Appendix to “The Cross Section of Labor Leverage and Equity Returns”*

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*The views, opinions, and conclusions expressed in here are those of the authors and do not necessarily represent those of the Federal Reserve Bank of Chicago, the Federal Reserve System, or the U.S. Census Bureau. All results have been reviewed to ensure that no confidential information is disclosed.

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A. Data Sources

A.1. Compustat and CRSP Data

A.1.1. Sample Construction

Accounting and financial data comes from the Compustat and CRSP merged data set. We include firms with common shares (shrcd= 10 and 11) and stocks traded on NYSE, AMEX, and NASDAQ (exchcd=$1, 2$, and 3). The sample is from January 1973 to June 2016 so that it is represented throughout by firms from all three exchanges.[1] As standard, we omit firms whose primary standard industry classification (SIC) is between 4900 and 4999 (regulated firms) or between 6000 and 6999 (financial firms).

Size is defined as the market value of equity (Compustat variables prc times shrout). The book-to-market ratio is defined as book value of equity over size. Book value is defined as shareholders' equity (Compustat SEQ) divided by the market value of equity. Firm-year observations with missing monthly returns in the current or previous year or with missing or negative lagged or twice-lagged measures of labor share, size, book-to-market, investment-to-assets, or return on equity are excluded from the the sample associated to the measure of labor share. Firm-year observations in which the measure of labor share is negative or greater than one are also excluded. Variables are Winsorized at the 0.5% level in each sample year to reduce the potential influence of outliers.

A.1.2. More details about the construction of the LS and ELS measures

The *LS* and *ELS* are measures of labor share based on publicly available Compustat data. The definitions of both measures are consistent with the theoretical definition of labor share presented in Section 1. The measures are based on the ratio of a proxy for labor costs and a proxy for value

[1]NASDAQ was founded in 1971 but enters the CRSP sample in December of 1972.

added, all at firm level.

The proxy for labor costs used in the *LS* measure is the *XLR* variable from Compustat. The proxy for labor costs used in the *ELS* measure is imputed and defined as the product of $(\mathrm{EMP}_{t-1}+\mathrm{EMP}_t)/2$ and the average of the ratio of *XLR* to *EMP* in the industry as defined by the 17-industry classification from Kenneth French's data library.

The proxies for value added in the *LS* and *ELS* measures attempt to emulate the theoretical definition of value added used in Section 1. With ideal data and in a neoclassical frictionless setting, we would construct an empirical measure of value added simply as sales minus the cost of materials. The first challenge is that the Compustat data set does not disentangle the costs of materials from other operating costs. We address this challenge by using a proxy for the cost of materials as operating costs minus the proxy for labor costs. Note that sales minus the proxy for the cost of materials equals operating profits plus the proxy for labor costs. The second challenge is that value added is created at the time of production and not at the time of sale of a good. To address this second challenge, we add to sales the growth in the inventory of final goods. This last adjustment takes into account the fact that some of goods sold in a given year were not produced in that year and were instead in inventory and also that some goods that were produced in a given year were not sold in that year and were incorporated into the inventory instead.

A.2. Census Data

We use confidential establishment-level data from the Annual Survey of Manufactures (ASM) collected by the U.S. Census Bureau and supplement them with the *KLEMS Multifactor Productivity Tables by Industry for Manufacturing Industries* provided by the Bureau of Labor Statistics (BLS).

The U.S. Census Bureau collects annual data on manufacturing establishments in the ASM since 1972. The 50 to 60 thousand establishments comprise large establishments sampled with certainty every year (about half of the overall sample) and small establishments sampled with a lower probability but for five consecutive years. The sampling design of the ASM is aimed at a data set that accurately reflects manufacturing employment and output while maintaining a representative age distribution: Exiting establishments are replaced with a set of new and incumbent establishments so that the age distribution reflects that in the business register. We follow conventional practice and drop all observations that are imputed from administrative records (*AR*=1) or that are not part of the *tabbed sample* (i.e., observations with such a low quality and relevance that the U.S. Census bureau ignores them when publishing official tabulations). These establishments are also small enough to be considered unimportant for economic aggregates.

Following common practice among researchers working with the ASM, we correct industry codes as proposed in Davis, Haltiwanger, and Schuh (1998), transform the 1972-based SIC codes into 1987-based codes and turn all pre-1997 SIC codes into 6-digit NAICS codes to obtain a consistent industry classification for each establishment. This will be necessary to merge industry-level data by the BLS as explained later.

