, because $s(v_t)$ is a decreasing function, the stopping time (IA.1) can be rewritten as $T(v_t) = \inf\{\tau : v_\tau \leq \widehat{v}\}$. We note that $T(v_t) \geq \widehat{T}$, where $\widehat{T}$ is the minimal time required to get from v_t to $\widehat{v}$, which is the time when $\Delta w_t = -\sqrt{\Delta t}$ and $\Delta j_t = 1$ along the path. Time $\widehat{T}$ is found from the condition $v_t + (\widehat{T}/\Delta t)(\mu_v \Delta t - \sigma_v \sqrt{\Delta t} + J_v) = \widehat{v}$, where $J_v < 0$ and $\mu_v \Delta t - \sigma_v \sqrt{\Delta t} < 0$ for small Δt. Hence, $\widehat{T} \to +\infty$ as $v_t \to +\infty$, and hence $T(v_t) \to +\infty$. We also note that $\mathbb{E}_t[\sum_{\tau=t}^{\infty} e^{-r(\tau-t)} D_\tau^{1-\gamma_A} \Delta t] < +\infty$ by condition (15). Therefore, for a sufficiently large v_t we obtain from inequality (IA.2) that $\Phi_A(v_t) < 0$, which contradicts initial assumption that $\Phi_A(v_t) \geq 0$ for all v_t. Hence, there exists $\overline{v}$ such that $\Phi_A(\overline{v}) = 0$. ■

Lemma IA.1 (Unconstrained optimization). Consider an infinitesimal unconstrained investor with risk aversion γ_i and labor income $l_i D_t$, $i = A, B$, who lives in the economy in which the state price density is given by (32). *The investor's value function is given by*

$$V_i^{unc}(W_t, v_t) = \frac{(W_t + l_i/(1 - l_A - l_B)S_t)^{1-\gamma_i}}{1 - \gamma_i} h_i(v_t)^{\gamma_i}, \tag{IA.3}$$

where $h(v_t)$ is a uniformly bounded wealth-consumption ratio, given by:

$$h_i(v_t) = \mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \Big(\frac{\xi_{i\tau}}{\xi_{it}} \Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \Delta t \Big]. \tag{IA.4}$$

The investor's optimal consumption is given by $c_{i\tau}^* = \ell(\xi_{i\tau} e^{\rho(\tau-t)})^{-1/\gamma_i}$, where ℓ is a constant. Moreover, for all feasible consumptions c_t the following inequalities are satisfied:

$$\mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_\tau) \Delta t \Big] \leq \mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_\tau^*) \Delta t \Big] = V_i^{unc}(W_t, v_t), \tag{IA.5}$$

$$\lim_{T\to\infty} \sup e^{-\rho T} \mathbb{E}_t^i \Big[V_i^{unc}(W_T, v_T) \Big] \leq 0. \tag{IA.6}$$

Proof of Lemma IA.1. We solve the problem using the martingale method. The static budget constraint is given by:

$$\mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} c_\tau^* \Big] = W_t + \frac{l_i S_t}{1 - l_A - l_B}, \tag{IA.7}$$

where the last term is the value of the labor income. Because the dividends and labor incomes are collinear, the value of the labor income is given by:

$$\mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} (l_i D_\tau) \Big] = \frac{l_i S_t}{1 - l_A - l_B}.$$

The first order condition gives the optimal consumption $c^*_\tau = \ell(\xi_{i\tau}/\xi_{it} e^{\rho(\tau-t)})^{-1/\gamma_i}$, where ℓ is the Lagrange multiplier that can be found by substituting c^*_τ into (IA.7). Finding the multiplier ℓ and substituting c^*_τ into the objective function, we obtain the value function (IA.3), where $h(v_t)$ is given by (IA.4).

Next, we show that $h(v_t)$ is uniformly bounded. First, we consider the case $\gamma_i \geq 1$. Using Eq. (IA.4) and Hölder's inequality, we obtain:

$$h_i(v_t) = \mathbb{E}^i_t\Big[\sum_{\tau=t}^{\infty}\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Big] \leq \Big(\mathbb{E}^i_t\Big[\sum_{\tau=t}^{\infty}\frac{\xi_{i\tau}D_\tau}{\xi_{it}D_t}\Big]\Big)^{1-1/\gamma_i}\Big(\mathbb{E}^i_t\Big[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]\Big)^{1/\gamma_i}.$$

We note that both multipliers on the right-hand side of the latter inequality are bounded. The first multiplier equals the price-dividend ratio and is bounded by Proposition 2. The second multiplier is bounded due to condition (15) on the model parameters. Consider now the case $\gamma_i \leq 1$. From the FOCs (29) and the fact that $\underline{s} \leq s \leq \overline{s}$, we obtain:

$$\frac{\xi_{iT}}{\xi_{it}} \geq e^{-\rho(T-t)}\Big(\frac{c^*_T}{c^*_t}\Big)^{-\gamma_i} \geq e^{-\rho(T-t)}\Big(\frac{D_T}{D_t}\Big)^{-\gamma_i}\Big(\frac{\overline{s}}{\underline{s}}\Big)^{-\gamma_i}.$$

From the latter inequality it follows that

$$\mathbb{E}^i_t\Big[\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Big] \leq \Big(\frac{\overline{s}}{\underline{s}}\Big)^{1-\gamma_i}\mathbb{E}^i_t\Big[e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]. \tag{IA.8}$$

The inequality (IA.8) and condition (15) imply that the infinite series in (IA.4) converges and function $h_i(v)$ is uniformly bounded. We also observe that $h_i(v) \geq \Delta t > 0$.

Now, we prove inequality (IA.5). We consider feasible consumption streams satisfying condition $W_t + l_i/(1-l_A-l_B)S_t \geq 0$ for all t, which means that investor's aggregate wealth is nonnegative at all times so that investor does not go bankrupt. From the investor's budget constraint and the latter inequality, for all feasible consumptions we obtain:

$$W_t + \frac{l_iS_t}{1-l_A-l_B} \geq \mathbb{E}^i_t\Big[\sum_{\tau=t}^{T}\frac{\xi_{i\tau}}{\xi_{it}}c_\tau\Delta t\Big] + \mathbb{E}^i_t\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T + \frac{l_iS_T}{1-l_A-l_B}\Big)\Big] \geq \mathbb{E}^i_t\Big[\sum_{\tau=t}^{T}\frac{\xi_{i\tau}}{\xi_{it}}c_\tau\Delta t\Big]. \tag{IA.9}$$

Consider the weighting function w_t given by

$$w_\tau = \frac{\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}}{\widehat{h}_{iT}(v_t)}, \quad \text{where } \widehat{h}_{iT}(v_t) = \mathbb{E}^i_t\Big[\sum_{\tau=t}^{T}\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Delta t\Big]. \tag{IA.10}$$

We note that $\mathbb{E}_t^i[\sum_{\tau=t}^T w_\tau \Delta t] = 1$. Using Jensen's inequality and inequality (IA.9), we obtain:

$$\mathbb{E}_t^i\Big[\sum_{\tau=t}^T \frac{e^{-\rho(\tau-t)}c_\tau^{1-\gamma_i}}{1-\gamma_i}\Delta t\Big] = \mathbb{E}_t^i\Big[\sum_{\tau=t}^T \frac{\Big((\xi_{i\tau}/\xi_{it})^{1/\gamma_i} e^{\rho(\tau-t)/\gamma_i} c_\tau\Big)^{1-\gamma_i} w_\tau \Delta t}{1-\gamma_i}\Big]\widehat{h}_{iT}(v_t)$$

$$\le \frac{\Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^T (\xi_{i\tau}/\xi_{it})^{1/\gamma_i} e^{\rho(\tau-t)/\gamma_i} c_\tau w_\tau \Delta t\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t) \tag{IA.11}$$

$$= \frac{\Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^T (\xi_{i\tau}/\xi_{it}) c_\tau \Delta t\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t)^{\gamma_i} \le \frac{\Big(W_t + \frac{l_i S_t}{1-l_A-l_B}\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t)^{\gamma_i}.$$

Taking limit $T \to \infty$ in (IA.11), and noting that $\widehat{h}_{iT}(v_t) \to h_i(v_t)$, we obtain (IA.5).

