Abstract

This Internet Appendix presents additional results. Section A specializes Result 3 using specific functional forms for utility. Section B presents the approach used to approximate integrals that are used to compute various moments. Sections C and D demonstrate conditions under which Assumptions 1 and 2 are satisfied when investors have CRRA or HARA utility, respectively. Section E presents figures that display (i) risk-neutral moments and truncated risk-neutral moments, and (ii) the bounds on expected excess returns.

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A. Remarks related to Result 3

Proof of Remark 1. When \( u[x] = \log x, \ u'[x] = \frac{1}{x}, \) and

\[
\frac{u'[W_t R_{f \rightarrow T}]}{u'[W_t R_{M \rightarrow T}]} = \frac{R_{M \rightarrow T}}{E_t^*(R_{M \rightarrow T})}.
\]

Replace this expression in equation (11). This ends the proof. ■

Proof of Remark 2. When \( u[x] = \frac{x^{1-\alpha}}{1-\alpha}, \ u'[x] = x^{-\alpha}, \) and

\[
\frac{u'[W_t R_{f \rightarrow T}]}{u'[W_t R_{M \rightarrow T}]} = \frac{R_{M \rightarrow T}^\alpha}{E_t^*(R_{M \rightarrow T}^\alpha)}.
\]

Replace this expression in equation (11). This ends the proof. Next, we derive a closed-form expression of \( E_t^*(R_{M \rightarrow T}^\alpha) \) which is necessary to compute the expected excess return. To do so, the spanning formula allows us to write

\[
R_{M \rightarrow T}^\alpha = \left( \frac{S_T}{S_t} \right)^\alpha = 1 + \alpha \frac{1}{S_t} (S_T - S_t) + \frac{\alpha(\alpha - 1)}{S_t^2} \int_{S_t}^{\infty} \left( \frac{K}{S_t} \right)^{\alpha - 2} (S_T - K)^+ dK
\]

\[
\quad \quad + \frac{\alpha(\alpha - 1)}{S_t^2} \int_0^{S_t} \left( \frac{K}{S_t} \right)^{\alpha - 2} (K - S_T)^+ dK.
\]

Thus, the expected value of the above quantity is

\[
E_t^*[R_{M \rightarrow T}^\alpha] = 1 + \alpha (R_{f \rightarrow T} - 1) + \frac{\alpha(\alpha - 1)}{S_t^2} \int_{S_t}^{\infty} \left( \frac{K}{S_t} \right)^{\alpha - 2} C_t[K] dK
\]

\[
\quad \quad + \frac{\alpha(\alpha - 1)}{S_t^2} \int_0^{S_t} \left( \frac{K}{S_t} \right)^{\alpha - 2} P_t[K] dK.
\]

■
Proof of Remark 3. When \( u[x] = 1 - e^{-\alpha t} \), \( u'[x] = \alpha e^{-\alpha t} \), and

\[
\frac{u'[W_{R,f \rightarrow T}]}{u[W_{R,M \rightarrow T}]} = e^{\tilde{\alpha} R_{M \rightarrow T}} \frac{E^* \left( u'[W_{R,f \rightarrow T}] \right)}{E^* \left( e^{\tilde{\alpha} R_{M \rightarrow T}} \right)} \text{ with } \tilde{\alpha} = \alpha W_t.
\]

Replace this expression in equation (11) and obtain the final result. Next, we derive closed-form expressions of quantities \( E^* \left( R_{M \rightarrow T} e^{\tilde{\alpha} R_{M \rightarrow T}} \right) \) and \( E^* \left( e^{\tilde{\alpha} R_{M \rightarrow T}} \right) \), which are necessary to compute the bound. To proceed, observe that

\[
R_{M \rightarrow T} e^{\tilde{\alpha} R_{M \rightarrow T}} = \frac{S_T}{S_t} e^{\tilde{\alpha} \frac{S_T}{S_t}} = e^{\tilde{\alpha}} + \left( \frac{S_T}{S_t} - 1 \right) (1 + \tilde{\alpha}) e^{\tilde{\alpha}} + \int_{S_t}^{\infty} \left[ \frac{2 \tilde{\alpha} + \tilde{\alpha}^2 K}{S_t^2} \right] e^{\tilde{\alpha} \frac{S_T}{S_t}} (S_T - K)^+ dK + \int_{S_t}^{S_T} \left[ \frac{2 \tilde{\alpha} + \tilde{\alpha}^2 K}{S_t^2} \right] e^{\tilde{\alpha} \frac{S_T}{S_t}} (K - S_T)^+ dK.
\]