The labor share is generally defined as the ratio of labor expenses and value added. The following section describes how we measure both of these.

A.2.1. Measuring labor costs

We define the following items as labor costs: Salaries and wages (item *SW*); involuntary labor costs (item *ILC*) such as unemployment insurance or social security contributions netted out from wages; and voluntary labor costs (item *VLC*) such as health, retirement and other benefits paid to employees. Labor input comprises both full-time employees and temporary workers, a distinction visible in Census data since 2002. Before that year, instructions to establishments on how to report the workforce and labor cost are ambiguous, but studying the development of employment and labor expenses before and after 2002 suggests most establishments included all worker compensation (both permanent and temporary) in their reported statistics.

Our labor expenses measure lacks any non-monetary compensation or ownership rights that have monetary value to an employee. Stock options, for example, are counted as labor income for tax purposes once a manager exercises the option but not at the point in time when the manager acquires the option. This downward bias of our labor share measure is likely uncorrelated with two-years ahead stock returns and thus shouldn't impact our results.

A.2.2. Measuring value added

Value added in the Census data is measured as sales less inventory investment for final and work-in-progress goods, resales, material inputs, and energy expenditures.[2] Unlike in industry-level BLS data, purchased services, another intermediate input, are not reported in the Census data. To account for that, we reduce the value of an establishment's production by the industry-year-specific share of purchased services in sales computed from the 3-digit NAICS industry-level BLS data.

A.2.3. Constructing firm labor shares

Merging the ASM data with the Compustat panel requires aggregating establishments to the firm level. Our definition of a firm is based on the employer identification number (*EIN*) which we obtain from the Standard Statistical Establishment List (SSEL). In a given year, we sum labor expenses and value added of all establishments with the same *EIN* and take the ratio of these two variables to obtain the firm labor share:

$$CLS = \frac{\text{labor expenses}}{\text{value added}}. \tag{A.1}$$

[2] We subtract resales so that an establishment's value added is defined by its production activities (as opposed to its trading activities).

Since the ASM is not a comprehensive panel of all establishments, we do not observe all labor costs and the full value added of a firm. Only if establishments not sampled in the ASM have (1) a considerably different labor share than the establishments in the ASM *and* (2) are large enough to change the firm-level labor share would our labor share measure be distorted. Fundamentally, the second point is already moot because, by construction, all large, economically significant establishments are covered in the ASM. So even significantly different labor shares of the not sampled establishments would not make a difference. Lastly, mismeasured labor shares at the firm level would not impact our analysis unless this measurement error was correlated with objects in our analysis such as stock market returns or operating profits. Following common practice, we truncate firms with a negative labor share and those in the top decile of labor shares. Negative labor shares usually arise when gross profits are negative, i.e., when value added does not suffice to pay for labor expenses let alone capital costs. Unreasonably large labor shares arise if a firm's value added is close to zero. While such firms carry essentially no relevance for aggregate outcomes because their weight in aggregate value added is negligible, their huge labor shares could bias firm-level analysis based on linear regressions and averages by bins. Like other researchers we smooth any hiccups in 1997 (transition to 1997 NAICS industry classification) and 2002 (switch in the permanent establishment identifier) by replacing their values for the labor share with the average of the labor shares in that and the preceding year. Our resulting panel of Census firms has labor shares in the unit interval like in the Compustat panel, and the distribution of *CLS* closely resembles that of *LS*. Abiding with Census disclosure rules compels us to limit attention to observations with non-missing values in any variable needed for analysis in any of the statistical tables and regressions.[3] This means we are left with a panel of about 572,700 year-firm observations over the period 1972-2009 where we observe the levels and growth rates of the labor share, non-labor operating costs, total operating costs and profits. That sample underlies the analysis using *CLS* as

[3]Census disclosure requires that only rounded number of observations should be disclosed.

the labor share measure in Tables 3 (Panel B), 4, and 5 (Panel B, right half).

We have also verified that, after aggregating our firm-level labor shares at the manufacturing sector level, it is very close in level and dynamic properties to the manufacturing labor share calculated from sectoral data in the BLS and BEA.