Finally, we prove inequality (IA.6). Because $c_\tau \ge 0$, from inequality (IA.9), we obtain:

$$\mathbb{E}_t^i\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T + \frac{l_i S_T}{1-l_A-l_B}\Big)\Big] \le W_t + \frac{l_i S_t}{1-l_A-l_B}. \tag{IA.12}$$

Using Jensen's inequality following the same steps as in inequality (IA.11), we obtain:

$$\frac{\mathbb{E}_t^i\Big[\Big(W_T + \frac{l_i S_T}{1-l_A-l_B}\Big)^{1-\gamma_i}\Big]}{1-\gamma_i} \le \frac{\Big(\mathbb{E}_t^i\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T + \frac{l_i S_T}{1-l_A-l_B}\Big)\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}}\Big]\Big)^{\gamma_i}$$

$$\le \frac{\Big(W_t + \frac{l_i S_t}{1-l_A-l_B}\Big)^{1-\gamma_i}}{1-\gamma_i}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}}\Big]\Big)^{\gamma_i}.$$

The above inequality and the boundedness of $h_i(v_t)$ then imply the following inequality:

$$e^{-\rho(\tau-t)}\mathbb{E}_t^i[V_{iT}^{unc}] \le Const \times V_{it}^{unc}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}} e^{-\rho(\tau-t)/\gamma_i}\Big]\Big)^{\gamma_i}. \tag{IA.13}$$

Inequality (IA.13) also holds for $\gamma_i = 1$ if CRRA preferences are replaced with logarithmic preferences. Suppose, $\gamma_i > 1$. Then, inequality (IA.6) is satisfied because $V_i^{unc} < 0$. Suppose, $\gamma_i \le 1$. Then, using inequalities (IA.8), (IA.13), and condition (15), we obtain:

$$e^{-\rho(\tau-t)}\mathbb{E}_t^i[V_{iT}^{unc}] \le Const \times \Big(\mathbb{E}_t^i\Big[e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]\Big)^{\gamma_i} \to 0, \text{ as } T \to \infty. \blacksquare$$

Lemma IA.2. Let $\mathcal{P}(V)$ be a point-wise monotone operator such that for all point-wise bounded functions V_1 and V_2 such that $V_1 \le V_2 \Rightarrow \mathcal{P}(V_1) \le \mathcal{P}(V_2)$. Suppose further there *exist point-wise bounded functions* $\underline{V}$ *and* $\overline{V}$ *such that* $\underline{V} \leq \overline{V}$*,* $\mathcal{P}(\underline{V}) \geq \underline{V}$*, and* $\mathcal{P}(\overline{V}) \leq \overline{V}$*. Then, there exists a point-wise bounded function* V^* *such that:* 1) $\underline{V} \leq V^* \leq \overline{V}$; 2) $V^* \leq \mathcal{P}(V^*)$; 3) $\mathcal{P}^n(\underline{V}) \to V^*$ *point-wise as* $n \to \infty$*.*

Proof of Lemma IA.2. From the monotonicity of the operator $\mathcal{P}(V)$ and the definitions of $\underline{V}$ and $\overline{V}$, we obtain:

$$\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}(\overline{V}) \leq \overline{V}. \tag{IA.14}$$

Applying the operator $\mathcal{P}$ to inequalities (IA.14), and then using the definitions of $\underline{V}$ and $\overline{V}$, we obtain: $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \overline{V}$. Proceeding in the same way n times we obtain $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \ldots \leq \mathcal{P}^n(\underline{V}) \leq \overline{V}$. Consequently, $\mathcal{P}^n(\underline{V})$ is point-wise increasing and bounded, and hence, converges to some function V^* such that $\underline{V} \leq V^* \leq \overline{V}$ and $\mathcal{P}^n(\underline{V}) \leq V^*$. Applying operator to both sides of the latter inequality, we find that $\mathcal{P}^{n+1}(\underline{V}) \leq \mathcal{P}(V^*)$. Taking limit, we find that $V^* \leq \mathcal{P}(V^*)$. ■

Proposition IA.2 (Verification of optimality). Consider an infinitesimal investor i *who lives in an economy in which the state price density is given by Eq.* (32)*. Suppose, this investor maximizes expected discounted utility* (6) *subject to a self-financing budget constraint and the collateral constraint* (9)*. Then, there exists unique bounded value function* V_i^* *satisfying the dynamic programming Eq.* (A4) *and the transversality condition, such that for all feasible consumptions*

$$V_{it}^* \geq \mathbb{E}_t^i\Big[\sum_{\tau=t}^{+\infty} u(c_{i\tau})\Delta t\Big], \tag{IA.15}$$

and, moreover,

$$V_{it}^* = \mathbb{E}_t^i\Big[\sum_{\tau=t}^{+\infty} u(c_{i\tau}^*)\Delta t\Big], \tag{IA.16}$$

for the optimal consumptions given by FOCs (29)*.*

Proof of Proposition IA.2. Consider the following operator:

$$\mathcal{P}_i(V) = \max_{c_t}\Big\{u_i(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i[V_{i,t+\Delta t})]\Big\}, \quad i = A, B \tag{IA.17}$$

where maximization is subject to budget constraint (A5) and collateral constraint (A6).

Consider the following functions:

$$\underline{V}_{it} = \begin{cases} 0, & \gamma_i < 1, \\ \mathbb{E}^i_t\Big[\sum\limits_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_\tau)\Delta t\Big], & \gamma_i \geq 1, \end{cases} \qquad \overline{V}_{it} = \begin{cases} V^{unc}_{it}, & \gamma_i \leq 1, \\ 0, & \gamma_i > 1, \end{cases} \tag{IA.18}$$

where V^{unc}_t is given by (IA.3).

We observe that for $\gamma_i \geq 1$ function $\underline{V}_i$ is bounded due to condition (15) imposed on model parameters. Because $c_t = l_i D_t$ is feasible, we obtain that

$$\mathcal{P}(\underline{V}_i) \geq u_i(l_i D_t) + e^{-\rho\Delta t}\mathbb{E}^i_t\Big[\sum_{\tau=t+\Delta t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_t)\Delta t\Big] = \underline{V}_i.$$

For $\gamma_i < 1$ it is easy to see that $\mathcal{P}(\underline{V}_i) \geq \underline{V}_i$ because $u_i(c) > 0$. Next, we prove that $\mathcal{P}_i(\overline{V}_i) \leq \overline{V}_i$. The latter inequality is straightforward for $\gamma_i > 1$ because $\mathcal{P}_i(0) \leq 0$. Suppose now, $\gamma_i \leq 1$. Consider operator $\widetilde{\mathcal{P}}_i(V_i)$ given by Eq. (IA.17), where the maximization is subject to the budget constraint (A5), but without the collateral constraint (A6). Hence, $\mathcal{P}_i(V_i) \leq \widetilde{\mathcal{P}}_i(V_i)$. By Lemma IA.1, $\overline{V}^{unc}_i$ is the solution of the unconstrained optimization, and hence $\overline{V}_i = \widetilde{\mathcal{P}}_i(\overline{V}_i)$. Therefore, $\mathcal{P}_i(\overline{V}_i) \leq \widetilde{\mathcal{P}}_i(\overline{V}_i) = \overline{V}_i$.