Thus,

\[
E^* \left( R_{M \rightarrow T} e^{\tilde{\alpha} R_{M \rightarrow T}} \right) = e^{\tilde{\alpha}} + (R_{f,f \rightarrow T} - 1) (1 + \tilde{\alpha}) e^{\tilde{\alpha}} + \frac{R_{f,f \rightarrow T}}{S_t^2} \int_{S_t}^{\infty} \left[ \frac{2 \tilde{\alpha} + \tilde{\alpha}^2 K}{S_t^2} \right] e^{\tilde{\alpha} \frac{S_T}{S_t}} C_t[K] dK + \frac{R_{f,f \rightarrow T}}{S_t^2} \int_{0}^{S_t} \left[ \frac{2 \tilde{\alpha} + \tilde{\alpha}^2 K}{S_t^2} \right] e^{\tilde{\alpha} \frac{S_T}{S_t}} p_t[K] dK.
\]

Furthermore,

\[
e^{\tilde{\alpha} R_{M \rightarrow T}} = e^{\tilde{\alpha} \frac{S_T}{S_t}} = e^{\tilde{\alpha}} + \frac{\tilde{\alpha}}{S_t} (S_T - S_t) e^{\tilde{\alpha}} + \frac{\tilde{\alpha}^2}{S_t^2} \int_{S_t}^{\infty} e^{\tilde{\alpha} \frac{S_T}{S_t}} (S_T - K)^+ dK + \frac{\tilde{\alpha}^2}{S_t^2} \int_{0}^{S_t} e^{\tilde{\alpha} \frac{S_T}{S_t}} (K - S_T)^+ dK.
\]

Thus,

\[
E^* \left[ e^{\tilde{\alpha} R_{M \rightarrow T}} \right] = e^{\tilde{\alpha}} + \tilde{\alpha} e^{\tilde{\alpha}} (R_{f,f \rightarrow T} - 1) + \frac{\tilde{\alpha}^2 R_{f,f \rightarrow T}}{S_t^2} \left( \int_{S_t}^{\infty} e^{\tilde{\alpha} \frac{S_T}{S_t}} C_t[K] dK + \int_{0}^{S_t} e^{\tilde{\alpha} \frac{S_T}{S_t}} p_t[K] dK \right). \quad (A1)
\]
Proof of Remark 4. In the case of hyperbolic absolute risk aversion (HARA) utility,

\[ u[x] = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1 - \gamma} + b \right)^\gamma \]

with \( a > 0 \) and \( \frac{ax}{1 - \gamma} + b > 0 \). (A2)

Denote the relative risk aversion by \( R \) evaluated at \( W_{t\rightarrow T}R_{f,t\rightarrow T} \). The relative risk tolerance is

\[ \frac{1}{R} = \frac{1}{1 - \gamma} + \frac{b}{aW_{t\rightarrow T}R_{f,t\rightarrow T}}. \]

From the risk tolerance expression,

\[ \frac{(1 - \gamma)}{R} = 1 + \frac{b(1 - \gamma)}{aW_{t\rightarrow T}R_{f,t\rightarrow T}} \]

and hence

\[ \left( \frac{(1 - \gamma)}{R} - 1 \right)^{-1} = \frac{aW_{t\rightarrow T}R_{f,t\rightarrow T}}{b(1 - \gamma)}. \]

For further simplification, we use the following notation

\[ a^* = \left( \frac{(1 - \gamma)}{R} - 1 \right)^{-1}. \] (A4)

Assume that \( 0 < \gamma < 1 \) and \( R > 1 \). It follows that \( a^* \) is negative and \( b \) is negative.

