Online Appendix for
“Competition, Profitability, and Discount Rates”

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A Price Wars Concern Investors

Price wars represent a prominent form of intensified competition in product markets. To show that price wars present a serious concern for investors, we give a few headline quotes in Online Appendix A.1, a few examples of analyst reports in Online Appendix A.2, and a case study in Online Appendix A.3.

A.1 Headline Quotes for Price Wars and Stock Returns

We cite a few examples of media headlines that show how price wars can depress firms’ stock returns.

- “Eastman Kodak Co. reported a 72% rise in third-quarter earnings, but its stock plunged 13% as investors focused on weaker sales and renewed fears that the company’s price war with rival Fuji Photo Film Co. is far from over.” – The Wall Street Journal on October 14th of 1998.

- “Best Buy Co. shares plunged 11% Tuesday, after the electronics chain warned investors about price war fears.” – The Wall Street Journal on November 20th of 2013.

- “US telecoms price war takes a toll on profits. Shares in the group fell by 3.9 per cent in early New York trading on Tuesday.” – Financial Times on December 9th of 2014.


- “Target shares dive as it shifts to cut-price strategy.” – Financial Times on February 28th of 2017.

- “Price war eats into the profits of pharmaceutical wholesalers and manufacturers alike and erases billions of dollars of the market value in recent days.” – The Wall Street Journal on August 5th of 2017.


- “Investors purge Infinera stock on price war concerns, ignore Q1 results.” – SDxCentral on May 10th of 2018.

- “The exchange-traded fund price war is intensifying and spreading, with BlackRock chainsawing the costs of $50bn worth of ETFs and investors continuing to flood into the cheapest options in the passive investment industry.” – Financial Times on June 11th of 2018.


- “Coffee price war takes jolt out of Dunkin’ results.” – Financial Times on September 27th of 2018.


A.2 Analyst Report Coverage on Price Wars

We cite a few analyst reports taken from the Investext (Thomson ONE) database that provide coverage and comments on price wars.

- Figure A.1 shows Credit Suisse’s coverage on the food retail industry.

- Figure A.2 shows Salomon Smith Barney’s coverage on Compaq.
• Figure A.3 shows Indigo Equity Research’s coverage on AT&T.
• Figure A.4 shows Cowen’s coverage on Dick’s Sporting Goods.

Figure A.1: Credit Suisse’s coverage on the food retail industry.

A.3 A Case Study: 1992 Airline Price War

Figure A.5 shows an example of an actual price war. On April 9th, 1992, American Airline, the U.S.’ largest carrier, launched the “value pricing” strategy, which triggered a price war in the airline industry (see, e.g., Michael and Silk, 1993a,b,c; Besanko, 2002). The price war ended in October 1992 when American Airline officially abandoned this strategy (see, e.g., McDowell, 1992). Investors were seriously concerned about the price war. Stock prices of airline
Compaq Computer (CPQ)"

**CPQ: PRICE WAR ROLLS ON - REITERATE 3H RATING WITH $24 TARGET**

**SUMMARY**

- Compaq reported 1Q01 results yesterday after the close. Revenue of $9.2B (-3% yoy) was slightly below our estimate of $9.3B, but in line with revised guidance of $9.0-9.2B. Diluted EPS of $0.12 (excluding one time charges), were one penny below consensus but within revised guidance of $0.12-0.14.

- Consumer demand remains weak worldwide and price pressure is intense in PCs and Intel-based servers. Mgmt also cited price pressure in enterprise storage and even business critical server (Alpha and Himalaya).

- The company now plans to reduce headcount by 7,000 employees worldwide versus prior guidance of 5,000. These reductions come from closure of manufacturing plants, the combination of the commercial and consumer PC divisions and cuts in administrative areas. Mgmt hopes to reduce op ex by $500M annually.

- 2Q revenue is now expected to decline to $9.0B and earnings are expected to decline to $0.05 (versus prior consensus of $0.17). The downward revision reflects mgmt’s intent to 1) reduce PC and Intel-based server inventories by $500M, 2) price more aggressively, and 3) spend more on demand generation in core markets to defend mkt share against Dell and IBM.

Figure A.2: Salomon Smith Barney’s coverage on Compaq.

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AT&T

**Overview of Q4 2014 - Telecoms price war deepens**

- Q4 revenue growth of 4% accelerated 1% from last Q. EBIT* fell 8%, as margins fell 1.5% to 14.0%. EPS’ growth of 4% (to $0.55), was driven by: lower interest payments, 2% share repurchases and a lower tax rate* (down 4% to 29%).

- Wireless revenue growth of 8% was driven by 0% more subscribers; Blended ARPU fell. GoQ Net Additions of 1.9 m subscribers, was driven by PostPaid (+0.8 m), and Devices (+1.3 m); of these Devices 0.8m were connected cars. Also, prepaid subscribers fell 0.2m.

- A key issue remains deteriorating EBIT margins as the wireless price war continues. Staff competition was also evident in high bids for wireless spectrum at the AWS-3 auction. News flow suggests that this price competition will not let up and will pressurize ARPU. Continued large charges for pensions & asset write-downs also limit upside stock potential.

- Low fundamental valuation $24 (-28%) - margin risk from price war

We expect the wireless price war to worsen and weak US economic growth to drag down AT&T’s margins. In turn, this will hit EPS and pressurize valuation multiples. AT&T’s strategic repositioning with major acquisitions and asset disposals are wild cards. Also, we believe equity markets are fundamentally overvalued (mostly due to low interest rates & QE) and expect >50% correction over time (see our 2015 Investment Outlook).

Figure A.3: Indigo Equity Research’s coverage on AT&T.
companies plummeted after the initiation of the price war and continued to drop. The three largest U.S. airlines (American Airline, Delta Airline, and United Airline) lost more than 20% of their market value before their stock prices rebounded in response to the end of the price war.

Figure A.5: 1992 Airline Price War.

B Data

B.1 The Industry Concentration Ratio

We use the U.S. Census concentration ratio data from 1987, 1992, 1997, 2002, 2007, and 2012 to compute the time-series maximal and mean revenue shares for the top four firms (CR4) and top eight firms (CR8) in each four-digit SIC industry.
The concentration ratios are given at the six-digit NAICS level after 1997. We follow Ali et al. (2008) and convert the ratios to the four-digit SIC level. Figure A.6 plots the histogram of the max CR4 (panel A1), max CR8 (panel B1), mean CR4 (panel A2), and mean CR8 (panel B2) in all four-digit SIC industries. Red vertical lines represent the cross-sectional mean values.

![Histogram of Revenue Share](image)

**Note:** This figure plots the histogram of the total revenue share of the top four and top eight firms in four-digit SIC industries. We use the U.S. Census concentration ratio data from 1987, 1992, 1997, 2002, 2007, and 2012 to compute the time-series maximal and mean revenue shares for the top four firms (CR4) and the top eight firms (CR8) in each four-digit SIC industry. The concentration ratios are given at the six-digit NAICS level after 1997. We follow Ali et al. (2008) and convert the ratios to the four-digit SIC level. We plot the histogram of the max CR4 (panel A1), max CR8 (panel B1), mean CR4 (panel A2), and mean CR8 (panel B2) in all four-digit SIC industries. Red vertical lines represent the cross-sectional mean values.