We drop subscript and superscript i for convenience. Consider the sequence $V_{n+1} = \mathcal{P}(V_n)$, with $V_0 = \underline{V}$, where $\underline{V}$ is given in (IA.18). Then, by Lemma IA.2, $V_n \to V^*$ point-wise as $n \to \infty$. Next, we show that V^* is the value function and $\mathcal{P}(V^*) = V^*$. By the definition operator $\mathcal{P}(V)$ in (IA.17), for all feasible consumption streams

$$\begin{aligned} V_{n+1} &\geq u(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t\left[V_n(W_{t+\Delta t}; v_{t+\Delta t})\right] \\ &\geq \mathbb{E}_t\Big[\sum_{\tau=t}^{n\Delta t} e^{-\rho(\tau-t)} u(c_\tau)\Delta t\Big] + e^{-\rho n\Delta t}\mathbb{E}_t[\underline{V}]. \end{aligned} \tag{IA.19}$$

Taking point-wise limit $n \to \infty$ in (IA.19) and taking into account that $\mathbb{E}_t[\underline{V}]$ is point-wise bounded, we obtain inequality (IA.15).

By Lemma IA.2, $V^* \leq \mathcal{P}(V^*)$ and $V^* \leq \overline{V}$, where $\overline{V}$ is given in (IA.18), and hence

$$\begin{aligned} V^*(W_t, v_t) &\leq u(c^*_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t\left[V^*(W_{t+\Delta t}; v_{t+\Delta t})\right] \\ &\leq \mathbb{E}_t\Big[\sum_{\tau=t}^{T} u(c^*_\tau)\Delta t\Big] + e^{-\rho T}\mathbb{E}_t[V^*(W_T, v_T)] \\ &\leq \mathbb{E}_t\Big[\sum_{\tau=t}^{T} u(c^*_\tau)\Delta t\Big] + e^{-\rho T}\mathbb{E}_t[\overline{V}(W_T, v_T)], \end{aligned} \tag{IA.20}$$

where c^* is the optimal consumption that solves optimization in Eq. (IA.17).

We note that $\overline{V} = 0$ for $\gamma > 1$ and $\limsup e^{-\rho T}\mathbb{E}_t[\overline{V}(W_T, v_T)] \leq 0$ as $T \to \infty$ for $\gamma \leq 1$, by Lemma IA.1. Taking the limit $T \to \infty$ in (IA.20) we find that $V^* \leq \mathbb{E}_t\Big[\sum_{\tau=t}^{+\infty} u(c^*_\tau)\Delta t\Big]$, which along with inequality (IA.15) yields (IA.16). Eq. (IA.16) along with inequality (IA.20) also imply that $V^* = \mathcal{P}(V^*)$. Moreover, V^* is point-wise bounded because $\underline{V} \leq V^* \leq \overline{V}$. Then, given the existence of the value function, the optimal consumptions are given by (29). Finally, we show that V^* satisfies the transversality condition. We note that $e^{-\rho(T-t)}\mathbb{E}_t[\underline{V}_T] \leq e^{-\rho(T-t)}\mathbb{E}_t[V^*_T] \leq e^{-\rho(T-t)}\mathbb{E}_t[\overline{V}_T]$. Taking limit $T \to 0$ we find that the upper and lower bound in the latter equation converge to 0, and hence the transversality condition is satisfied for V^*. ■

Proposition IA.3 (Closed-form solutions).

1) *In the limit* $\Delta t \to 0$ *the price-dividend ratio* Ψ *and wealth-consumption ratios* Φ_i *are given by Eq.* (34)-(35), *where function* $\widehat{\Psi}(v;\theta)$ *is given by:*

$$\widehat{\Psi}(v;\theta) = \int_{\underline{v}}^{v} s(y)^\theta \widehat{\psi}(v-y)dy + \frac{\int_{\underline{v}}^{\overline{v}} s(y)^\theta \left[\widehat{\psi}'(\overline{v}-y) - \widehat{\psi}(\overline{v}-y)\right] dy}{1 + H\left(\widehat{\psi}(\overline{v}-\underline{v}) - \int_0^{\overline{v}-\underline{v}} \widehat{\psi}(y)dy\right)} \left(1 - H\int_0^{v-\underline{v}} \widehat{\psi}(y)dy\right), \tag{IA.21}$$

where $s(y)$ *solves Eq.* (14),[1] and $\widehat{\psi}(x)$, H *and some auxiliary variables are given by:*

$$\widehat{\psi}(x) = \frac{2}{\widehat{\sigma}_v^2}\sum_{n=0}^{\infty}\Bigg[\left(\frac{2\lambda_A(1+J_D)^{1-\gamma_A}}{\widehat{\sigma}_v^2}\right)^n \frac{\exp\big((\zeta_+ + \zeta_-)(x+n\widehat{J}_v)/2\big)}{(\zeta_+ - \zeta_-)^{2n+1}n!} \tag{IA.22}$$

$$\times\, Q_n\left(\frac{(\zeta_+ - \zeta_-)(x+n\widehat{J}_v)}{2}\right)\mathbf{1}_{\{x+n\widehat{J}_v\geq 0\}}\Bigg], \tag{IA.23}$$

$$Q_n(x) = \exp(-x)\sum_{m=0}^{n}(2x)^{n-m}\frac{(n+m)!}{m!(n-m)!} - \exp(x)\sum_{m=0}^{n}(-2x)^{n-m}\frac{(n+m)!}{m!(n-m)!}, \tag{IA.24}$$

$$H = \lambda_A + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda_A(1+J_D)^{1-\gamma_A}, \tag{IA.25}$$

[1]Although $s(y)$ is not in closed form, we observe from Eq. (14) that its inverse is given by $s^{-1}(x) = \gamma_B\ln(x) - \gamma_A\ln(1-x)$. The change of variable $x = s(y)$ eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of $s(y)$ because $s(y)$ is intuitive and easily computable from (14).

$$\zeta_\pm = -\frac{\widehat{\mu}_v^A + (1-\gamma_A)\widehat{\sigma}_v\sigma_D \mp \sqrt{(\widehat{\mu}_v^A + (1-\gamma_A)\widehat{\sigma}_v\sigma_D)^2 + 2\widehat{\sigma}_v^2\left(\lambda_A + \rho - (1-\gamma_A)\mu_D^A + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)}}{\widehat{\sigma}_v^2}, \tag{IA.26}$$

$$\widehat{\mu}_v^A = \widehat{\mu}_v + \delta_A\widehat{\sigma}_v, \quad \mu_D^A = \mu_D + \delta_A\sigma_D. \tag{IA.27}$$

2) *Stock return volatility in normal times and the jump size* J_t *are given by:*

$$\sigma_t = \sigma_D + \left(\frac{\widehat{\Psi}'(v_t;-\gamma_A)}{\widehat{\Psi}(v_t;-\gamma_A)} - \frac{\gamma_A(1-s(v_t))}{\gamma_A(1-s(v_t)) + \gamma_B s(v_t)}\right)\widehat{\sigma}_v, \tag{IA.28}$$

$$J_t = \frac{(1+J_D)\widehat{\Psi}\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}; -\gamma_A\Big)s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)^{\gamma_A}}{\widehat{\Psi}(v_t;-\gamma_A)s(v_t)^{\gamma_A}} - 1. \tag{IA.29}$$

Numbers of shares $n^*_{i,st}$ *and leverage* $L_{it} = -b_{it}B_{it}$ *to market price* S_t *ratio are given by:*

$$n^*_{i,st} = \frac{\widehat{\Phi}_i(v_t)\sigma_D + \widehat{\Phi}'_i(v_t)\widehat{\sigma}_v}{\Psi(v_t)\sigma_t}, \quad \frac{L_{it}}{S_t} = n_{i,st} - \frac{\widehat{\Phi}_i(v_t)}{\Psi(v_t)(1-l_A-l_B)}, \tag{IA.30}$$

where $\widehat{\Phi}_A(v_t) = \Phi_A(v_t)s(v_t)$ *and* $\widehat{\Phi}_B(v_t) = \Phi_B(v_t)(1-s(v_t))$.