Next, the marginal utility can be written as

\[ u'[x] = a \left( \frac{ax}{1 - \gamma} + b \right)^{\gamma - 1} = a \left( \frac{a^*bx}{W_{t\rightarrow T}R_{f,t\rightarrow T}} + b \right)^{\gamma - 1} \]

Using equation (8), we simplify the inverse of the SDF as

\[ \frac{\mathbb{E}_t(M_{t\rightarrow T})}{M_{t\rightarrow T}} = \frac{(a^*b(R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) + b)^{1-\gamma}}{\mathbb{E}_t\left((a^*b(R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) + b)^{1-\gamma}\right)} \]

\[ = \frac{(-a^*(R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}}{\mathbb{E}_t\left((-a^*(R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}\right)} \] (A6)

We simplify the expression in the second line above by factoring out \(-b\) from both the numerator and denominator in the first line.
The physical moment $M_t^{(n)}$ is

\[
M_t^{(n)} = \mathbb{E}_t (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^n
\]

\[
= \mathbb{E}_t^* \left( \frac{\mathbb{E}_t M_{t\rightarrow T}}{M_{t\rightarrow T}} (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^n \right) \quad \text{(using the Radon-Nikodym Theorem)}
\]

\[
= \mathbb{E}_t^* \left( \frac{(-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}}{\mathbb{E}_t^* (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}} (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^n \right)
\]

\[
= \text{COV}_t^* \left( \frac{(-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}}{\mathbb{E}_t^* (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}} (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^n \right) + M_t^{(n)}.
\]

This expression simplifies to

\[
M_t^{(n)} - M_t^{*(n)} = \frac{\text{COV}_t^* \left( (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma}, (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^n \right)}{\mathbb{E}_t^* \left( (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma} \right)}.
\]

Next, we derive closed-form expressions needed to compute the bound when $n = 1$:

\[
\mathbb{E}_t (R_{M,t\rightarrow T} - R_{f,t\rightarrow T}) = \frac{\mathbb{E}_t^* \left( (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma} R_{M,t\rightarrow T} \right)}{\mathbb{E}_t^* \left( (-a^* (R_{M,t\rightarrow T}/R_{f,t\rightarrow T}) - 1)^{1-\gamma} \right)} - R_{f,t\rightarrow T}.
\]

Since $R_{M,t\rightarrow T} = \frac{S_T}{S_f}$, we denote

\[
g [S_T] = \left( -a^* \frac{S_T}{S_f} - 1 \right)^{1-\gamma} \frac{S_T}{S_f} \quad \text{and} \quad f [S_T] = \left( -a^* \frac{S_T}{S_f} - 1 \right)^{1-\gamma}
\]

It follows that

\[
g [S_f R_{f,t\rightarrow T}] = \left( -a^* - 1 \right)^{1-\gamma} R_{f,t\rightarrow T},
\]

\[
g_{SS} [K] = (1-\gamma) (-\gamma) \left( \frac{a^*}{S_f R_{f,t\rightarrow T}} \right)^2 \left( \frac{-a^*}{S_f R_{f,t\rightarrow T} K - 1} \right)^{-\gamma-1} \frac{K}{S_f} - 2 a^* (1-\gamma) \left( \frac{-a^*}{S_f^2 R_{f,t\rightarrow T}} K - 1 \right)^{-\gamma}
\]

\[
f [S_f R_{f,t\rightarrow T}] = \left( -a^* - 1 \right)^{1-\gamma},
\]

\[
f_{SS} [K] = (1-\gamma) (-\gamma) \left( \frac{a^*}{S_f R_{f,t\rightarrow T}} \right)^2 \left( \frac{-a^*}{S_f R_{f,t\rightarrow T} K - 1} \right)^{-\gamma-1}.
\]
Thus, the Carr and Madan (2001) spanning formula can be used to show that

\[
g[S_T] = g[S_tR_{f,T} + (S_T - S_tR_{f,T})gS[S_tR_{f,T}]
+ \int_{S_tR_{f,T}}^{\infty} g_{SS}[K](S_T - K)^+ dK + \int_0^{S_tR_{f,T}} g_{SS}[K](K - S_T)^+ dK
\]

and

\[
f[S_T] = f[S_tR_{f,T} + (S_T - S_tR_{f,T})fS[S_tR_{f,T}]
+ \int_{S_tR_{f,T}}^{\infty} f_{SS}[K](S_T - K)^+ dK + \int_0^{S_tR_{f,T}} f_{SS}[K](K - S_T)^+ dK
\]