**Figure A.6: Revenue share of the top four and top eight firms in four-digit SIC industries.**

### B.2 Matching PatentsView with CRSP/Compustat/Capital IQ

In this appendix, we detail the procedure for matching the data from PatentsView, CRSP/Compustat, and Capital IQ. We first drop patent assignees that are classified by PatentsView as individuals and government agencies, because these assignees are not associated with any particular industry. We then clean assignee names in PatentsView and firm names in CRSP/Compustat and Capital IQ following the approach of Hall et al. (2001). To elaborate, we remove punctuation and clean special characters. We then transform the names into upper case and standardize them. For example, "INDUSTRY" is standardized to "IND" and "RESEARCH" to "RES"; and corporate form words (e.g., "LLC" and "CORP") are dropped.

\footnote{The PatentsView data are available at \url{http://www.patentsview.org/download/}.}
Matching PatentsView with CRSP/Compustat. We match patent assignees in PatentsView with firms in CRSP/Compustat based on standardized names. We use the fuzzy name-matching algorithm (matchit command in Stata), which generates matching scores (Jaccard index) for all name pairs of patent assignees in PatentsView and firms in CRSP/Compustat.\(^2\) We obtain a pool of potential matches based on two criteria: (1) the matching score must be higher than 0.6; and (2) the first three letters of the patent assignee name must be the same as those of firms in CRSP/Compustat.\(^3\) We then manually identify exact matches from all potential matches.\(^4\)

As pointed out by Lerner and Seru (2017), one major challenge in linking patent data to CRSP/Compustat is that some patent assignees are subsidiaries of firms in CRSP/Compustat. For these assignees, we cannot directly match them with CRSP/Compustat based on firm names. To deal with this challenge, the NBER patent data (Hall et al., 2001) use the 1989 edition of the Who Owns Whom directory (now known as the D&B WorldBase - Who Owns Whom) to match subsidiaries to parent companies. Kogan et al. (2017) purged the matches identified from the NBER patent data, and extended the matching between patent data and CRSP/Compustat to 2010. For those patent assignees that are subsidiaries of firms in CRSP/Compustat, we augment our matches by incorporating the data of Kogan et al. (2017) for patents granted before 2010. For patents granted after 2010, we use the subsidiary-parent link table from the 2017 snapshot of the Orbis data to match subsidiaries in PatentsView to their parent firms in CRSP/Compustat.

Matching PatentsView with Capital IQ. We match the remaining patent assignees in PatentsView with firms in Capital IQ following the same matching procedure. To keep the workload manageable, we drop firms in Capital IQ whose assets are worth less than $100 million (in 2017 dollars). Because our focus is on the US product market, we also drop foreign firms whose asset value is below the 90th percentile of the asset value distribution among firms in the CRSP/Compustat sample in each year. This is because small foreign firms are less likely to have a material impact on the competition environment of the US product market. We match PatentsView to Capital IQ directly using the information on subsidiaries provided by Capital IQ.

C Mathematical Details of the Extended Model

In this appendix section, we provide mathematical details for the extended model with endogenous jumps developed in Appendix A.1. Denote by \(V^{C}_{i,t} \equiv V^{C}_i(x_t)\) firm \(i\)'s value in the equilibrium where collusion and non-collusion are chosen optimally, pinned down by

\[
0 = \begin{cases} 
\Lambda_t \left[ \Pi_i(\theta^{C}_{i,t}, \theta^{C}_{j,t})M_{i,t} - \lambda V^{C}_{i,t} \right] dt + \mathbb{E}_t \left[ d \left( \Lambda_t V^{C}_{i,t} \right) \left| \theta^{C}_{i,t}, \theta^{C}_{j,t} \right| \right], & \text{if } \Gamma_{i,t} \geq \nu \text{ for all } i, \quad (I) \\
\max_{\theta_{i,t}} \Lambda_t \left[ \Pi_i(\theta^{N}_{i,t}, \theta^{N}_{j,t})M_{i,t} - \lambda V^{C}_{i,t} \right] dt + \mathbb{E}_t \left[ d \left( \Lambda_t V^{C}_{i,t} \right) \left| \theta^{N}_{i,t}, \theta^{N}_{j,t} \right| \right], & \text{otherwise}, \quad (II)
\end{cases}
\]

where \(\theta^{C}_{i,t} \equiv \theta^{C}_i(x_t)\) with \(i = 1, 2\) in \((I)\) are collusive profit margins and \(\theta^{N}_{i,t} \equiv \theta^{N}_i(x_t)\) with \(i = 1, 2\) in \((II)\) are non-collusive profit margins, which are the solution to the joint problems of \((II)\)\(^5\), and \(\Gamma_{i,t}\) is the benefit intensity of the union of the sample sets, and is defined as the size of the intersection divided by the size of the union of the sample sets. The index ranges between 0 and 1, reflecting zero to perfect similarity.

\(^2\)The Jaccard index measures the similarity between finite sample sets, and is defined as the size of the intersection divided by the size of the union of the sample sets.

\(^3\)These two matching criteria are sufficiently conservative to ensure that exact matches are included in the pool of potential matches. For example, among all the exact matches in the first quarter of 2016, 98% satisfy the two matching criteria and are included in our pool of potential matches.

\(^4\)We rely on assignee names in PatentsView and firm names in CRSP/Compustat, in addition to location information in both datasets, to identify matches.

\(^5\)The equilibrium profit margin here, \(\theta^{N}_i(x_t)\), is different from those in the non-collusive equilibrium that solve HJB Eq. (18) in the main text because of the difference in continuation value.
collusion relative to non-collusion, defined by

$$\Gamma_{i,t} \equiv \Pi_i(\Theta^C_i) - \Pi_i(\Theta^N_i) + \frac{\mathbb{E}_t \left[ \Lambda_t V^C_{i,t+dt} \right] - \mathbb{E}_t \left[ \Lambda_t V^C_{i,t+dt} \right] \Theta^N_i}{\Lambda_t M_i dt}.$$

where $\Theta^C_i = (\theta^C_{1,t}, \theta^C_{2,t})$ and $\Theta^N_i = (\theta^N_{1,t}, \theta^N_{2,t})$ are the collusive and non-collusive profit margin vectors respectively. The intuition behind characterizations (I) and (II) of $V_i^C(x_i)$ is straightforward: firms within the industry would choose to collude on the profit-margin scheme $\Theta^C(\cdot)$ over $[t, t + dt]$ if the benefit of collusion exceeds the non-pecuniary cost (i.e., $\Gamma_{i,t} \geq \nu$ for both $i = 1, 2$). In this case, the relationship between $V_i^C(x_i)$ and $V_i^C(x_{t+dt})$ is characterized by (I).

Alternatively, firms within the industry would choose not to collude over $[t, t + dt]$ if the benefit of collusion is smaller than the non-pecuniary cost (i.e., $\Gamma_{i,t} < \nu$ for some $i$). In this case, the relationship between $V_i^C(x_t)$ and $V_i^C(x_{t+dt})$ is characterized by (II).

Firm $i$’s deviation value $V^D_i(x_t) = V^D_i(x_{t+dt})$ is given by the following HJB equations:

$$0 = \begin{cases} \max_{\theta_{i,t}} \Lambda_t \left[ \Pi_i(\theta_{i,t}, \theta^C_{j,t}) M_i - \lambda V^D_i - \xi \left( V^D_i - V^N_i \right) \right] dt + \mathbb{E}_t \left[ d(\Lambda_t V^D_i) \right]_{\theta_{i,t}, \theta^C_{j,t}}, & \text{if } \Gamma_{i,t} \geq \nu \text{ for all } i, \quad (III) \\ \max_{\theta_{i,t}} \Lambda_t \left[ \Pi_i(\theta_{i,t}, \theta^N_{j,t}) M_i - \lambda V^D_i \right] dt + \mathbb{E}_t \left[ d(\Lambda_t V^D_i) \right]_{\theta_{i,t}, \theta^N_{j,t}}, & \text{if deviation is not punished} \\ \max_{\theta_{i,t}} \Lambda_t \left[ \Pi_i(\theta_{i,t}, \theta^C_{j,t}) M_i - \lambda V^D_i \right] dt + \mathbb{E}_t \left[ d(\Lambda_t V^D_i) \right]_{\theta_{i,t}, \theta^C_{j,t}}, & \text{otherwise}, \quad (IV) \end{cases}$$

where $\theta^N_{i,t} \equiv \theta^N_i(x_t)$ with $i = 1, 2$ are the non-collusive profit margins that solve the maximization problems in (IV), and $V^N_i(x_t)$ with $i = 1, 2$ are firm values in the non-collusive equilibrium.