A.2.4. Matching Census and Compustat data

The key analysis is to relate the Census labor share to the Compustat stock market returns. Of the roughly 100 thousand year-firm observations in the Compustat panel, we can match about 14 thousand observations to firms in our panel of Census firms using the *EIN*. Using that variable gives us a higher match rate than using the Compustat bridge, which extends only until 2005. The unmatched observations consist of years outside the ASM sample (before 1973/after 2009), non-manufacturing firms, firms with foreign ownership but no physical manufacturing operation in the U.S. and missing information on the matching variables. This matched panel underlies Tables 5 (Panel B, left half), 6, 7, and 8. Regressing the growth rate of operating income before depreciation growth on labor shares creates missing values, so that the sample underlying Table 6 (using *CLS* in Panel A) only comprises about 10 thousand observations. Due to Census disclosure requirements, this is also the sample underlying the summary statistics in Panel C of Table 1 and Panel B (left half) of Table 5.

In the matched Compustat Census sample, we have several measures for the labor share. *LS* is directly measured in the Compustat data but only available for a few observations; *CLS* is directly measured in the Census data and available for nearly all observations. *ELS*, in contrast, was imputed for those observations in the Compustat sample where we could not compute *LS*. The two natural questions are (a) how well do the two direct measures of the labor share line up and (b) how valid is the imputation procedure to obtain *ELS*. These questions are important for researchers that want to work with the labor share in Compustat but do not have access to the Census data.

Table A–1: Correlation between labor share measures

	CLS	No. obs.
LS	0.561	1,700
	(0.020)	
ELS	0.552	14,100
	(0.007)	

Table A–1 displays the pooled correlation coefficients between the labor share measures. To convey a sense of economic importance, we weigh observations with their value added in a given year. The results show that both the directly measured *LS* and the imputed *ELS* are positively and significantly related to the labor share measured in the Census data. Given that the high quality of employment and labor compensation in the Census data, we hence conclude that analyzing labor shares in Compustat data using our imputation methods is a good approximation of the true labor share.

A.3. Macroeconomic Data

The GDP growth series is taken from Table 1.1.3 of the National Income and Product Accounts of the Bureau of Economic Analysis (www.bea.gov). The real wage series and total factor productivity growth series are annualized, based on the quarterly seasonally adjusted series from the Bureau of Labor Statistics Major Sector Productivity and Costs program (www.bls.gov/lpc). The series cover the non-farm business sector. Following Arias, Hansen, and Ohanian (2007), we compute TFP growth as $\Delta \text{Log}\,[TFP] = \Delta \text{Log}\,[Y] - \frac{2}{3}\Delta \text{Log}\,[H]$, where $\Delta \text{Log}\,[Y]$ is the real output series and $\text{Log}\,[H]$ is the hours of all persons series. The real wage series is real hourly compensation. This measure is based on the BEA estimates for labor compensation, and it includes benefits. As a result, our measures of real wages and productivity are comparable in sectoral coverage and in construction.

A.4. Portfolio Sorts Details

For each of our labor share measures, we use two different sets of yearly breakpoints that define labor share portfolios. The first set of breakpoints, which defines equally-weighted portfolios, is denoted *all-but-micro breakpoints*. These breakpoints are defined as the 20th, 40th, 60th, and 80th labor share measure percentiles of the stocks with non-missing values for the labor share measure that have market values above the 20th size percentile among all NYSE-listed stocks. The second set of breakpoints, which defines value-weighted portfolios, is denoted *NYSE breakpoints*. These breakpoints are defined as the 20th, 40th, 60th, and 80th labor share measure percentile among NYSE-listed stocks and with non-missing labor share measure observations. The labor share and market size values used in the construction of the breakpoints for year t are measured at the end of calendar year t-1 (CRSP/Compustat samples) or year t-2 (merged CRSP/Census sample). Due to the sampling design of the Census data, *CLS* is lagged by two years to avoid the use of information not available to market participants on a given date. After constructing the breakpoints, portfolios are formed with the following procedures. For portfolios based on the merged CRSP/Compustat sample, we use the timing convention from Fama and French (1993). We use the breakpoints at year t to classify stocks in our sample based on the labor share measured at the end calendar year t-1 (i.e., the timing of the *LS* and *ELS* measures is the same as the one used to construct the breakpoints). We then hold each portfolio from July of year t until the end of June of year t+1. For portfolios based on the merged CRSP/Census sample, we use the breakpoints at year t to classify stocks in our sample based on the labor share measured at the end calendar year t-2 (i.e., the timing of the *CLS* measure is the same as the one used to construct the breakpoints). We then hold each portfolio from the end of December of year t until the end of December of year t+1.[4]

[4] The results of the portfolio sorts are qualitatively unchanged when we use the *CLS* measure at the end of year $t-1$ (i.e., when we follow the same procedure used in the portfolio sort tests based on CRSP/Compustat data).