Hence,

\[
\mathbb{E}_t^* g[S_T] = g[S_tR_{f,T} + R_{f,T} \left( \int_{S_tR_{f,T}}^{\infty} g_{SS}[K]C_t[K] dK + \int_0^{S_tR_{f,T}} g_{SS}[K]P_t[K] dK \right)],
\]

\[
\mathbb{E}_t^* f[S_T] = f[S_tR_{f,T} + R_{f,T} \left( \int_{S_tR_{f,T}}^{\infty} f_{SS}[K]C_t[K] dK + \int_0^{S_tR_{f,T}} f_{SS}[K]P_t[K] dK \right)].
\]

B. Approximation of integrals used to compute the risk-neutral moments

B.1. Risk-neutral moments of simple returns

We discretize expressions for the risk-neutral moments of simple return (B16) as follows:

Let \( \{K_i\} \) represent all available out-of-the-money strikes for a particular date and maturity where \( i \in \{1, \ldots, N\} \). Define the step size as follows:

\[
\Delta I(K_i) \equiv \begin{cases} 
\frac{K_{i+1} - K_{i-1}}{2}, & \text{for } 0 \leq i \leq N \text{ (with } K_{-1} \equiv 2K_0 - K_1 \text{ and } K_{N+1} \equiv 2K_N - K_{N-1}) \\
0, & \text{else}
\end{cases}
\]

Then the risk-neutral moments can be approximated as
\[
E_t^\ast \left[ (R_{M,t\to T} - R_{f,t\to T})^n \right] \approx \frac{n(n-1)R_{f,t\to T}}{S_t^2} \left( \sum_{K_i \leq R_{f,t\to T}S_t} \left( \frac{K_i}{S_t} - R_{f,t\to T} \right)^{n-2} P_t[K_i] \Delta I(K_i) \right) + \sum_{K_i > R_{f,t\to T}S_t} \left( \frac{K_i}{S_t} - R_{f,t\to T} \right)^{n-2} C_t[K_i] \Delta I(K_i) \right).
\]

B.2. Risk-neutral moments in down market

In this subsection, we show how we approximate the integrals in Equation (B27). To approximate (B27), we first approximate \( \text{Prob}^\ast \left[ \frac{S_T}{S_t} \leq k_0 \right] \). To compute \( \text{Prob}^\ast \left[ \frac{S_T}{S_t} \leq k_0 \right] \), we first observe that, for \( K = S_t k_0 \),
\[
\left( k_0 - \frac{S_{t+1}}{S_t} \right)^+ = \left( \frac{S_t k_0 - S_{t+1}}{S_t} \right) 1_{S_t < S_{t+1}} = \left( \frac{K - S_{t+1}}{S_t} \right) 1_{K > S_{t+1}}.
\]

Thus,
\[
\frac{\partial}{\partial K} \left( K - S_{t+1} \right)^+ = 1_{K > S_{t+1}} \text{ and } \frac{1}{S_t} \frac{\partial}{\partial K} \left( K - S_{t+1} \right)^+ = \frac{1}{S_t} E_t^\ast \left[ 1_{K > S_{t+1}} \right].
\]