### D Illustration of Equilibrium Concepts

In this appendix, we illustrate the dynamic game-theoretic equilibrium within an industry in our baseline model. We start by illustrating the non-collusive equilibrium in Section D.1. In Section D.2, we illustrate the collusive equilibrium that naturally arises from the dynamic repeated interaction between the two firms. The collusive equilibrium is a subgame perfect Nash equilibrium that is endogenously sustained when the non-collusive equilibrium is used as punishment. In Section D.3, we illustrate the incentive-compatibility constraints and the determination of collusive profit margins in the collusive equilibrium.

#### D.1 The Non-Collusive Equilibrium

In the non-collusive equilibrium, the two firms simultaneously set profit margins, taking the other firm’s profit margin as given. Thus, the equilibrium profit margins are determined by the intersection of the two firms’ optimal profit margins as a function of the other firm’s profit margin. Denote by $\hat{\theta}^N_1(M_{1,t}/M_t; \theta_{2,t})$ firm 1’s optimal profit margin as a function of its share of the customer base $M_{1,t}/M_t$ and firm 2’s profit margin $\theta_{2,t}$. Similarly, denote by $\hat{\theta}^N_2(M_{1,t}/M_t; \theta_{1,t})$ firm 2’s optimal profit margin as a function of firm 1’s share of the customer base $M_{1,t}/M_t$ and profit margin $\theta_{1,t}$.

In panel A of Figure A.7, the blue solid line plots firm 1’s optimal profit margin as a function of firm 2’s profit margin $\theta_{2,t}$, when the two firms have equal shares of the customer base (i.e., $M_{1,t}/M_t = 0.5$). The black dash-dotted line plots firm 2’s optimal profit margin as a function of firm 1’s profit margin $\theta_{1,t}$ for the same customer base share.

The intersection of the two curves (the blue solid circle) determines the equilibrium profit margins, i.e., $\theta^N_1(0.5)$ and $\theta^N_2(0.5)$:

$$\theta^N_1(0.5) = \hat{\theta}^N_1(0.5; \theta^N_2(0.5)) \quad \text{and} \quad \theta^N_2(0.5) = \hat{\theta}^N_2(0.5; \theta^N_1(0.5)). \quad (A.1)$$
The two firms set exactly the same profit margins when their customer base shares are identical. Both curves are upward sloping, indicating that there exists strategic complementarity in setting profit margins in the non-collusive equilibrium: both firms tend to set lower profit margins when the other firm’s profit margin is lower. This is because when the other firm’s profit margin is lower, the price elasticity of demand endogenously increases, motivating the firm to lower its own profit margin. Because of such strategic complementarity, the non-collusive equilibrium features low profit margins for both firms. To see this clearly, suppose firm 2 sets $\theta_2, t = 0.46$. Then firm 1’s best response is to set $\theta_1, t = 0.36$ ($A_1$). Given that firm 1’s profit margin is now lower than that of firm 2, firm 2 will further lower its profit margin to $\theta_2, t = 0.29$ ($A_2$). But now that firm 2’s profit margin is lower than that of firm 1, firm 1 will again lower its profit margin to $\theta_1, t = 0.22$ ($A_3$). This cycle continues until the profit margins reach equilibrium values. Such profit margin adjustments occur instantaneously in a rational expectation equilibrium.6

In panel B, we investigate how firms adjust their profit margins when their customer base shares change. The blue solid line and the black dash-dotted line represent the same benchmark case (i.e., $M_1, t / M_t = 0.5$) as in panel A. The red dashed line and the red dotted line refer to the profit margins set by the two firms when firm 1’s customer base share $M_1, t / M_t$ increases from 0.5 to 0.8 (thus firm 2’s customer base share decreases from 0.5 to 0.2 accordingly). It is shown that firm 1’s optimal profit margin function shifts upward and that of firm 2 shifts to the left, implying that both firms tend to set higher profit margins when their respective shares of the customer base increase. Intuitively, the firm’s influence on the equilibrium price index increases with its customer base share (see Eq. (2) in the main text). Therefore, a larger share increases the firm’s market power and lowers the price elasticity of demand, resulting in higher

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6The dynamics of profit margin adjustment is related to the old tradition that used tâtonnement or cobweb dynamics to capture the off-equilibrium adjustment of prices in Walrasian economies.
We now turn to the illustration of the collusive equilibrium. In the benchmark equilibrium \((N_0)\), the profit margins are \(\theta_{1,N_0}\) and \(\theta_{2,N_0}\). A larger customer base share \(M_{1,t}/M_t\) shifts the equilibrium to \(N_2\), and the new equilibrium profit margins satisfy \(\theta_{1,N_2} > \theta_{1,N_0}\) and \(\theta_{2,N_2} < \theta_{2,N_0}\). However, if firm 2 were to keep its profit margin decisions unchanged (represented by the black dash-dotted line), the new equilibrium would be \(N_1\), with \(\theta_{1,N_1} > \theta_{1,N_2}\), indicating that firm 1 would raise its profit margin more in response to the increase in its customer base share \(M_{1,t}/M_t\). Therefore, firm 1’s profit margin is less responsive because it anticipates that firm 2 would lower its profit margin \(\theta_{2,t}\) (as captured by the red dotted line). Such strategic concerns result in a smaller increase in firm 1’s profit margin \(\theta_{1,t}\), which helps prevent too big a reduction in the demand for its goods.

Panel C shows that when firm 1’s customer base share increases, it sets higher profit margins (blue solid line) while firm 2 sets lower profit margins (black dashed line) symmetrically in equilibrium. This is because firm 1 is gaining whereas firm 2 is losing market power (see Eq. (2) in the main text). Moreover, both firms have higher value when their customer base shares increase (see panel D).

### D.2 The Collusive Equilibrium

We now turn to the illustration of the collusive equilibrium. In the collusive equilibrium, both firms set profit margins according to the collusive profit-margin scheme \(\theta_{2}^C(M_{1,t}/M_t, \gamma_t)\).

In panel A of Figure A.8, we compare the firm’s profit margins in the collusive and the non-collusive equilibria. As the two firms are symmetric, we only focus on illustrating firm \(i\)'s profit margin. The black dashed line plots firm \(i\)'s profit margin in the non-collusive equilibrium (as in panel C of Figure A.7) and the blue solid line in the collusive equilibrium. It can be seen that due to collusion, firm \(i\) sets higher profit margins than it would in the non-collusive equilibrium. When firm \(i\)'s customer base share increases, firm \(i\)'s collusive profit margin increases because of its increasing market power.