Proof of Proposition IA.3. 1) First, we solve the differential-difference equation in Lemma 2. We denote $g(x) = \widehat{\Psi}(x+\underline{v};\theta)$ and apply the following changes of variables:

$$\begin{aligned} &x = v - \underline{v}, \quad \tilde{\sigma} = \widehat{\sigma}_v, \quad \tilde{\mu} = \widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v, \quad \tilde{J} = -\widehat{J}_v, \\ &\tilde{\lambda} = \lambda_A(1+J_D)^{1-\gamma_A}, \quad \tilde{\rho} = \lambda_A + \rho - (1-\gamma_A)(\mu_D + \delta_A\sigma_D) + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2. \end{aligned} \tag{IA.31}$$

Eq. (36)-(37) with new variables now become:

$$\frac{\tilde{\sigma}^2}{2}g''(x) + \tilde{\mu}g'(x) - \tilde{\rho}g(x) + \tilde{\lambda}g(\max\{x-\tilde{J},0\}) + s(x+\underline{v})^\theta = 0, \tag{IA.32}$$

$$g'(0) = 0, \quad g(\overline{v}-\underline{v}) - g'(\overline{v}-\underline{v}) = 0. \tag{IA.33}$$

Let $\mathcal{L}[g(x)] = \int_0^\infty e^{-zx}g(x)dx$ be the Laplace transform of $g(x)$, and similarly for other functions. The Laplace transforms of $g'(x)$, $g''(x)$ and $g(\max\{x-\tilde{J},0\})$ are given by:

$$\begin{aligned} \mathcal{L}[g'(x)] &= z\mathcal{L}[g(x)] - g(0), \\ \mathcal{L}[g''(x)] &= z^2\mathcal{L}[g(x)] - zg(0) - g'(0), \\ \mathcal{L}\Big[g(\max\{x-\tilde{J},0\})\Big] &= \int_0^\infty e^{-zx}g(\max\{x-\tilde{J},0\})dx \\ &= \int_0^{\tilde{J}} e^{-zx}g(0)dx + \int_{\tilde{J}}^\infty e^{-zx}g(x-\tilde{J})dx \\ &= \frac{1}{z}(1-e^{-\tilde{J}z})g(0) + e^{-\tilde{J}z}\mathcal{L}[g(x)]. \end{aligned} \tag{IA.34}$$

Applying the transform to Eq. (IA.32), we arrive at the following equation:

$$\frac{\tilde{\sigma}^2}{2}\left(z^2\mathcal{L}\left[g(x)\right] - zg(0) - g'(0)\right) + \tilde{\mu}\left(z\mathcal{L}\left[g(x)\right] - g(0)\right) - \tilde{\rho}\mathcal{L}\left[g(x)\right] + \tilde{\lambda}\left(e^{-\tilde{J}z}\mathcal{L}\left[g(x)\right] + \frac{1}{z}(1-e^{-\tilde{J}z})g(0)\right) + \mathcal{L}\left[s(x+\underline{v})^{\theta}\right] = 0. \tag{IA.35}$$

Applying boundary condition $g'(0) = 0$ and solving for $\mathcal{L}\left[g(x)\right]$, we obtain:

$$\mathcal{L}\left[g(x)\right] = \frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} + g(0)\left(\frac{1}{z} - \frac{\tilde{\rho}-\tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right). \tag{IA.36}$$

Now define a new function $\widehat{\psi}(x)$ through inverse Laplace transform

$$\widehat{\psi}(x) = \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right]. \tag{IA.37}$$

Next, we apply inverse transform to each term in (IA.36). Noting that $\mathcal{L}^{-1}[1/z] = 1$ and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right] &= \int_0^x s(y+\underline{v})^{\theta} \cdot \widehat{\psi}(x-y)dy,\\ \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right] &= \int_0^x \mathbf{1}_{\{y\geq 0\}} \cdot \widehat{\psi}(x-y)dy = \int_0^x \widehat{\psi}(y)dy.\end{aligned} \tag{IA.38}$$

The linearity of the Laplace transform gives the following equation:

$$g(x) = \mathcal{L}^{-1}\left[\mathcal{L}\left[g(x)\right]\right] = \int_0^x s(y+\underline{v})^{\theta} \cdot \widehat{\psi}(x-y)dy + g(0)\left[1 - \left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^x \widehat{\psi}(y)dy\right]. \tag{IA.39}$$

We calculate $g(0)$ below, and then after changing the variable back from x to $v = x + \underline{v}$, substituting in expressions for $\tilde{\rho}$ and $\tilde{\lambda}$ from (IA.31), we obtain (IA.21).

Next, we solve for $\widehat{\psi}(x)$ in closed form. We expand $\mathcal{L}\left[\widehat{\psi}(x)\right]$ as series, and sum up the inverse transforms of each term in the summation to get $\widehat{\psi}(x)$.

$$\begin{aligned}\mathcal{L}\left[\widehat{\psi}(x)\right] &= \frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\\ &= (\tilde{\rho} - \tilde{\mu}z - \tfrac{\tilde{\sigma}^2}{2}z^2)^{-1} \cdot (1 - \frac{\tilde{\lambda}e^{-\tilde{J}z}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2})^{-1}\\ &= \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}.\end{aligned} \tag{IA.40}$$

The above series converges for z such that $|\tilde{\rho} - \tilde{\mu}z - (\tilde{\sigma}^2/2)z^2| > |\tilde{\lambda}\exp(-\tilde{J}z)|$. This holds if the real part of z is sufficiently large, e.g., $\Re(z) > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho} + \tilde{\lambda}}$. The inverse Laplace transform can then be calculated along the line $(\overline{z} - i\infty, \overline{z} + i\infty)$ in the complex domain where $\overline{z} > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho} + \tilde{\lambda}}$, and hence, the inequality for $\Re(z)$ is satisfied.