Therefore,
\[
\frac{1}{S_t} \frac{\partial}{\partial K} \left( K - S_{t+1} \right)^+ = \frac{1}{S_t} E_t^\ast \left[ 1_{K > S_{t+1}} \right] = \frac{1}{S_t} \text{Prob}^\ast \left[ S_{t+1} < K \right] \quad (A8)
\]

and
\[
\frac{\partial P_t[K]}{\partial K} = \frac{1}{R_{f,t}} \text{Prob}^\ast \left[ S_{t+1} < K \right],
\]

where \( P_t[K] \) is the price of a put option with maturity \( K \). We can use the center difference as a proxy for \( \frac{\partial P_t[K]}{\partial K} \):
\[
\frac{\partial P_t[K]}{\partial K} = \left( \frac{P_t[K + \Delta] - P_t[K - \Delta]}{2\Delta} \right).
\]

Hence,
\[
\text{Prob}^\ast \left[ S_{t+1} < K \right] = R_{f,t} \left( \frac{P_t[K + \Delta] - P_t[K - \Delta]}{2\Delta} \right) .
\]

Finally,
\[
\text{Prob}^\ast \left[ S_{t+1} < k_0 S_t \right] = R_{f,t} \left( \frac{P_t[k_0 S_t + \Delta] - P_t[k_0 S_t - \Delta]}{2\Delta} \right).
\]

We discretize the expressions as follows:

Let \( K_- \) and \( K_+ \) be the first out-of-the-money put strike below and above \( k_0 S_t \), respectively. Define the
step size $\Delta I(K_i)$ as above. Let $\{K_i\}$ be the set of strikes available that are below $k_0S_t$ on a given date and for a given maturity. We approximate the risk-neutral probability as

$$Prob_{t}^*[S_{t+1} < k_0S_t] \approx R_{f,t} \left( \frac{P_t[K_+] - P_t[K_-]}{K_+ - K_-} \right).$$

We approximate the price of a put with strike $k_0S_t$ as the linearly interpolated price of the nearest two straddling puts:

$$P_t[k_0S_t] \approx \left( \frac{k_0S_t - K_-}{K_+ - K_-} \right) P_t[K_-] + \left( \frac{K_+ - k_0S_t}{K_+ - K_-} \right) P_t[K_+].$$

Finally, we approximate the integral above using the discretization:

$$\int_{0}^{k_0S_t} \left( \frac{K}{S_t} - R_{f,t} \right)^{n-2} P_t[K] dK \approx \sum_{K_i \leq k_0S_t} \left( \frac{K}{S_t} - R_{f,t} \right)^{n-2} P_t[K_i] \Delta I(K_i).$$

C. Are Assumptions 1 and 2 satisfied in the case of CRRA utility?

Assumption 1 In case of the CRRA utility $u'[x] = x^{-\alpha}$. To check whether Assumption 1 is verified, we set $n = 2k + 1$ and show that the following physical moment

$$N_t^{(\alpha,k)} = E_t \left( R_{M,t \rightarrow T}^{-\alpha} (R_{M,t \rightarrow T} - R_{f,t \rightarrow T})^{2k+1} \right)$$

can be expressed in terms of risk neutral quantities and is negative. Then, we will show that the risk neutral moment $M_t^{(n)}$ is positively related to $N_t^{(\alpha,k)}$. To proceed, note that

$$N_t^{(\alpha,k)} = E_t \left( \frac{M_{t \rightarrow T}}{E_t[M_{t \rightarrow T}]} \frac{E_t[M_{t \rightarrow T}]}{M_{t \rightarrow T}} R_{M,t \rightarrow T}^{-\alpha} (R_{M,t \rightarrow T} - R_{f,t \rightarrow T})^{2k+1} \right).$$

Thus, the Radon-Nikodym Theorem allows us to move from the physical measure to the risk neutral measure. It follows that

$$N_t^{(\alpha,k)} = E_t^* \left( \frac{E_t[M_{t \rightarrow T}]}{M_{t \rightarrow T}} R_{M,t \rightarrow T}^{-\alpha} (R_{M,t \rightarrow T} - R_{f,t \rightarrow T})^{2k+1} \right).$$
The inverse of the SDF takes the form

\[ u^\prime \left[ W_t R f, t \rightarrow T \right] \Rightarrow u^\prime \left[ W_t RM, t \rightarrow T \right] \]

\[ E_t^* (u^\prime \left[ W_t R f, t \rightarrow T \right] u^\prime \left[ W_t RM, t \rightarrow T \right]) \] (see equation (8)). With CRRA utility, equation (8) reduces to

\[ \frac{E_t (M_{t \rightarrow T})}{M_{t \rightarrow T}} = \frac{1}{\zeta_t^{\alpha}} R_{M_{t \rightarrow T}}^{\alpha} \] with \( \zeta_t = E_t^* (R_{M_{t \rightarrow T}}^{\alpha}) \).