Interestingly, panel B shows that the ability to collude on higher profit margins, as reflected by the difference between the collusive and the non-collusive profit margin exhibits an inverted U shape. Profit margins increase the most due to collusion when the two firms have comparable customer base shares (i.e., \(M_{1,t}/M_t = 0.5\)). Intuitively, collusion allows both firms to set higher profit margins than they would be able to in the non-collusive equilibrium. However, the collusive profit-margin scheme has to be chosen such that both firms have no incentive to deviate given their current customer base shares. When firm \(i\) is dominating the market (i.e., with high \(M_{1,t}/M_t\)), it has little reason to form a collusive equilibrium as it already has high market power, which allows it to set a high profit margin in the non-collusive equilibrium anyway (see the black dashed line). On the other hand, when firm \(i\) has a low customer base share \(M_{1,t}/M_t\), firm \(j\) has little reason to form a collusive equilibrium because it already has high market power and can set a high profit margin in the non-collusive equilibrium anyway. Thus, it is easier to collude on higher profit margins when the two firms have more comparable customer base shares.

The above intuition is more clearly seen in two extreme cases. When firm \(i\)'s customer base share \(M_{1,t}/M_t = 1\), panel A shows that it sets a profit margin close to \(\frac{1}{\epsilon}\). This is the profit margin that firm \(i\) would choose when facing a price elasticity of demand \(\epsilon\). In this case, firm \(i\) essentially acts almost as an industry monopoly and sets profit margins to compete with firms in other industries. Thus, the constant industry-level price elasticity of demand is what determines firm \(i\)'s optimal profit margin in both the collusive and the non-collusive equilibria. By contrast, when firm \(i\)'s customer base share \(M_{1,t}/M_t = 0\), panel A shows that it sets a profit margin close to \(\frac{1}{\eta}\). This is the profit margin that firm \(i\) would choose when facing a short-run price elasticity of demand \(\eta\). In this case, firm 1 essentially acts almost as a price taker in the industry because it has little market power to influence the industry’s price index. Thus, the constant within-industry elasticity of substitution is what determines firm \(i\)'s optimal profit margin in both the collusive and the non-collusive equilibria.
Panel C compares firm $i$’s value in the collusive and non-collusive equilibria. Colluding on higher profit margins leads to higher firm value. Not surprisingly, due to the inverted-U-shaped pattern of collusive profit margins, the difference in firm value displays a similar inverted U shape (panel D) as the customer base share $M_{i,t}/M_t$ varies.

### D.3 Determination of Collusive Profit Margins

In this section, we clarify how the collusive profit margins are determined in equilibrium. In panel A of Figure A.8, the red dash-dotted line plots the optimal profit margin that firm $i$ would choose conditional on its deviation from the collusive profit-margin scheme.\(^7\) The panel shows that the optimal deviation profit margin is always lower than the collusive profit margin. This is intuitive because firms collude on higher profit margins than what they would set in the non-collusive equilibrium, and thus both firms have the incentive to undercut each other in order to increase contemporaneous demand. Whether firm $i$ would deviate depends on the gains from setting the optimal deviation profit margin. Intuitively, there are countervailing forces that determine the gains from deviation. If deviation is not punished by firm $j$, then firm $i$ would gain by stealing demand from firm $j$ through setting lower profit margins. However, if deviation is punished by firm $j$, then the equilibrium will switch to the non-collusive one which features low profit margins for both firms.

Whether the collusive equilibrium can be sustained depends on the level of collusive profit margins. A higher collusive profit margin would make deviation less attractive for firms.

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\(^7\)Here, we follow the standard game theory by considering one-shot deviation. That is, we consider what deviation profit margin firm $i$ would choose conditional on firm $j$ not deviating from the collusive equilibrium. The one-shot deviation property ensures a necessary and sufficient condition — that no player would profit from one-shot deviations — for a strategy profile of a finite extensive-form game to form a subgame perfect Nash equilibrium.
profit margin increases the profits from deviation and is more difficult to sustain in equilibrium. The collusive profit margins we choose are the profit margins that lie on the “Pareto-efficient frontier”, subject to the incentive-compatibility constraint that neither firm has the incentive to deviate in the collusive equilibrium. In panel C of Figure A.8, the red dash-dotted line plots the deviation value that firm $i$ would obtain by setting the optimal deviation profit margin (the red dash-dotted line in panel A). It is shown that firm $i$’s deviation value is exactly the same as its collusive value, indicating that it makes no difference to firm $i$ whether it sets the collusive profit margin or deviates from the collusive equilibrium. In other words, firm $i$’s incentive-compatibility constraints are binding. Because the collusive and deviation values are equal for any customer base share, firm $j$ also has no preference between collusion and deviation.

The incentive-compatibility constraint is violated if firms choose collusive profit margins above the blue solid line in panel A. We illustrate this in Figure A.9. To obtain a stark comparison, we assume that the collusive profit margin is set equal to $\frac{1}{\epsilon}$ (as shown by the blue solid line in panel A), which is the profit margin that maximizes the contemporaneous demand if the two firms can perfectly collude with each other and act like a monopoly.

The red dash-dotted line indicates that when firm $i$’s customer base share $M_{i,t}/M_t$ is lower than 0.72, it would set a significantly lower profit margin in an attempt to steal firm $j$’s customer base share. As a result, firm $i$’s deviation value is strictly larger than its collusion value (see the red dash-dotted line in panel B) when $M_{i,t}/M_t < 0.72$, indicating that the incentive-compatibility constraint is violated. Thus, requiring the two firms to collude on a higher profit margin as is done here does not generate a subgame perfect Nash equilibrium because one or both firms will deviate by setting a lower profit margin.

E Illustration of the Key Mechanism

We emphasize that the decrease in profit margins after positive discount rate shocks is caused by intensified competition rather than weakening aggregate demand.

To elaborate on this point, in panel A of Figure A.10, we plot the supply and demand curves for firm $i$’s product in collusive equilibrium. Fixing firm $i$’s customer base $M_{i,t}$, the supply curve (blue solid line) is flat because firm $i$ agrees to sell its product at collusive price $P_{C,i}$ irrespective of its contemporaneous demand. The demand curve (black dashed line) is downward-sloping, and represents Eq. (2) in the main text. The initial equilibrium is given by point $O_0$.

An increase in the discount rate (i.e., $\gamma_t$) reduces collusion incentives and weakens market power, shifting the supply curve downward to the blue dotted line. If the demand curve were unchanged, the new equilibrium would feature a much lower price and a much higher demand for firm $i$’s goods (point $O_1$). However, the demand curve also shifts downward to the black dash-dotted line, because the industry’s price index $P_t$ endogenously declines dramatically due to the self-fulfilling price undercutting. The new equilibrium is given by point $O_1$, featuring a much lower equilibrium...
price and a slightly higher equilibrium demand for firm i’s goods.

As illustrated in panel D, the intensified competition driven by positive discount rate shocks is caused initially by the downward shift in firm i’s supply curve owing to the self-fulfilling decline in its market power. The shift in the supply curve reduces firm i’s relative price $P_{i,t}/P_t$, which in turn increases firm i’s price elasticity of demand. When firm i’s price elasticity of demand is higher, forming a collusion is more difficult from competitor j’s perspective, because now firm i has more incentives to undercut prices. The diminished collusion incentive further induces a downward shift in firm j’s supply curve, which further reduces firm j’s relative price $P_{j,t}/P_t$, increasing firm j’s price elasticity of demand. Such a feedback loop leads to self-fulfilling weakened market power and lower profit margins.