Let $\zeta_- < \zeta_+$ be roots of $\tilde{\rho} - \tilde{\mu}z - \tilde{\sigma}^2 z^2/2 = 0$, given by (IA.26). We use the following inversion formula for $1/[(z-\zeta_+)(z-\zeta_-)]^{n+1}$ from Gradshteyn and Ryzhik (2007, p. 1117):

$$\mathcal{L}^{-1}\left[\frac{1}{[(z-\zeta_+)(z-\zeta_-)]^{n+1}}\right] = \frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{x^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}x} I_{n+\frac{1}{2}}\left(\frac{\zeta_+ - \zeta_-}{2}x\right). \quad \text{(IA.41)}$$

Function $e^{-n\tilde{J}z}$ in the complex domain corresponds to a shift from x to $x - n\tilde{J}$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}\right] = \tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\mathbf{1}_{x\geq n\tilde{J}} \times\frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x-n\tilde{J})} I_{n+\frac{1}{2}}\left(\frac{(\zeta_+ - \zeta_-)(x-n\tilde{J})}{2}\right). \quad \text{(IA.42)}$$

Consequently, the explicit expression for $\widehat{\psi}(x)$ is given by:

$$\widehat{\psi}(x) = \sum_{n=0}^{\infty}\tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\frac{\mathbf{1}_{\{x\geq n\tilde{J}\}}\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x-n\tilde{J})} I_{n+\frac{1}{2}}\left(\frac{(\zeta_+ - \zeta_-)(x-n\tilde{J})}{2}\right), \quad \text{(IA.43)}$$

where function $I_{n+\frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_- < \zeta_+$ are given by (IA.26) and $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\lambda}$, and $\tilde{J}$ are defined in (IA.31). Bessel function $I_{n+\frac{1}{2}}(\cdot)$ is given by (see Eq. 8.467 in Gradshteyn and Ryzhik (2007)):

$$I_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^z\sum_{m=0}^{n}\frac{(-1)^m(n+m)!}{m!(n-m)!(2z)^m} + (-1)^{n+1}e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}\right]. \quad \text{(IA.44)}$$

Substituting (IA.44) into (IA.43), after minor algebra, we obtain expression (IA.23) for $\widehat{\psi}(x)$. The infinite series (IA.43) has only a finite number of nonzero terms because for a fixed x indicators $1_{\{x\geq n\tilde{J}\}}$ vanish for sufficiently large n, and hence, (IA.43) is well-defined.

To find $g(0)$ in Eq. (IA.39), we first evaluate $\widehat{\psi}(0)$. From the above formula (IA.43), because $\mathbf{1}_{\{0\geq n\tilde{J}\}} = 0$ for all $n > 0$, we obtain

$$\widehat{\psi}(0) = -\frac{2}{\tilde{\sigma}^2}\cdot\frac{e^{\zeta_+\cdot 0} - e^{\zeta_-\cdot 0}}{\zeta_+ - \zeta_-} = 0. \quad \text{(IA.45)}$$

Differentiating (IA.39) and using $\widehat{\psi}(0) = 0$, we find:

$$g'(x) = \int_0^x s(y + \underline{v})^\theta \cdot \widehat{\psi}'(x - y)dy - g(0) \cdot \left(\tilde{\rho} - \tilde{\lambda}\right) \widehat{\psi}(x), \tag{IA.46}$$

We solve for $g(0)$ from the boundary condition $g(\overline{v} - \underline{v}) - g'(\overline{v} - \underline{v}) = 0$ and obtain:

$$g(0) = \frac{\displaystyle\int_0^{\overline{v}-\underline{v}} s(y + \underline{v})^\theta \cdot \left[\widehat{\psi}'(\overline{v} - \underline{v} - y) - \widehat{\psi}(\overline{v} - \underline{v} - y)\right] dy}{1 - \left(\tilde{\rho} - \tilde{\lambda}\right) \displaystyle\int_0^{\overline{v}-\underline{v}} \widehat{\psi}(y)dy + \left(\tilde{\rho} - \tilde{\lambda}\right) \widehat{\psi}(\overline{v} - \underline{v})}. \tag{IA.47}$$

Substituting (IA.47) into (IA.39), we derive Eq. (IA.21) for $\widehat{\Psi}(v; \theta)$.

2) Eq. (IA.28) for σ_t and Eq. (IA.29) for J_t are the same as Eq. (A72) for σ_t and Eq. (A73) in the Appendix of the paper. Next, we find the trading strategies. Eq. (8) for $W_{i,t+\Delta t}$, implies the following expressions for $n^*_{i,st}$ and b^*_{it}:

$$n^*_{i,st} = \sqrt{\frac{\text{var}_t[W_{i,t+\Delta t} - W_{it}|\text{normal}]}{\text{var}_t[\Delta S_t + (1 - l_A - l_B)D_t\Delta t|\text{normal}]}},$$
$$b^*_{it} = \mathbb{E}_t[W_{i,t+\Delta t}|\text{normal}] - n_{it}\mathbb{E}_t[S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t|\text{normal}].$$

Taking limit $\Delta t \to 0$ in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in Eq. (IA.30). ■

Proposition IA.4. Consider two economies with the following restrictions on model parameters: 1) $\lambda_A = \lambda_B = 0$ *and* 2) $\lambda_A = \lambda_B \equiv \lambda > 0$, $\gamma_A = \gamma_B \equiv \gamma > 0$. *Then, the differential-difference equation* (36) *is an ordinary differential equation. Moreover, there exist constants* $C_+ > 0$ *and* $C_- > 0$ *such that the solution of Eq.* (36) *satisfying boundary conditions* (37) *is given by*

$$\widehat{\Psi}(v) = C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \widehat{\Psi}^{unc}(v), \tag{IA.48}$$

where function $\widehat{\Psi}^{unc}(v)$ *corresponds to an unconstrained model, constants* $C_\pm$ *are given by*

$$\begin{aligned} C_+ &= \frac{(1 - \varphi_-)e^{\varphi_- \overline{v}}(\widehat{\Psi}^{unc})'(\underline{v}) - \varphi_- e^{\varphi_- \underline{v}}((\widehat{\Psi}^{unc})(\overline{v}) - (\widehat{\Psi}^{unc})'(\overline{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \overline{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \overline{v} + \varphi_- \underline{v}}}, \\ C_- &= \frac{(\varphi_+ - 1)e^{\varphi_+ \overline{v}}(\widehat{\Psi}^{unc})'(\underline{v}) + \varphi_+ e^{\varphi_+ \underline{v}}((\widehat{\Psi}^{unc})(\overline{v}) - (\widehat{\Psi}^{unc})'(\overline{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_- \overline{v} + \varphi_+ \underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+ \overline{v} + \varphi_- \underline{v}}}, \end{aligned} \tag{IA.49}$$

and $\varphi_\pm$ *are a positive and a negative solutions, respectively, of equation* $h(\varphi) = 0$*, where* $h(\varphi)$ *is a quadratic characteristic polynomial given by:*

$$h(\varphi) = \begin{cases} \rho - (1-\gamma_A)(\mu_D + \delta_A\sigma_D) + \dfrac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 \\[2ex] \quad -(\widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v)\varphi - \dfrac{\widehat{\sigma}_v^2\varphi^2}{2}, \quad \textit{if} \quad \lambda_A = \lambda_B = 0, \\[2ex] \rho - (1-\gamma)(\mu_D + \delta_A\sigma_D) + \dfrac{(1-\gamma)\gamma}{2}\sigma_D^2 + \lambda(1 - (1+J_D)^{1-\gamma}) \\[2ex] \quad -(\widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1-\gamma)\sigma_D\widehat{\sigma}_v)\varphi - \dfrac{\widehat{\sigma}_v^2\varphi^2}{2}, \quad \textit{if} \quad \lambda_i = \lambda, \ \gamma_i = \gamma. \end{cases} \tag{IA.50}$$

Proof of Proposition IA.4. From Eq. (40), we observe that the term involving jump $\widehat{J}_v$ vanishes when $\lambda_A = \lambda_B = 0$, and the jump $\widehat{J}_v$ is zero when $\lambda_A = \lambda_B > 0$ and $\gamma_A = \gamma_B$. Hence, the differential-difference equation (36) becomes a linear ODE. Solution $\widehat{\Psi}^{unc}(v)$ corresponding to the unconstrained model is a particular solution of (36). The general solution is then given by (IA.48). Constants $C_\pm$ can be easily found from the boundary conditions (37) by solving the following system of linear equations:

$$\begin{aligned} C_+\varphi_+ e^{\varphi_+\underline{v}} + C_-\varphi_- e^{\varphi_-\underline{v}} &= -(\Psi^{unc})'(\underline{v}), \\ C_+(\varphi_+ - 1)e^{\varphi_+\overline{v}} + C_-(\varphi_- - 1)e^{\varphi_+\overline{v}} &= \Psi^{unc}(\overline{v}) - (\Psi^{unc})'(\overline{v}). \end{aligned} \tag{IA.51}$$