(A9)

Using this expression for the inverse of the SDF in (A9), we obtain

\[ N^{(\alpha,k)} = \frac{1}{\zeta_t^{\alpha}} \left( 1 - \frac{R_{M_{t \rightarrow T}}^{\alpha} R_{M_{t \rightarrow T}}^{-\alpha} (R_{M_{t \rightarrow T}} - R_{f_{t \rightarrow T}})^{2k+1}}{\zeta_t^{\alpha}} \right) = 1 - \frac{M_{t}^{\alpha(2k+1)}}{\zeta_t^{\alpha} \zeta_t^{\alpha}}. \]

Since \( \zeta_t > 0 \), the right hand side of the above expression has the same sign as the model-free quantity \( M_{t}^{\alpha(2k+1)} \). Empirically, this term is negative when \( k \geq 1 \). Therefore, \( N^{(\alpha,k)} \leq 0 \) for any \( k \geq 1 \). This shows that the CRRA utility satisfies Assumption 1. This is mainly due to the no-arbitrage conditions that guarantee the existence of the risk neutral measure and, hence, allow us to apply the Radon-Nikodym Theorem using the CRRA-based SDF.

**Assumption 2** To verify whether CRRA satisfies Assumption 2, we first compute the parameters \( \tau, \rho, \) and \( \kappa \) implied by CRRA utility:

\[ \tau = \frac{1}{\alpha}, \rho = \frac{1}{2} \frac{(\alpha + 1)}{\alpha}, \text{ and } \kappa = \frac{(\alpha + 2)(\alpha + 1)}{6\alpha^2}. \] (A10)

Therefore,

\[ 1 - \rho = 1 - \frac{1}{2} \frac{(\alpha + 1)}{\alpha} = \frac{1}{2} \left( 1 - \frac{1}{\alpha} \right). \]

Consequently, \( 1 - \rho \leq 0 \) if \( 0 < \alpha \leq 1 \). Further

\[ 1 - 2\rho + \kappa = 1 - \frac{(\alpha + 1)}{\alpha} + \frac{(\alpha + 2)(\alpha + 1)}{6\alpha^2} = \frac{(\alpha - 1)(\alpha - 2)}{6\alpha^2} \]

Thus, \( 1 - 2\rho + \kappa \geq 0 \) if \( \alpha \in (-\infty, 1] \cup [2, +\infty) \). As a result, the CRRA utility satisfies

\[ \theta_1 > 0, \theta_2 \leq 0 \text{ and } \theta_3 \geq 0 \]
if $0 < \alpha < 1$.

We therefore conclude that the CRRA utility satisfies Assumptions 1 and 2 if $0 < \alpha < 1$.

D. Are Assumptions 1 and 2 satisfied in the case of HARA utility?

**Assumption 1** To check whether Assumption 1 is satisfied, we set $n = 2k + 1$ and show that the following physical moment

$$N(a^*, b, k) = \mathbb{E}_t \left( (a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1} \right)$$

can be expressed in terms of risk neutral quantities and is negative for any $k \geq 1$. To accomplish this, we observe that

$$N(a^*, b, k) = \mathbb{E}_t \left( \frac{M_{t \rightarrow T}}{M_{t \rightarrow T}} \mathbb{E}_{t} (a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1} \right).$$

The Radon-Nikodym Theorem allows us to move from the physical measure to the risk neutral measure. It follows that

$$N(a^*, b, k) = \mathbb{E}_t^* \left( \frac{M_{t \rightarrow T}}{M_{t \rightarrow T}} (a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1} \right).$$