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**Figure A.10: Impact of various shocks on equilibrium prices and quantities.**

By contrast, panels B and C highlight that short-run shocks cannot lead to self-fulfilling declines in firms’ profit margins. Panel B shows that a negative short-run demand shock (i.e., a decline in $M_t$) only generates a downward shift in the demand curve without affecting the supply curve. As a result, the change in equilibrium price and demand depends purely on the price elasticity of supply. Given a flat supply curve (infinite price elasticity of supply), firm i’s price in the new equilibrium (point $O_1$) is exactly the same as the initial equilibrium price (point $O_0$). Panel C shows that a negative short-run supply shock (i.e., an increase in $\omega$) only generates an upward shift in the supply curve without affecting the demand curve. As a result, the change in equilibrium demand and supply purely depends on the price elasticity of demand. As the demand curve is downward-sloping, the new equilibrium (point $O_1$) has a higher price and a lower demand for firm i’s goods.

Thus, the intensified competition caused by positive discount rate shocks creates shifts in both the demand and the supply curves, unlike the effect of short-run demand or supply shocks.

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8If the marginal cost of production increases with output, the supply curve would be upward-sloping. Then, a negative short-run demand shock would result in a lower equilibrium price and a lower equilibrium demand for firm i’s good. This is the standard negative effect of demand shocks on equilibrium prices in models with decreasing-return-to-scale production technology. We intentionally assume a constant marginal cost of production $\omega$ to eliminate this effect and cleanly present the effect from endogenous competition in the collusive equilibrium.
F Discussions on Model Elasticities

The parameters $\eta$ and $\epsilon$ capture the elasticities of substitution of goods produced within the same industry and across different industries respectively. In this section, we discuss the role of the two elasticities in collusion incentives and profit margins.

In our baseline calibration, we set $\eta > \epsilon$ to be consistent with empirical estimates. As we vary $\eta$ and $\epsilon$, the model can capture different degrees of within- and cross-industry competition. As we have shown in the main text, the price elasticity of demand for firm $i$ depends on both the within-industry elasticity $\eta$ and the cross-industry elasticity $\epsilon$ because firm $i$ simultaneously faces within-industry competition from firm $j$ and cross-industry competition from firms in other industries.

With $\eta > \epsilon$, within-industry competition is more fierce than cross-industry competition due to the higher elasticity of substitution among goods produced in the same industry. Thus, essentially the within-industry elasticity $\eta$ gives the upper bound of competition and hence determines the lower bound of profit margins, whereas the cross-industry elasticity $\epsilon$ gives the lower bound of competition and hence determines the upper bound of profit margins.

In particular, firm $i$ faces the highest level of competition when it becomes atomistic in the industry (i.e., $M_{i,t}/M_t = 0$). In this case, firm $i$ would set the profit margin close to $\frac{1}{\eta}$, determined by the within-industry elasticity $\eta$. However, when firm $i$ is atomistic, firm $j$ is essentially the monopoly in the industry, facing the minimal level of competition due to the absence of within-industry competition. Thus, firm $j$ would set the profit margin close to $\frac{1}{\epsilon}$, determined by the cross-industry elasticity $\epsilon$. Because firm $j$ is already setting a very high profit margin, it has no incentive to collude with firm $i$, although firm $i$ wants to collude due to its low profit margin.
Thus, the two firms have the incentive to collude with each other only when neither firm monopolizes the industry. In this case, collusion benefits both firms by alleviating within-industry competition so that profit margins increase, reflecting the cross-industry elasticity $\epsilon$ to a greater extent. Therefore, the existence of collusion incentive crucially depends on the assumption that $\eta > \epsilon$. If $\eta = \epsilon$, the level of competition does not change with the customer base share, and the firm would always set the profit margin close to $\frac{1}{\epsilon}$, determined by the cross-industry elasticity $\epsilon$.

Specifically, if $\eta = 1.6$ ($= \epsilon$), panel B of Figure A.11 shows that firm 1 always sets its profit margin close to $\frac{1}{\epsilon}$. In this case, achieving the collusive equilibrium does not further increase the two firms’ profit margins because they are already setting a very high profit margin consistent with what is implied by cross-industry competition. Firm $i$’s value increases linearly with its customer base share $M_i$, $t/M_t$ (see panel E). In panels C and F, we further increase $\eta = \epsilon = \infty$ to mimic an economy with perfect competition. The infinite elasticity results in zero profit margins (see panel C). Both firms attain zero value (see panel F) in equilibrium regardless of their customer base shares.

G The Extended Model

In this section, we incorporate heterogeneous loadings on aggregate cash-flow shocks to the baseline model with endogenous jumps in Appendix A.1. We show that our extended model is able to simultaneously rationalize both the cross-industry gross profitability premium and value premium.

Suppose the variable $x_t$ represents the growth rate of aggregate cash flows in the corporate sector, which follows a mean-reverting process

$$dx_t = -\kappa (x_t - \bar{x}) + \sigma_x dZ_{x,t}. \quad (A.2)$$

The industry-level expected growth rate of customer base is related to the growth rate of aggregate cash flows in the following way:

$$g_t = \psi(x_t - \bar{x}), \quad (A.3)$$

where the coefficient $\psi$ captures how sensitive the expected growth rate of the industry’s customer base is to variations in aggregate cash flows.

The evolution of firm $i$’s customer base (i.e., Eq. (6) in the main text) in the industry is modified to incorporate the exposure to aggregate cash-flow risk $dZ_{x,t}$ as follows:

$$dM_{i,t}/M_{i,t} = \left[\alpha (C_{i,t}/M_{i,t})^h + g_t - \rho\right] dt + \zeta dZ_t + \sigma_M dW_{i,t}, \quad (A.4)$$

implying that the firm loads on aggregate cash-flow risk $dZ_{x,t}$ due to the term $g_t$.

The SDF in our baseline model is modified to incorporate the aggregate cash-flow risk $dZ_{x,t}$:

$$\frac{d\Lambda_t}{\Lambda_t} = -r_f dt - \gamma_t dZ_t - \zeta dZ_{\gamma,t} - \chi dZ_{x,t}, \quad (A.5)$$

where the time-varying market prices of risk $\gamma_t$ evolves as in Eq. (17) of the main text and the Brownian motion $dZ_{\gamma,t}$ is the aggregate discount rate shock. The aggregate cash-flow shock $dZ_{x,t}$ is independent of $dZ_t$, $dZ_{\gamma,t}$, and $dW_{i,t}$ for $i = 1, 2$ in any industry.

To capture the value and growth firms, we assume that the parameter $\psi$ is heterogeneous across industries, which means that industries are exposed differently to aggregate cash-flow risk. This modeling approach is consistent with empirical evidence that value stocks are more exposed to aggregate cash-flow risk (Cohen et al., 2003; Campbell and Vuolteenaho, 2004; Bansal et al., 2005; Parker and Julliard, 2005; Hansen et al., 2008; Campbell et al., 2010; Santos and Veronesi, 2010).

In summary, industries are different from each other in both the exposure to aggregate cash-flow risk captured
by $\psi$, and the market leadership turnover rate captured by $\lambda$. Denote by $G(\psi, \lambda)$ the joint distribution of $\psi$ and $\lambda$ across industries. We specify the joint distribution using a flexible parametric method. That is, we specify the two marginal distributions and the dependence of the two variables using a copula. Specifically, the marginal (cumulative) distributions of $\psi$ and $\lambda$ are denoted by $F_\psi(\psi)$ and $F_\lambda(\lambda)$, respectively. We capture the correlation between the two industry characteristics $\lambda$ and $\psi$ using the Frank copula:

$$C(x_1, x_2) = \mathbb{P}(F_\lambda(\lambda) \leq x_1, F_\psi(\psi) \leq x_2) = -\frac{1}{\vartheta} \log \left[ 1 + \frac{(e^{-\vartheta x_1} - 1)(e^{-\vartheta x_2} - 1)}{e^{-\vartheta} - 1} \right],$$  \hspace{1cm} (A.6)

where the single parameter $\vartheta \in \mathbb{R}$ governs the dependence between two marginal distributions, $F_\lambda(\lambda)$ and $F_\psi(\psi)$. A higher $\vartheta$ implies that $\lambda$ and $\psi$ are more positively associated with each other. When $\vartheta = 0$, the two variables $\lambda$ and $\psi$ are independent.