It remains to prove that constants $C_\pm$ are positive. First, we show that they have the same sign. Because the state price density in the unconstrained economy is given by $\xi_\tau = \ell(s(v_\tau)D_\tau)^{-\gamma_A}$, where ℓ is a constant, the P/D ratio in this economy is given by:

$$\Psi^{unc}(v) = \mathbb{E}_t^A\left[\int_t^\infty e^{-\rho(\tau-t)}\Big(\frac{s(v_\tau)D_\tau}{s(v)D_t}\Big)^{-\gamma_A}\frac{D_\tau}{D_t}d\tau\right], \tag{IA.52}$$

where $s(v)$ is the consumption share satisfying Eq. (14). Define function $\widehat{\Psi}^{unc}(v) = \Psi^{unc}(v)s(v)^{-\gamma_A}$, which is given by:

$$\widehat{\Psi}^{unc}(v) = \mathbb{E}_t^A\left[\int_t^\infty e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_A}s(v_\tau)^{-\gamma_A}d\tau\right]. \tag{IA.53}$$

Using Itô's Lemma it can be easily shown that $\widehat{\Psi}^{unc}(v)$ in Eq. (IA.53) is a particular solution of the differential equation (36). Proposition 1 implies that in the unconstrained economy $v_\tau = v + \widehat{\mu}_v(\tau - t) + \widehat{\sigma}_v(w_\tau - w_t)$, and hence v_τ is a linear function of v. Differentiating Eq. (14) for the consumption share we find that $s'(v) = -1/(\gamma_A/s(v) + \gamma_B/(1-s(v)))$. Using $s'(v)$ and differentiating (IA.53) w.r.t. v, we obtain:

$$
\begin{aligned}
(\widehat{\Psi}^{unc})'(v) &= -\gamma_A \mathbb{E}_t^A \left[\int_t^\infty e^{-\rho(\tau-t)} \Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_A} s(v_\tau)^{-\gamma_A - 1} s'(v_\tau) d\tau \right]. \\
&= \mathbb{E}_t^A \left[\int_t^\infty e^{-\rho(\tau-t)} \Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_A} \frac{\gamma_A/s(v_\tau)}{\gamma_A/s(v_\tau) + \gamma_B/(1-s(v_\tau))} s(v_\tau)^{-\gamma_A} d\tau \right].
\end{aligned}
\tag{IA.54}
$$

From (IA.54) it can be easily observed that

$$
(\widehat{\Psi}^{unc})'(v) > 0, \quad \widehat{\Psi}^{unc}(v) - (\widehat{\Psi}^{unc})'(v) > 0. \tag{IA.55}
$$

From the latter two inequalities it follows that constants $C_\pm$ given by (IA.49) have the same sign if $\varphi_+ > 0 > \varphi_-$ and $\varphi_+ > 1$. The latter inequalities are proven in Lemma IA.3 below, and hence, C_+ and C_- have the same sign. The fact that $C_\pm > 0$ follows from Proposition 2 which shows that the P/D ratio in the unconstrained economy is lower than in the constrained economy, which implies $\widehat{\Psi}(v) = C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \widehat{\Psi}^{unc}(v) \geq \widehat{\Psi}^{unc}(v)$, and hence $C_\pm > 0$. ■

Lemma IA.3. Consider characteristic polynomials $h(\varphi)$ given by (IA.50). *Denote by φ_- and φ_+ the smallest and the largest solutions of equation $h(\varphi) = 0$. Then, $\varphi_+ > 0 > \varphi_-$ and $\varphi_+ > 1$.*

Proof Lemma IA.3. We note that $h(0) > 0$, which is a continuous-time analogue of condition (15) for investor A imposed for the existence of equilibrium. Next, we show that $h(1) > 0$. For simplicity, we only look at the case when $\lambda_A = \lambda_B = 0$ and risk aversions γ_A and γ_B are arbitrary. The other case of Proposition IA.4 in which $\lambda_A = \lambda_B = \lambda$ and $\gamma_A = \gamma_B$ can be proven analogously. Substituting $\widehat{\mu}_v$ and $\widehat{\sigma}_v$ from Eq. (38)-(39) into Eq. (IA.50), after some algebra, we obtain:

$$
h(1) = \rho - (1-\gamma_B)(\mu_D + \delta_B \sigma_D) + \frac{(1-\gamma_B)\gamma_B}{2}\sigma_D^2.
$$

The inequality $h(1) > 0$ is a continuous-time analogue of inequality (15) for investor B.

Then, because $h(-\infty) = h(+\infty) = -\infty$, $h(0) > 0$ and $h(1) > 0$ it follows that there exist roots $\varphi_\pm$ such that $\varphi_+ > 1$ and $0 > \varphi_-$. ■

Proposition IA.5. (U-shaped P/D ratios) Let Ψ and Ψ^{unc} be the price-dividend ratios in the constrained and unconstrained economies, respectively.

1) *Ψ is a decreasing (increasing) function of consumption share s near the boundary $\underline{s}$ ($\overline{s}$).*

2) *Consider two economies with the following parameters: a) $\lambda_A = \lambda_B = 0$, $\gamma_A \geq \gamma_B \geq 1$ and b) $\lambda_A = \lambda_B \equiv \lambda > 0$, $\gamma_A = \gamma_B \equiv \gamma \geq 1$. Then, $\Psi - \Psi^{unc}$ is a positive and convex function of s, and is a decreasing (increasing) function of s near the boundary $\underline{s}$ ($\overline{s}$).*

3) *Suppose $\gamma_A = \gamma_B > 1$. Then, Ψ^{unc} is a convex function of s.*

Proof Proposition IA.5.

1) Lemma 2 shows that $\Psi = \widehat{\Psi} s^{\gamma_A}$, where $\widehat{\Psi}(v)$ satisfies boundary conditions (37). Now, consider $\widehat{\Psi}(v)$ as a function of s and differentiate w.r.t. to s:

$$\frac{\partial \Psi}{\partial s} = \widehat{\Psi}'(v)\frac{\partial v}{\partial s} s^{\gamma_A} + \gamma_A \widehat{\Psi}(v) s^{\gamma_A - 1} = \Big(-\widehat{\Psi}'(v)\Big(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\Big) + \widehat{\Psi}(v)\frac{\gamma_A}{s}\Big) s^{\gamma_A}, \qquad \text{(IA.56)}$$

where $\partial v/\partial s$ is obtained by differentiating Eq. (14) for the consumption share. Using boundary conditions (37) and the fact that $\underline{s} = s(\overline{v})$ and $\overline{s} = s(\underline{v})$ (because $s(v)$ is a decreasing fundtion of v), we obtain:

$$\frac{\partial \Psi}{\partial s}\Big|_{s=\underline{s}} = -\widehat{\Psi}(\overline{v})\frac{\gamma_B \underline{s}_A^{\gamma}}{1-\underline{s}} < 0, \qquad \frac{\partial \Psi}{\partial s}\Big|_{s=\overline{s}} = \widehat{\Psi}(\underline{v})\gamma_A \overline{s}^{\gamma_A - 1} > 0.$$