Now, we replace $\frac{M_{t \rightarrow T}}{M_{t \rightarrow T}}$ by (A5) and obtain

$$N(a^*, b, k) = \mathbb{E}_t^* \left( \frac{M_{t \rightarrow T}}{M_{t \rightarrow T}} (a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1} \right).$$

$$= \mathbb{E}_t^* \left( \frac{(a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1}}{(a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1}} \right)$$

$$= \frac{M_{t \rightarrow T}^* (R_{M \rightarrow T} / R_{f \rightarrow T})^{2k+1}}{(a^* b (R_{M \rightarrow T} / R_{f \rightarrow T}) + b)^{2k+1}}$$

$$= \frac{1}{\xi_t} M_{t \rightarrow T}^* (n)$$
with $\xi_t > 0$. Since the model free quantity $E^*_t \left( (R_{M,t\rightarrow T} - R_{f,t\rightarrow T})^{2k+1} \right)$ is negative and $\xi_t$ is positive, it follows that $\mathcal{N}^{(a^*,b,k)}$ is negative for any $k \geq 1$. This shows that the HARA utility satisfies Assumption 1. This is mainly due to the no-arbitrage conditions that guarantee the existence of the risk neutral measure and, hence, allow us to apply the Radon-Nikodym Theorem using the HARA-based SDF.

**Assumption 2** We begin by deriving the relevant preference parameters under HARA utility.

\[
\begin{align*}
    u'[x] &= a \left( \frac{ax}{1-\gamma} + b \right)^{\gamma-1} \\
    u''[x] &= -a^2 \left( \frac{ax}{1-\gamma} + b \right)^{\gamma-2} \\
    u'''[x] &= \frac{a(2-\gamma)(ax}{1-\gamma} + b)^{\gamma-3} \\
    u'''[x] &= \frac{a(2-\gamma)(\gamma-3)}{(1-\gamma)^2} \left( \frac{ax}{1-\gamma} + b \right)^{\gamma-4}.
\end{align*}
\]

Thus, the risk tolerance is

\[
\tau = \frac{1}{(1-\gamma)} + \frac{b}{a(W_t R_{f,t\rightarrow T})} = \frac{1}{(1-\gamma)} + \frac{1}{(1-\gamma)} \frac{b(1-\gamma)}{aW_tR_{f,t\rightarrow T}}.
\]

Using the definition (A3), the risk tolerance is $\tau = \frac{1}{k}$. Since $R^*_t > 0$, $\theta_1 > 0$. Next, we derive closed-form expressions for $\rho$ and $\kappa$.

\[
\rho = \frac{1}{2} \frac{(2-\gamma)}{1-\gamma} \quad \text{and} \quad \kappa = \frac{1}{6} \frac{(2-\gamma)(3-\gamma)}{(1-\gamma)^2}.
\]

Thus, it follows that $1 - \rho \leq 0$ if $\gamma \in [0,1)$. Further, $1 - 2\rho + \kappa \geq 0$ if $\gamma \in (-\infty, -1] \cup [0,1) \cup (1, +\infty)$. We conclude that if $\gamma \in [0,1)$, then $\theta_1 > 0$, $\theta_2 \leq 0$ and $\theta_3 \geq 0$. Thus, Assumption 2 is satisfied under these conditions. To summarize, Assumptions 1 and 2 are satisfied in case of HARA utility if $\gamma \in [0,1)$.