A.5. Beta Construction Details

Conditional betas are estimated with monthly returns for the tradable factors MKT, ME, IA, and ROE over rolling one-year windows. Conditional betas are estimated with quarterly returns rates for the macroeconomic growth series of TFP and GDP over rolling five-year windows. We correct conditional betas for nonsynchronous price movements using the methodology proposed by Dimson (1979) and discussed by Lewellen and Nagel (2006), i.e., we include one-period lagged factors in the regressions and define betas as the sum of the slopes on the contemporaneous and lagged factor return or growth.

B. Theoretical Motivation Details

B.1. Derivation of Propositions 1 through 4 and Corollary 1

Capital is fixed, so the firm maximizes its profits solely by optimizing labor as given by

$$\Pi[X,W] = \max_{L}\{XF[K,L] - WL\}. \tag{B.2}$$

After applying the envelope theorem to Equation (B.2), we obtain

$$\frac{\partial \mathrm{Log}\,[\Pi]}{\partial \mathrm{Log}\,[X]} = \frac{1}{1-S}, \tag{B.3a}$$

$$\frac{\partial \mathrm{Log}\,[\Pi]}{\partial \mathrm{Log}\,[W]} = -\frac{S}{1-S}, \tag{B.3b}$$

where $S \equiv {}^{WL}\!/_{Y}$ is the labor share. The overall effect of a change in productivity (i.e., taking into account the effect on the wage) is given by

$$\frac{d\,\mathrm{Log}\,[\Pi]}{d\,\mathrm{Log}\,[X]} = \frac{\partial \mathrm{Log}\,[\Pi]}{\partial \mathrm{Log}\,[X]} + \frac{\partial \mathrm{Log}\,[\Pi]}{\partial \mathrm{Log}\,[W]}\frac{\partial \mathrm{Log}\,[W]}{\partial \mathrm{Log}\,[X]} = \frac{1-S\theta}{1-S}, \tag{B.4}$$

where $\theta \equiv {}^{\partial \text{Log}[W]}/_{\partial \text{Log}[X]}$ is the elasticity of the wage to productivity.

Optimal labor $L[X,W]$ is defined implicitly by the first-order condition:

$$XF_L[K, L[X,W]] - W = 0. \tag{B.5}$$

Differentiating Equation (B.5) leads to

$$\frac{\partial \text{Log}[L]}{\partial \text{Log}[X]} = \frac{\gamma}{1-S}, \tag{B.6a}$$
$$\frac{\partial \text{Log}[L]}{\partial \text{Log}[W]} = -\frac{\gamma}{1-S}, \tag{B.6b}$$

where $\gamma \equiv \frac{F_K[K,L]F_K[K,L]}{F[K,L]F_{KL}[K,L]}$ is the elasticity of substitution between capital and labor. Equations (B.6a) and (B.6b) follow from the constant return to scale property of the function F.

The overall effect of a productivity shock on labor demand is hence

$$\frac{d\,\text{Log}[L]}{d\,\text{Log}[X]} = \gamma\frac{1-\theta}{1-S}. \tag{B.7}$$

The effects on output, which are obtained by differentiation of $Y = XF[K,L]$, are given by

$$\frac{\partial \text{Log}[Y]}{\partial \text{Log}[X]} = 1 + \frac{\gamma S}{1-S}, \tag{B.8a}$$
$$\frac{\partial \text{Log}[Y]}{\partial \text{Log}[W]} = -\frac{\gamma S}{1-S}, \tag{B.8b}$$

so that the overall effect is given by

$$\frac{d\,\text{Log}[Y]}{d\,\text{Log}[X]} = 1 + \frac{\gamma S(1-\theta)}{1-S}. \tag{B.9}$$

Combining Equations (B.4) and (B.9) yields the expression in Proposition 1:

$$\ell \equiv \frac{\frac{d\,\mathrm{Log}[\Pi]}{d\,\mathrm{Log}[X]}}{\frac{d\,\mathrm{Log}[Y]}{d\,\mathrm{Log}[X]}} - 1 = \frac{\frac{(1-\gamma)S(1-\theta)}{1-S}}{1+\frac{\gamma S(1-\theta)}{1-S}}. \tag{B.10}$$

Proposition 2 and Corollary 1 follow directly from Equation (B.10).