### G.1 Discussions on the Model Mechanism

In an investment-based asset pricing model, value and growth firms are distinguished by their relative value of assets in place and their growth options (e.g., Berk et al., 1999; Gomes et al., 2003; Papanikolaou, 2011; Ai and Kiku, 2013; Kogan and Papanikolaou, 2014; Dou, 2017). The firms whose values depend more on assets in place tend to have a lower market-to-book or price-earnings ratio, and they are referred to as value firms. Thus, the value premium can be computed based on the long-short portfolio sorted on the market-to-book or price-earnings ratio. Lettau and Wachter (2007) expand the evidence on value premium by showing that portfolios, formed not only on the market-to-book ratio but also on the price-earnings ratio, the price-dividend ratio, and the price-cashflow ratio, generate expected excess returns that cannot be attributed to market beta.

In the models that do not explicitly account for firms’ strategic investment (e.g., Lettau and Wachter, 2007; Santos and Veronesi, 2010), the market-to-book ratio is not well defined. Thus, the price-earnings ratio can be used as an alternative empirical proxy to characterize value and growth firms and industries (Graham and Dodd, 1934; Basu, 1977, 1983; Ball, 1978; Rosenberg et al., 1985; Jaffe et al., 1989; Fama and French, 1992; Cochrane, 1999).

To gain more intuitions, let us start with the baseline model in Section A.1 of the main text, where $\lambda$ captures the only ex-ante heterogeneous characteristic across industries. In the baseline model, all else being equal, a higher $\lambda$ increases the price-earnings ratio but reduces the gross profitability of firms within the industry. It leads to an endogenous negative correlation between gross profitability and the price-earnings ratio in the model. The negative correlation implies that value firms tend to have high profitability. Thus, in the baseline model, the value premium sorted on the price-earnings ratio would be weakened after controlling for the gross profitability, and the gross profitability premium sorted on the gross profitability would be weakened after controlling for the price-earnings ratio. These predictions are opposite to what we observe in data.

By introducing the cash-flow shock and industry’s heterogeneous cash-flow loading $\psi$, our extended model can generate a positive correlation between gross profitability and the price-earnings ratio by properly calibrating a positive correlation between $\lambda$ and $\psi$ (i.e., calibrating $\vartheta$ in the Frank copula). The heterogeneous exposure to the cash-flow shock has been proven to play a first-order role in explaining the value premium (e.g., Campbell and Vuolteenaho, 2004; Lettau and Wachter, 2007; Santos and Veronesi, 2010). The extended model can simultaneously explain the value premium, the gross profitability premium, the positive relation between the gross profitability and the price-earnings ratio, the more pronounced value premium after controlling for the gross profitability, and the more pronounced gross profitability premium after controlling for the price-earnings ratio. This is because: (i) the profitability premium is mainly attributed to industries’ differential exposure to discount-rate shocks (i.e., $dZ_{\gamma,t}$), which is determined by

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9 Simultaneously modeling strategic investment and pricing decisions in a dynamic duopoly game would make the model very difficult to solve. This is because the collusion decision on prices also depends on the strategic decisions on investment, resulting in a high-dimensional fixed point problem.
heterogeneous \( \lambda \); (ii) the value premium is mainly attributed to industries’ differential exposure to the aggregate cash-flow risk (i.e., \( dZ_{x,t} \)), which is determined by heterogeneous \( \psi \); and (iii) the relation between the value premium and the profitability premium is attributed to the positive correlation between the price-earnings ratio and gross profitability, which is determined by the positive correlation between \( \lambda \) and \( \psi \).

G.2 Calibration

Regarding the parameters that also belong to the baseline model, we set the same values as in Table A.1 of the main text. Below, we discuss the calibration of parameters that solely belong to the extended model, summarized in panel A of Table A.1. We set \( \pi = 0.0189 \) so that the average growth rate of aggregate cash flows in the corporate sector is about 1.89%, roughly equal to the average growth rate of aggregate consumption. We set \( \kappa = 0.262 \) and \( \sigma_x = 0.0015 \) so that the implied persistence and predictable component of aggregate cash flows in the average industry are consistent with those of consumption growth calibrated by Bansal et al. (2012).

Note that in the baseline calibration of Section A.2, the values of \( \lambda \) across industries are bounded between \( \lambda \) and \( \bar{\lambda} \) with \( \lambda = 0 \) and \( \bar{\lambda} = 0.18 \). When conducting simulations, we assume that \( \lambda \) follows a uniform distribution and discretize \( [\lambda, \bar{\lambda}] \) into \( N = 10 \) grids with equal spacing, so that \( \lambda_1 = \lambda \) and \( \lambda_N = \bar{\lambda} \). Similarly, in the extended model, we assume that the values of \( \psi \) across industries follow a uniform distribution (unconditionally) and are bounded between \( \psi \) and \( \bar{\psi} \). We also discretize \( [\psi, \bar{\psi}] \) into \( N = 10 \) grids with equal spacing, so that \( \psi_1 = \psi \) and \( \psi_N = \bar{\psi} \). We normalize the lowest loading to be \( \psi = 0 \). We set \( \psi = 7 \) to generate an unconditional cross-industry value premium of 3.34%. The market price of aggregate cash-flow risk is set at \( \chi = 0.4 \). Finally, we calibrate the parameter \( \vartheta = 0.65 \) governing the correlation between \( \psi \) and \( \lambda \) so that the rank correlation between gross profitability and the price-earnings ratio is 0.13, as in the data.

G.3 Quantitative Results

Panel B of Table A.1 presents the expected returns of portfolios sorted on industries’ gross profitability. The cross-industry gross profitability premium implied by the model is 4.02%, roughly in line with the premium of 5.06% in the data. Panel C shows the expected returns of portfolios sorted on industries’ price-earnings ratio. The model generates a value premium of 3.04% compared with 3.90% in the data. Panel D presents the results of double sorts on gross profitability and the price-earnings ratio. The correlation between the two is 0.13 in both the model and the data. In the model, the gross profitability premium is 5.19% after controlling for the price-earnings ratio, which is higher than the unconditional gross profitability premium of 4.02%, reported in panel B. Moreover, the value premium is 3.93% after controlling for gross profitability, which is higher than the unconditional value premium of 3.04%, reported in panel C. These double-sort results are consistent with those in the data.

H The Numerical Algorithm

In this section, we detail the numerical algorithm that solves the model under the risk-neutral measure. To give an overview, our algorithm proceeds in the following steps:

1. We solve the non-collusive equilibrium, which requires solving the static Nash equilibrium of the dynamic game played by two firms. The simultaneous-move dynamic game requires us to solve the intersection of the two firms’ best response (i.e., optimal profit margin) functions, which themselves are optimal solutions to coupled partial differential equations (PDEs).