2) $\Psi = \widehat{\Psi} s^{\gamma_A}$, $\Psi^{unc} = \widehat{\Psi}^{unc} s^{\gamma_A}$. Using Eq. (IA.48) for $\widehat{\Psi}$, the fact that in the latter equation $C_\pm > 0$ (Proposition IA.4), and the Eq. (14) for v in terms of s, we obtain:

$$\Psi - \Psi^{unc} = C_-(1-s)^{\varphi_- \gamma_B} s^{(1-\varphi_-)\gamma_A} + C_+(1-s)^{\varphi_+ \gamma_B} s^{(1-\varphi_+)\gamma_A}.$$

Differentiating the latter equation, we obtain:

$$\begin{aligned}(\Psi - \Psi^{unc})' = C_-\Big(\frac{(1-\varphi_-)\gamma_A}{s} - \frac{\varphi_- \gamma_B}{1-s}\Big)(1-s)^{\varphi_- \gamma_B} s^{(1-\varphi_-)\gamma_A} \\ + C_+\Big(\frac{(1-\varphi_+)\gamma_A}{s} - \frac{\varphi_+ \gamma_B}{1-s}\Big)(1-s)^{\varphi_+ \gamma_B} s^{(1-\varphi_+)\gamma_A}.\end{aligned} \qquad \text{(IA.57)}$$

Differentiating one more time, we obtain:

$$(\Psi-\Psi^{unc})''=C_-\Big(\frac{(1-\varphi_-)^2\gamma_A^2-(1-\varphi_-)\gamma_A}{s^2}+\frac{\varphi_-^2\gamma_B^2-\varphi_-\gamma_B}{(1-s)^2}-\frac{2\gamma_A\gamma_B\varphi_-(1-\varphi_-)}{s(1-s)}\Big)\frac{(1-s)^{\varphi_-\gamma_B}}{s^{(\varphi_--1)\gamma_A}}$$

$$+\ C_+\Big(\frac{(1-\varphi_+)^2\gamma_A^2-(1-\varphi_+)\gamma_A}{s^2}+\frac{\varphi_+^2\gamma_B^2-\varphi_+\gamma_B}{(1-s)^2}-\frac{2\gamma_A\gamma_B\varphi_+(1-\varphi_+)}{s(1-s)}\Big)\frac{(1-s)^{\varphi_+\gamma_B}}{s^{(\varphi_+-1)\gamma_A}}.$$

Using the facts that $\varphi_+>0>\varphi_-$, $\varphi_+>1$, and $\gamma_i\geq 1$ from the above equation we find that $(\Psi-\Psi^{unc})''>0$, and hence, $\Psi-\Psi^{unc}$ is convex.

Next, we verify that $\Psi-\Psi^{unc}$ is decreasing (increasing) around $\underline{s}$ ($\overline{s}$). Taking into account Eq. (14) for v as a function of s, we rewrite $(\Psi-\Psi^{unc})'$ in (IA.57) as follows:

$$(\Psi-\Psi^{unc})'=\bigg(C_-\Big(\frac{(1-\varphi_-)\gamma_A}{s}-\frac{\varphi_-\gamma_B}{1-s}\Big)e^{\varphi_-v}+C_+\Big(\frac{(1-\varphi_+)\gamma_A}{s}-\frac{\varphi_+\gamma_B}{1-s}\Big)e^{\varphi_+v}\bigg)s^{\gamma_A}$$

$$=\bigg(\frac{\gamma_A}{s}\Big(C_-(1-\varphi_-)e^{\varphi_-v}+C_+(1-\varphi_+)e^{\varphi_+v}\Big)-\frac{\gamma_B}{1-s}\Big(C_-\varphi_-e^{\varphi_-v}+C_+\varphi_+e^{\varphi_+v}\Big)\bigg)s^{\gamma_A}. \tag{IA.58}$$

Using Eq. (IA.51) for constants $C_\pm$, inequalities (IA.55) for the derivatives of $\widehat{\Psi}^{unc}$ and $C_\pm>0$ (Proposition IA.4), and the fact that $\underline{s}=s(\overline{v})$ and $\overline{s}=s(\underline{v})$, from (IA.58) we obtain:

$$(\Psi-\Psi^{unc})'(\underline{s})=\frac{\gamma_A}{\underline{s}}((\widehat{\Psi}^{unc})'(\overline{v})-(\widehat{\Psi}^{unc})(\overline{v}))+\frac{\gamma_B}{1-\underline{s}}((\widehat{\Psi}^{unc})'(\overline{v})-\widehat{\Psi}^{unc}(\overline{v})-C_+e^{\varphi_+\overline{v}}-C_-e^{\varphi_-\overline{v}})<0,$$

$$(\Psi-\Psi^{unc})'(\overline{s})=\frac{\gamma_A}{\overline{s}}((\widehat{\Psi}^{unc})'(\underline{v})+C_+e^{\varphi_+\underline{v}}+C_-e^{\varphi_-\underline{v}})+\frac{\gamma_B}{1-\overline{s}}((\widehat{\Psi}^{unc})'(\underline{v}))>0.$$

3) Suppose $\gamma_A=\gamma_B=\gamma$. Then, from Eq. (14) for variable v in terms of consumption share s we find that $s_t=1/(1+\exp(v_t/\gamma))$. Using the latter equation, we obtain:

$$s_\tau=\frac{s_t}{s_t+(1-s_t)e^{(v_\tau-v_t)/\gamma}}. \tag{IA.59}$$

Substituting s_τ into Eq. (IA.52) for the unconstrained P/D ratio, we obtain:

$$\Psi_t^{unc}=\mathbb{E}_t^A\bigg[\int_t^\infty e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma}\Big(s_t+(1-s_t)e^{(v_\tau-v_t)/\gamma}\Big)^\gamma d\tau\bigg]. \tag{IA.60}$$

We note that because state variable v_t in the unconstrained case follows an arithmetic Brownian motion (see Proposition 2), then $v_\tau-v_t$ does not depend on v_t or s_t. Then, twice differentiating (IA.60) w.r.t. s_t, we obtain:

$$(\Psi_t^{unc})''_{ss}=\gamma(\gamma-1)\mathbb{E}_t^A\bigg[\int_t^\infty e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma}\Big(s_t+(1-s_t)e^{(v_\tau-v_t)/\gamma}\Big)^{\gamma-2}\Big(1-e^{(v_\tau-v_t)/\gamma}\Big)^2d\tau\bigg].$$

We observe that $(\Psi_t^{unc})'' \geq 0$ when $\gamma \geq 1$, and hence, Ψ_t^{unc} is a convex function of s. ■

IA.2. Alternative parameter specifications

The results in our baseline analysis are derived assuming that the more risk averse investor is also more pessimistic about the output growth rate, and the main calibration in Section 4 shows equilibrium processes when investors have the same risk aversions but different beliefs. In this section, we demonstrate the robustness of our results on the effects of constraints on interest rates, Sharpe ratios, P/D ratios, and volatilities by considering two economies with alternative specifications of exogenous model parameters.

Fig. IA.1 shows the equilibrium processes when investor A is more risk averse but also more optimistic than investor B. Fig. IA.2 shows the equilibrium processes when the investors have different risk aversions and identical beliefs about the probabilities of states, that is, $\lambda_A = \lambda_B$ and $\delta_A = \delta_B = 0$. The exact values of model parameters are given in the legends of the figures. From both figures, we observe that interest rates, Sharpe ratios, P/D ratios, and volatilities are affected by the collateral constraints in the same way as in our baseline analysis in Section 4. In particular, interest rates go down, Sharpe ratios spike in bad times, P/D ratios are U-shaped and sensitive to small shocks, and volatilities are higher (lower) in good (bad) times. However, we note that the magnitudes of these effects are smaller than in our baseline calibration.