After differentiating labor share S, we obtain

$$\frac{\partial\mathrm{Log}[S]}{\partial\mathrm{Log}[X]} = \frac{\partial\mathrm{Log}[L]}{\partial\mathrm{Log}[X]} - \frac{\partial\mathrm{Log}[Y]}{\partial\mathrm{Log}[X]} = \gamma - 1, \quad \text{and} \tag{B.11a}$$

$$\frac{\partial\mathrm{Log}[S]}{\partial\mathrm{Log}[W]} = 1 + \frac{\partial\mathrm{Log}[L]}{\partial\mathrm{Log}[W]} - \frac{\partial\mathrm{Log}[Y]}{\partial\mathrm{Log}[W]} = 1 - \gamma. \tag{B.11b}$$

Equations (B.11a) and (B.11b) show that labor share S is increasing in productivity X (and decreasing in the wage W) if and only if $\gamma < 1$.

To derive Proposition 3, we first introduce the two assumptions discussed in the text: (1) There are two periods, $t = 0$ and $t = 1$, and (2) X_1, W_1, and M_1 are jointly log-normally distributed. For convenience, we also approximate log profit with its log linearized transformation, as given by

$$\mathrm{Log}[\Pi_1] = \mathrm{Log}[\Pi[X_1, W_1]] \tag{B.12a}$$

$$\simeq \mathrm{Log}[\Pi[X_0, W_0]] + \frac{\partial\mathrm{Log}[\Pi[X_0, W_0]]}{\partial\mathrm{Log}[X]}(\mathrm{Log}[X_1] - \mathrm{Log}[X_0]) + \frac{\partial\mathrm{Log}[\Pi[X_0, W_0]]}{\partial\mathrm{Log}[W]}(\mathrm{Log}[W_1] - \mathrm{Log}[W_0]). \tag{B.12b}$$

The gross asset return is

$$R_1 = \frac{\Pi_1}{P_0}, \tag{B.13}$$

where

$$P_0 \equiv \mathrm{E}_0[M_1\Pi_1] \tag{B.14}$$

is the price of the firm defined as the expected discounted profit. From Equations (B.13) and (B.14), we have that the expected excess asset return over the risk-free rate is given by

$$\frac{\mathrm{E}_0\left[R_1\right]}{r_{\mathrm{F}}} = \frac{\mathrm{E}_0\left[\Pi_1\right]\mathrm{E}_0\left[M_1\right]}{\mathrm{E}_0\left[M_1\Pi_1\right]}. \tag{B.15}$$

Using the log linear approximation in Equation (B.12b) and the assumption about the joint log-normality of X_1, W_1, and M_1, we obtain

$$\mathrm{Log}\left[\mathrm{E}_0\left[\Pi_1\right]\right] = \mathrm{E}_0\left[\mathrm{Log}\left[\Pi_1\right]\right] + \frac{1}{2}V\,\mathrm{Log}\left[\Pi_1\right], \tag{B.16a}$$

$$\mathrm{Log}\left[\mathrm{E}_0\left[M_1\right]\right] = \mathrm{E}_0\left[\mathrm{Log}\left[M_1\right]\right] + \frac{1}{2}V\,\mathrm{Log}\left[M_1\right], \tag{B.16b}$$

$$\mathrm{Log}\left[\mathrm{E}_0\left[M_1\Pi_1\right]\right] = \mathrm{E}_0\left[\mathrm{Log}\left[\Pi_1\right]\right] + \mathrm{E}_0\left[\mathrm{Log}\left[M_1\right]\right] + \frac{1}{2}V\,\mathrm{Log}\left[\Pi_1\right] + \frac{1}{2}V\,\mathrm{Log}\left[M_1\right] + \mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[\Pi_1\right]\right), \tag{B.16c}$$

hence the following standard expression for the excess return on the firm:

$$\mathrm{Log}\left[\frac{\mathrm{E}_0\left[R_1\right]}{R_f}\right] \simeq \mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[\Pi_1\right]\right). \tag{B.17}$$