2. We solve the collusive equilibrium using the value functions in the non-collusive equilibrium as the value of punishment. This requires solving a high-dimensional fixed-points problem, because we are interested in the
Table A.1: Calibration and model-implied moments.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average agg. cash-flow growth</td>
<td>( \bar{\gamma} )</td>
<td>0.0189</td>
<td>Persistence of expected growth</td>
<td>( \kappa )</td>
<td>0.262</td>
</tr>
<tr>
<td>Volatility of expected growth</td>
<td>( \sigma_{\gamma} )</td>
<td>0.0015</td>
<td>Market price of risk for cash-flow shock</td>
<td>( \chi )</td>
<td>0.4</td>
</tr>
<tr>
<td>Range of cash-flow risk loadings</td>
<td>([\bar{\psi}, \bar{\psi}])</td>
<td>([0, 7])</td>
<td>Copula parameter</td>
<td>( \vartheta )</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Panel B: Gross profitability premium (%)

<table>
<thead>
<tr>
<th>Gross profitability quintile</th>
<th>Q1</th>
<th>Q5</th>
<th>Q5 − Q1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>5.20</td>
<td>10.26</td>
<td>5.06</td>
</tr>
<tr>
<td></td>
<td>([3.69, 7.05])</td>
<td>([8.93, 11.58])</td>
<td>([2.74, 7.05])</td>
</tr>
<tr>
<td>Model</td>
<td>8.29</td>
<td>12.58</td>
<td>4.02</td>
</tr>
<tr>
<td></td>
<td>([6.54, 9.72])</td>
<td>([9.31, 14.69])</td>
<td>([2.89, 5.13])</td>
</tr>
</tbody>
</table>

Panel C: Value premium (%)

<table>
<thead>
<tr>
<th>Price-earnings ratio quintile</th>
<th>Q1</th>
<th>Q5</th>
<th>Q1 − Q5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>10.95</td>
<td>7.05</td>
<td>3.90</td>
</tr>
<tr>
<td></td>
<td>([8.60, 12.19])</td>
<td>([5.73, 8.62])</td>
<td>([0.90, 5.69])</td>
</tr>
<tr>
<td>Model</td>
<td>11.22</td>
<td>7.98</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>([9.35, 13.01])</td>
<td>([6.28, 9.62])</td>
<td>([2.05, 4.21])</td>
</tr>
</tbody>
</table>

Panel D: Double sorts on gross profitability (GP) and the price-earnings (P/E) ratio

<table>
<thead>
<tr>
<th>Correlation between GP and P/E ratio</th>
<th>GP premium controlling for P/E ratio</th>
<th>Value premium controlling for GP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.13</td>
<td>5.60</td>
</tr>
<tr>
<td></td>
<td>([0.12, 0.14])</td>
<td>([3.20, 7.61])</td>
</tr>
<tr>
<td>Model</td>
<td>0.13</td>
<td>5.19</td>
</tr>
<tr>
<td></td>
<td>([0.09, 0.17])</td>
<td>([3.87, 6.42])</td>
</tr>
</tbody>
</table>

Note: When constructing the model moments, we simulate a sample of 2,000 industries for 150 years with an 80-year burn-in period. We then compute the model-implied moments similar to the data. For each moment, the table reports the average value of 2,000 simulations and the 2.5th and 97.5th estimated percentiles of the simulated distribution (in brackets).

We highest collusive profit margins with binding incentive-compatibility constraints. We thus use an iteration method inspired by Abreu et al. (1986, 1990), Ericson and Pakes (1995), and Fershtman and Pakes (2000).

Note that standard methods for solving PDEs with free boundaries (e.g., finite difference or finite element) can easily lead to non-convergence of value functions. To mitigate such problems and obtain accurate solutions, we solve the continuous-time game using a discrete-time dynamic programming method, as in Dou et al. (2019), Chen et al. (2019), Dou and Ji (2020), Dou et al. (2020). In Appendix H.1, we present the discretized recursive formulation for the baseline model, including firms’ problems in the non-collusive equilibrium, the collusive equilibrium, and deviation. In Appendix H.2, we present the algorithm for the extended model. In Appendix H.3, we discuss how we discretize the stochastic processes, time grids, and state variables in the model. Finally, in Appendix H.4, we discuss the details of implementing our numerical algorithms, including finding the equilibrium profit margins in the non-collusive equilibrium and solving for the optimal collusive profit margins.

H.1 The Baseline Model

We solve the model in risk-neutral measure, where we have

\[
\begin{align*}
\text{d}Z_t & = -\gamma_t \text{d}t + \text{d}Z_t, \\
\text{d}Z_{\gamma,t} & = -\zeta \text{d}t + \text{d}Z_{\gamma,t}.
\end{align*}
\]

(A.7)

(A.8)

Because firm 1 and firm 2 are symmetric, one firm’s value and policy functions are obtained directly from those of the other firm. In this section, we illustrate firm 1’s problem in our baseline model. We first illustrate the non-collusive
equilibrium and then the collusive equilibrium.

H.1.1 The Non-Collusive Equilibrium

Below, we present the recursive formulation for the firm’s value in the non-collusive equilibrium. Then we exploit linearity to simplify the problem and present the recursive formulation for the normalized firm value. Finally, we present the conditions that determine the non-collusive (Nash) equilibrium.

Recursive Formulation for the Non-Collusive Firm Value. The industry’s state is characterized by three state variables: firm 1’s customer base $M_{1,t}$, firm 2’s customer base $M_{2,t}$, and the market price of risk $\gamma_t$. Denote the value functions in the non-collusive equilibrium by $\hat{V}^N_i(M_{1,t}, M_{2,t}, \gamma_t)$ for $i = 1, 2$.

To characterize the equilibrium value functions, it is more convenient to introduce two off-equilibrium value functions. Let $\hat{V}^N_i(M_{1,t}, M_{2,t}, \gamma_t; \theta_{j,t})$ be firm $i (= 1, 2)$’s value when its competitor $j$’s profit margin is set at any (off-equilibrium) value $\theta_{j,t}$.

Firm 1 solves the following problem:

$$\hat{V}^N_1(M_{1,t}, M_{2,t}, \gamma_t; \theta_{2,t}) = \max_{\theta_{1,t}} \omega^{1-\epsilon} \theta_{1,t}(1-\theta_{1,t})^{\eta-1}(1-\theta_{2,t})^{\gamma-\eta} M_{1,t} \Delta t + e^{-(\gamma_f + \lambda) \Delta t} \mathbb{E}_t \left[ \hat{V}^N_i(M_{1,t+\Delta t}, M_{2,t+\Delta t}, \gamma_{t+\Delta t}) \right],$$

(A.9)

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \theta_t = \left[ \frac{M_{1,t}(1-\theta_{1,t})^{\eta-1} + M_{2,t}(1-\theta_{2,t})^{\eta-1}}{M_t} \right]^{\frac{1}{\eta-1}} \text{ with } M_t = M_{1,t} + M_{2,t}. \quad (A.10)$$

(2) The customer base evolves according to

$$M_{i,t+\Delta t} = M_{i,t} + \left[ \alpha(1-\theta_{i,t})^{\phi h} (1-\theta_{t})^{(\epsilon-\eta) h \omega - \rho} \right] M_{i,t} \Delta t + \zeta M_{i,t} (-\gamma_t \Delta t + \Delta \tilde{Z}_t) + \sigma M_{i,t} \Delta W_{i,t}, \quad \text{for } i = 1, 2. \quad (A.11)$$

(3) The aggregate state $\gamma_t$ evolves according to

$$\gamma_{t+\Delta t} = \gamma_t - \varphi (\gamma_t - \bar{\gamma}) \Delta t - \pi (-\zeta \Delta t + \Delta \tilde{Z}_{\gamma,t}). \quad (A.12)$$