Finally, we discuss general conditions for the counter- and procyclicality of consumption share s. It is intuitive that share s is countercyclical in our baseline analysis in Section 4, in which we assume that the less risk averse investor is also more optimistic than the more risk averse investor because in good (bad) time the distribution of wealth and consumption shift toward the less (more) risk averse investor. Without restricting risk aversions and beliefs, consumption share s is countercyclical if the volatility of state variable v given by (39) is positive, $\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta_B - \delta_A > 0$, and is procyclical otherwise. The variable v follows a reflected Brownian motion, so that $v_{t+dt} = \max\Big\{ \underline{v}; \min\{ \overline{v}; v_t + \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t + \widehat{J}_v dj_t\}\Big\}$. Hence, if $\widehat{\sigma}_v > 0$ then positive shocks $dw_t > 0$ increase the state variable by $\widehat{\sigma}_v dw_t$ and decrease consumption share s because $s(v)$ is a decreasing function of v, as can be seen

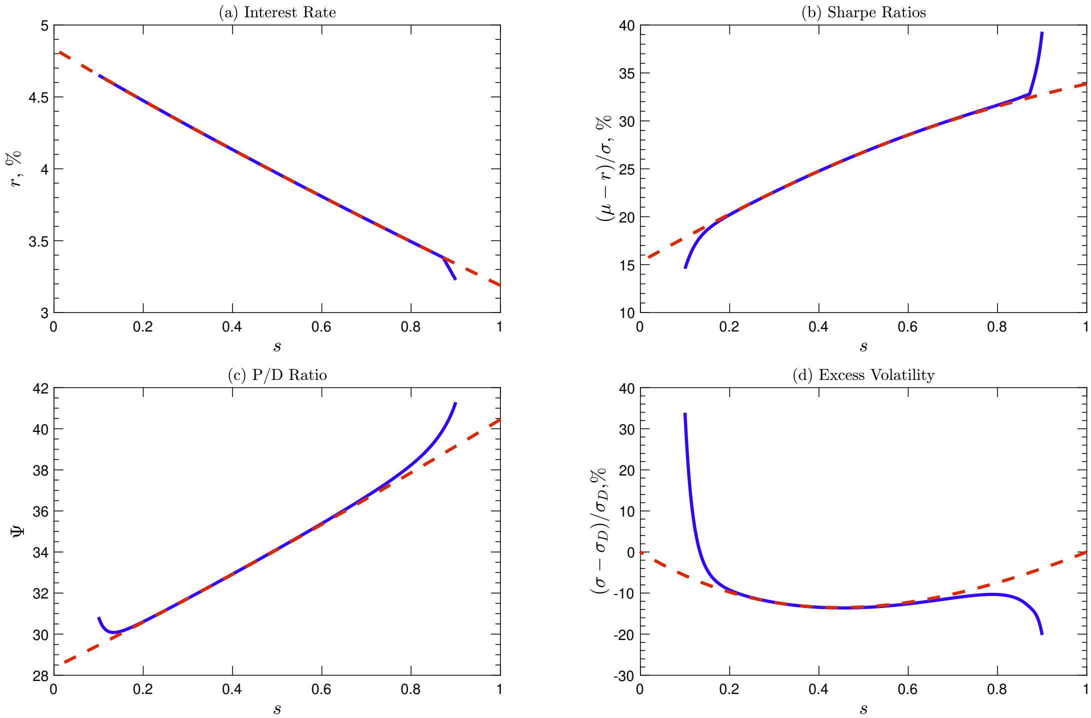

Fig. IA.1. Equilibrium processes
Panels (a)–(d) show interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are: $\gamma_A = 1.5$, $\gamma_B = 2$, $\delta_B = -\delta_A = 0.05$, $\lambda_A = 0.02$, $\lambda_B = 0.01$, $\rho = 0.02$, $\mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $J_D = -0.25$.

from Eq. (14). As a result, consumption share s is higher in bad (good) times, following periods of consecutive negative (positive) shocks dw_t. We note that condition $\widehat{\sigma}_v > 0$ is satisfied for the parameters used for the equilibrium processes on Fig. IA.1, and hence share s is countercyclical. All qualitative results remain the same when $\widehat{\sigma}_v < 0$.[2]

References

Gradshteyn, I.S., and I.M. Ryzhik, 2007. Table of Integrals, Series, and Products (6th edition). Academic Press, New York.

[2]We also remark that in the derivation of the differential-difference Eq. (36) we use the fact that $\widehat{J}_v < 0$, where $\widehat{J}_v$ is given by Eq. (40). The condition $\widehat{J}_v < 0$ is satisfied in our baseline analysis where $\gamma_A > \gamma_B$ and $\lambda_A > \lambda_B$. We note that this condition is without loss of generality because if it is violated it can be restored by relabeling investors A and B as investors B and A, respectively.

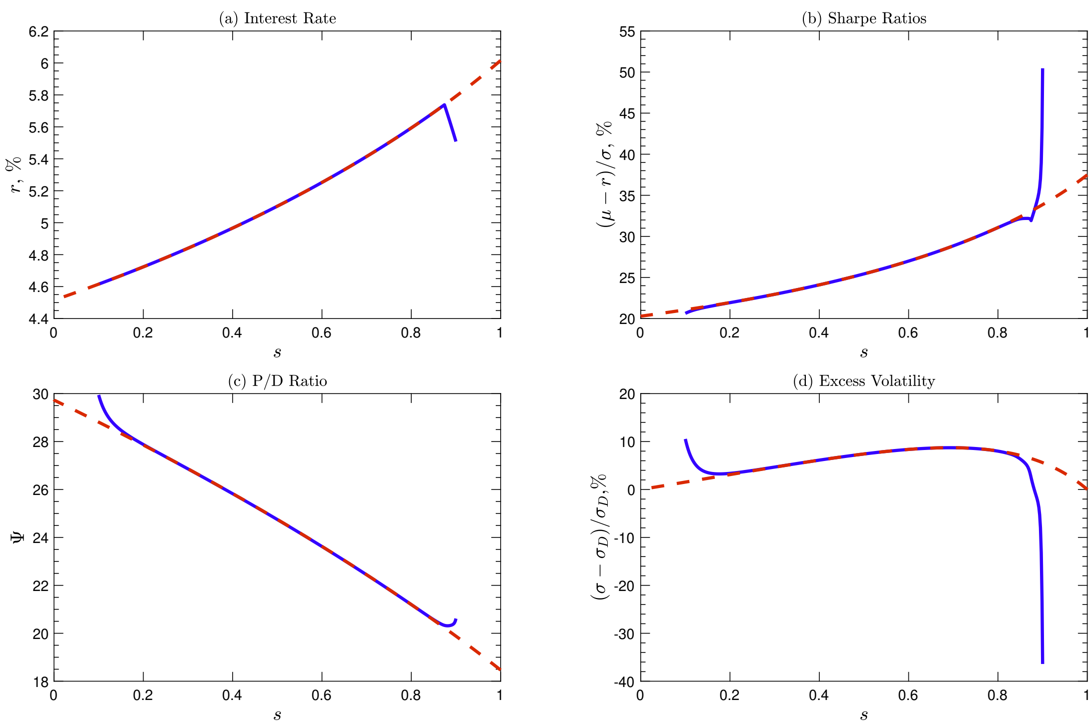

Fig. IA.2. Equilibrium processes
Panels (a)–(d) show interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are: $\gamma_A = 4$, $\gamma_B = 2$, $\delta_B = \delta_A = 0$, $\lambda_A = \lambda_B = 0.01$, $\rho = 0.02$, $\mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $J_D = -0.25$.