Proposition 3, which follows from Equations (B.3a) and (B.3b) and from the log linear approximation in Equation (B.12b), is given by

$$\begin{aligned}
\mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[\Pi_1\right]\right) &\simeq \frac{\partial\,\mathrm{Log}\left[\Pi[X_0, W_0]\right]}{\partial\,\mathrm{Log}\left[X\right]}\mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[X_1\right]\right) \\
&\quad + \frac{\partial\,\mathrm{Log}\left[\Pi[X_0, W_0]\right]}{\partial\,\mathrm{Log}\left[W\right]}\mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[W_1\right]\right), && \text{(B.18a)} \\
&\simeq \frac{1}{1-S}\beta_{\mathrm{X}} - \frac{S}{1-S}\beta_{\mathrm{W}}, && \text{(B.18b)} \\
&\simeq \beta_{\mathrm{X}} + \left(\beta_{\mathrm{X}} - \beta_{\mathrm{W}}\right)\frac{S}{1-S}, && \text{(B.18c)}
\end{aligned}$$

where $\beta_{\mathrm{X}} \equiv \mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[X_1\right]\right)$ and $\beta_{\mathrm{W}} \equiv \mathrm{Cov}\left(\mathrm{Log}\left[M_1\right], \mathrm{Log}\left[W_1\right]\right)$. Finally, Proposition 4 follows directly from Proposition 3 and from Assumption 3.

B.2. Extension with fixed costs and traditional operating leverage

In this section we incorporate fixed operating costs in the previous model to illustrate how traditional operating leverage affects our analysis.

Fixed operating costs are given by fK, so that optimized operating profits are now given by

$$\Pi^{\mathrm{f}} = \max_{L}\{XF[K,L] - LW - fK\}, \tag{B.19}$$

where the superscript f in Π^{f} denotes operating profits when under fixed operating costs. Note that we can define the share of fixed costs to profits net of labor costs, $S^{\mathrm{f}} \equiv {}^{fK}\!/_{Y(1-S)}$, where $S = {}^{WL}\!/_{XF[K,L]}$ is the labor share (not including fixed costs), so that

$$\begin{aligned} \Pi^{\mathrm{f}} &= Y(1-S)(1-S^{\mathrm{f}}) && \text{(B.20a)} \\ &= \Pi(1-S^{\mathrm{f}}), && \text{(B.20b)} \end{aligned}$$

where Π are operating profits in the otherwise identical case without fixed costs. Overall operating leverage includes components from labor leverage and from *traditional* operating leverage as given by

$$\begin{aligned} 1 + \text{Operating Leverage} &= \frac{d\,\mathrm{Log}[\Pi^{\mathrm{f}}]/d\,\mathrm{Log}[X]}{d\,\mathrm{Log}[Y]/d\,\mathrm{Log}[X]}, && \text{(B.21a)} \\ &= \frac{d\,\mathrm{Log}[\Pi]/d\,\mathrm{Log}[X]\left(1 + \frac{1}{1-S^{\mathrm{f}}}\right)}{d\,\mathrm{Log}[Y]/d\,\mathrm{Log}[X]}, && \text{(B.21b)} \\ &= \underbrace{\frac{1 + \frac{S}{1-S}(1-\theta)}{1 + \gamma\frac{S}{1-S}(1-\theta)}}_{\text{Labor Lev.}} \underbrace{\left(1 + \frac{1}{1 - \frac{fK}{\Pi}}\right)}_{\textit{Traditional}\text{ Op. Lev.}}. && \text{(B.21c)} \end{aligned}$$

Expression (B.21c) shows how labor leverage and traditional leverage interact and magnify each other.

C. Solution to the value of the firm

We start with the standard PDE for the value of the firm implied by the condition that the discounted value of the gains portfolio that reinvests the firm's dividends is a martingale. The solution to the value of a firm can be expressed as a function of its TFP X and its labor share S, $V[X_{\mathrm{t}}, S_{\mathrm{t}}]$. Given that operating profits (Equation (24)) are homogeneous of degree one in X and S, we conjecture and later verify that the value of the firm is also homogeneous of degree one in X and S. That is, we assume the existence of a function v such that $V[X,S] = XKv[S]$. Let $g[s] = v[S]$ where $s_{\mathrm{t}} \equiv \mathrm{Log}[S_{\mathrm{t}}]$. The homogeneity of the value of the firm allows us to simplify the PDE into the following ordinary differential equation (ODE):

$$h[s_{\mathrm{t}}] + c_0 g[s_{\mathrm{t}}] + c_1 g'[s_{\mathrm{t}}] + c_2 g''[s_{\mathrm{t}}] = 0, \tag{C.22}$$