**E. Additional figures**
Fig. E.1: Risk-neutral simple return moment measures at maturities of 30, 60, 90, 180, and 360 days.
Fig. E.2: Truncated moment measures using $k_0 = 0.8$ at maturities of 30, 60, 90, 180, and 360 days. Raw data is averaged over 20 calendar days to reduce noise.
Fig. E.3. Unrestricted bound measures using the 60-day maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (39) and (27), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.4. Restricted bound measures using the 60-day maturity, annualized. Top: Upper and lower bound measures implied by 41 and 31, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.5. Unrestricted bound measures using the 90-day maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (39) and (27), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.6. Restricted bound measures using the 90-day maturity, annualized. Top: Upper and lower bound measures implied by 41 and 31, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.7. Unrestricted bound measures using the 180-day maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (39) and (27), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.8. Restricted bound measures using the 180-day maturity, annualized. Top: Upper and lower bound measures implied by 41 and 31, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.9. Unrestricted bound measures using the 360-day maturity and estimated parameter values from Table 2, annualized. Top: Upper and lower bound measures implied by (39) and (27), respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.10. Restricted bound measures using the 360-day maturity, annualized. Top: Upper and lower bound measures implied by 41 and 31, respectively, and the lower bound implied by Martin (2017). Middle: Difference between upper and lower bounds. Bottom: Difference between the lower bound and the lower bound implied by Martin (2017). We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 business days.
Fig. E.11. $LBR_{1\rightarrow T}$ decomposition according to equation (68) with a 60-day horizon. $D_{1\rightarrow T}^{(1)}$ represents the contribution from the lower bound in Martin (2017). $D_{1\rightarrow T}^{(2)}$, $D_{1\rightarrow T}^{(3)}$, and $D_{1\rightarrow T}^{(4)}$ represent the additional contributions of risk-neutral variance, skewness, and kurtosis to $LBR_{1\rightarrow T}$, respectively.

Fig. E.12. $LBR_{1\rightarrow T}$ decomposition according to equation (68) with a 90-day horizon. $D_{1\rightarrow T}^{(1)}$ represents the contribution from the lower bound in Martin (2017). $D_{1\rightarrow T}^{(2)}$, $D_{1\rightarrow T}^{(3)}$, and $D_{1\rightarrow T}^{(4)}$ represent the additional contributions of risk-neutral variance, skewness, and kurtosis to $LBR_{1\rightarrow T}$, respectively.
Fig. E.13. $LBR_{t \rightarrow T}^{(1)}$ decomposition according to equation (68) with a 180-day horizon. $D_{t \rightarrow T}^{(1)}$ represents the contribution from the lower bound in Martin (2017). $D_{t \rightarrow T}^{(2)}$, $D_{t \rightarrow T}^{(3)}$, and $D_{t \rightarrow T}^{(4)}$ represent the additional contributions of risk-neutral variance, skewness, and kurtosis to $LBR_{t \rightarrow T}^{(1)}$, respectively.

Fig. E.14. $LBR_{t \rightarrow T}^{(1)}$ decomposition according to equation (68) with a 360-day horizon. $D_{t \rightarrow T}^{(1)}$ represents the contribution from the lower bound in Martin (2017). $D_{t \rightarrow T}^{(2)}$, $D_{t \rightarrow T}^{(3)}$, and $D_{t \rightarrow T}^{(4)}$ represent the additional contributions of risk-neutral variance, skewness, and kurtosis to $LBR_{t \rightarrow T}^{(1)}$, respectively.
Fig. E.15. Covariance bounds from inequality (49) at the 60-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

Fig. E.16. Covariance bounds from inequality (49) at the 90-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.
Fig. E.17. Covariance bounds from inequality (49) at the 180-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.

Fig. E.18. Covariance bounds from inequality (49) at the 180-day maturity. We use a value of using $k_0 = 0.8$ in estimating the upper bounds and average truncated moments over 20 calendar days.
Fig. E.19. Unrestricted Lower Bound Term Structures: This figure illustrates the term structure behavior implied by the unrestricted lower bound (27) during normal (left) and turbulent (right) times. In the figure on the left, we plot the implied term structure at four “normal” dates. These were chosen simply as the first date for which we have data during years that were not particularly turbulent. The bold dotted line represents the average lower bound (by maturity) over the three low-volatility periods identified in Sichert (2018). In the figure on the right, we plot the implied term structure at four “turbulent” dates. These include the Russian debt crisis, the September 11 terrorist attacks, the Lehman Brothers bankruptcy, and the Flash Crash, respectively. The bold dotted line represents the average lower bound (by maturity) over the two high-volatility periods identified in Sichert (2018). The unrestricted bounds are estimated using parameter values from Table 1. The bounds are annualized and reported in percentages.