where

$$h[s_t] \equiv \begin{cases} (1-\alpha)^{\frac{1}{\rho}} (1-\mathrm{Exp}[s_{\mathrm{t}}])^{1-\frac{1}{\rho}}, & \text{if} \quad \mathrm{Exp}[s_{\mathrm{t}}] < 1, \\ 0, & \text{if} \quad \mathrm{Exp}[s_{\mathrm{t}}] \geq 1, \end{cases}$$

$$c_0 \equiv -(r - \lambda + \eta\rho_{\mathrm{X}}\sigma_{\mathrm{X}} - \mu_{\mathrm{X}}),$$

$$c_1 \equiv \left(\eta(\rho_{\mathrm{W}}\sigma_{\mathrm{W}} - \rho_{\mathrm{X}}\sigma_{\mathrm{X}}) + \frac{1}{2}\left(2\mu_{\mathrm{X}} - 2\mu_{\mathrm{W}} - 2\rho_{\mathrm{X}}\rho_{\mathrm{W}}\sigma_{\mathrm{X}}\sigma_{\mathrm{W}} + \sigma_{\mathrm{X}}^2 + \sigma_{\mathrm{W}}^2\right)\right)\left(\frac{\rho}{1-\rho}\right),$$

$$c_2 \equiv \frac{\rho^2\left(\sigma_{\mathrm{W}}^2 + \sigma_{\mathrm{X}}^2 - 2\rho_{\mathrm{W}}\rho_{\mathrm{X}}\sigma_{\mathrm{W}}\sigma_{\mathrm{X}}\right)}{2(1-\rho)^2}.$$

The general solution to (C.22) is known and is given by

$$g[s_t] = v[S_t] = \begin{cases} C_1 S^{x_1} + C_2 S^{x_2} + \frac{(1-\alpha)^{1/\rho}}{c_2(x_1-x_2)}\left(-\frac{S\,{}_2F_1(1-\mathrm{x}1, {}^1\!/\!\rho; 2-\mathrm{x}1; S)}{x_1-1} + \frac{{}_2F_1(-x_1, {}^1\!/\!\rho; 1-x_1; S)}{x_1} + \frac{S\,{}_2F_1(1-x_2, {}^1\!/\!\rho; 2-x_2; S)}{x_2-1} - \frac{{}_2F_1(-x_2, {}^1\!/\!\rho; 1-x_2; S)}{x_2}\right), & \text{if} \quad S_{\mathrm{t}} < 1, \\ D_1 S^{x_1} + D_2 S^{x_2}, & \text{if} \quad S_{\mathrm{t}} \geq 1, \end{cases}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and x_1 and x_2 are the positive and negative roots of the fundamental polynomial of the ODE in Equation (C.22), respectively. Since the value of the homogeneous solution can not grow without bound as $S \to 0$ or as $S \to \infty$ the constants in the homogeneous solution associated with S^{x_2} (i.e., C_2) in the active region $S < 1$ and S^{x_1} (i.e., D_1) in the inactive region $S \geq 1$ must be zero. What remains is to find the constants C_1 and D_2 such that the smooth-pasting conditions at the active–inactive region threshold hold. The solution for these constants is given by:

$$C_1 = -\frac{(1-\alpha)^{1/\rho}}{c_2(x_1 - x_2)} \frac{G[2 - {}^1\!/\!_\rho]\, G[-x_2]}{G[2 - {}^1\!/\!_\rho - x_2]}, \tag{C.24a}$$

$$D_2 = -\frac{(1-\alpha)^{1/\rho}}{c_2(x_1 - x_2)} \frac{G[2 - {}^1\!/\!_\rho]\, G[-x_1]}{G[2 - {}^1\!/\!_\rho - x_{12}]}, \tag{C.24b}$$

where G represents the Gamma function.

References

Arias, A., Hansen, G. D., Ohanian, L. E., 2007. Why have business cycle fluctuations become less volatile? Economic Theory 32, 43–58.

Davis, S. J., Haltiwanger, J. C., Schuh, S., 1998. Job Creation and Destruction. MIT Press.

Dimson, E., 1979. Risk measurement when shares are subject to infrequent trading. Journal of Financial Economics 7, 197–226.

Fama, E. F., French, K. R., 1993. Common risk factors in the returns on stocks and bonds. Journal of Financial Economics 33, 3–56.

Lewellen, J., Nagel, S., 2006. The conditional CAPM does not explain asset-pricing anomalies. Journal of Financial Economics 82, 289–314.