Recursive Formulation for the (Normalized) Non-Collusive Firm Value. Exploiting the linearity, we normalize the firm’s value by $M_t = M_{1,t} + M_{2,t}$. Firm 1’s customer base share is $m_{1,t} = M_{1,t}/M_t$; firm 2’s customer base share is $m_{2,t} = M_{2,t}/M_t = 1 - m_{1,t}$. Define

$$v^N_1(m_{1,t}, \gamma_t) = \frac{V^N_1(M_{1,t}, M_{2,t}, \gamma_t)}{M_t} \quad (A.13)$$

$$v^N_1(m_{1,t}, \gamma_t; \theta_{j,t}) = \frac{\hat{V}^N_1(M_{1,t}, M_{2,t}, \gamma_t; \theta_{j,t})}{M_t}. \quad (A.14)$$

Firm 1 solves the following normalized problem:

$$\hat{v}^N_1(m_{1,t}, \gamma_t; \theta_{2,t}) = \max_{\theta_{1,t}} \omega^{1-\epsilon} \theta_{1,t}(1-\theta_{1,t})^{\eta-1}(1-\theta_{2,t})^{\gamma-\eta} m_{1,t} \Delta t + e^{-(\gamma_f + \lambda) \Delta t} \mathbb{E}_t \left[ \frac{M_{1,t+\Delta t}}{M_t} \hat{v}^N_1(m_{1,t+\Delta t}, \gamma_{t+\Delta t}) \right], \quad (A.15)$$

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \theta_t = \left[ m_{1,t}(1-\theta_{1,t})^{\eta-1} + (1-m_{1,t})(1-\theta_{2,t})^{\eta-1} \right]^{\frac{1}{\eta-1}}. \quad (A.16)$$

20
(2) Firm 1’s customer base share evolves according to
\[ m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1-\theta_{1,t})^{\eta h}(1-\theta_t)^{(e-\eta)h}\omega^{-\epsilon h} - \rho \right] m_{1,t}\Delta t + \varsigma m_{1,t}(-\gamma_t\Delta t + \Delta Z_t) + \sigma_MM_{1,t}\Delta W_{1,t}. \tag{A.17} \]

(3) The industry’s customer base evolves according to
\[ \frac{M_{t+\Delta t}}{M_t} = 1 + \varsigma(-\gamma_t\Delta t + \Delta Z_t) + \left[ \alpha(1-\theta_{1,t})^{\eta h}(1-\theta_t)^{(e-\eta)h}\omega^{-\epsilon h} - \rho \right] m_{1,t}\Delta t + \alpha(1-\theta_{2,t})\gamma_t(1-\theta_t)^{(e-\eta)h}\omega^{-\epsilon h} - \rho \right] (1-m_{1,t})\Delta t + \sigma_MM_{1,t}\Delta W_{1,t} + \sigma.MM(1-m_{1,t})\Delta W_{2,t}. \tag{A.18} \]

(4) The aggregate state \( \gamma_t \) evolves according to
\[ \gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \pi)\Delta t - \pi(-\gamma_t + \Delta Z_t). \tag{A.19} \]

**The Non-Collusive (Nash) Equilibrium.** Denote the equilibrium profit margin functions by \( \hat{\theta}_i^N(m_{1,t},\gamma_t) \) and the off-equilibrium profit margin functions by \( \hat{\theta}_i^N(m_{1,t},\gamma_t;\theta_{j,t}) \). Exploiting the symmetry between firm 1 and firm 2, we can obtain firm 2’s off-equilibrium value and policy functions as
\[
\hat{v}_i^N(m_{1,t},\gamma_t;\theta_{j,t}) = \hat{v}_2^N(1-m_{1,t},\gamma_t;\theta_{1,t}), \tag{A.20}
\hat{\theta}_i^N(m_{1,t},\gamma_t;\theta_{j,t}) = \hat{\theta}_2^N(1-m_{1,t},\gamma_t;\theta_{1,t}). \tag{A.21}
\]

Given firm \( j = 1, 2 \)’s profit margin \( \theta_{j,t} \), firm \( i \) optimally sets the profit margin \( \theta_{i,t} \). The non-collusive (Nash) equilibrium is derived from the fixed point—each firm’s profit margin is optimal given the other firm’s optimal profit margin:
\[
\hat{\theta}_i^N(m_{1,t},\gamma_t) = \hat{\theta}_1^N(m_{1,t},\gamma_t;\theta_2^N(m_{1,t},\gamma_t)), \tag{A.22}
\hat{\theta}_i^N(m_{1,t},\gamma_t) = \hat{\theta}_2^N(m_{1,t},\gamma_t;\theta_1^N(m_{1,t},\gamma_t)). \tag{A.23}
\]

The equilibrium value functions are given by
\[
\hat{v}_i^N(m_{1,t},\gamma_t) = \hat{v}_1^N(m_{1,t},\gamma_t;\theta_2^N(m_{1,t},\gamma_t)), \tag{A.24}
\hat{v}_i^N(m_{1,t},\gamma_t) = \hat{v}_2^N(m_{1,t},\gamma_t;\theta_1^N(m_{1,t},\gamma_t)). \tag{A.25}
\]

After finding the equilibrium value and solving the policy functions above, we can verify that the following conditions are satisfied due to symmetry:
\[
\hat{v}_1^N(m_{1,t},\gamma_t) = \hat{v}_2^N(1-m_{1,t},\gamma_t), \tag{A.26}
\hat{\theta}_1^N(m_{1,t},\gamma_t) = \hat{\theta}_2^N(1-m_{1,t},\gamma_t). \tag{A.27}
\]

**H.1.2 The Collusive Equilibrium**

Below, we present the recursive formulation for the firm’s value in the collusive equilibrium and then the one when the firm deviates from the collusive equilibrium. Finally, we present the incentive-compatibility constraints and the conditions that determine the optimal collusive profit margins.

**Recursive Formulation for the Collusive Firm Value.** In the collusive equilibrium, we can still exploit the linearity property and solve for firms’ value as a function of their customer base shares. Specifically, denote by \( \hat{v}_i^C(m_{1,t},\gamma_t;\tilde{\theta}^C(\cdot)) \)
firm $i$’s value in the collusive equilibrium with collusive profit margins $\tilde{\Theta}^C(\cdot)$. Note that because the two firms in the same industry are symmetric, the collusive profit margins satisfy $\tilde{\Theta}_1^C(m_{1,t}, \gamma_t) = \tilde{\Theta}_2^C(1-m_{1,t}, \gamma_t)$.

Firm 1 solves the following normalized problem:

$$\tilde{v}_1^C(m_{1,t}, \gamma_t; \tilde{\Theta}_1^C(\cdot)) = \omega^{1-\epsilon} \tilde{v}_1^C(m_{1,t}, \gamma_t)(1 - \tilde{\Theta}_1^C(m_{1,t}, \gamma_t))^{\eta - 1}(1 - \tilde{\Theta}_1^C)^{-\eta}m_{1,t} \Delta t$$

$$+ e^{-(r_f + \lambda) \Delta t} \left[ \frac{M_{t+\Delta t}}{M_t} \tilde{v}_1^C(m_{1,t+\Delta t}, \gamma_{t+\Delta t}; \tilde{\Theta}_1^C(\cdot)) \right],$$

(A.28)

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \tilde{\Theta}_1^C = \left[ m_{1,t}(1 - \tilde{\Theta}_1^C(m_{1,t}, \gamma_t))^{\eta - 1} + (1 - m_{1,t})(1 - \tilde{\Theta}_2^C(m_{1,t}, \gamma_t))^{\eta - 1} \right]^{\frac{1}{\eta - 1}}.$$  

(A.29)

(2) Firm 1’s customer base share evolves according to

$$m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1 - \tilde{\Theta}_1^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{(\epsilon - \eta)h}\omega^{-\epsilon h} - \rho \right] m_{1,t} \Delta t + \varsigma m_{1,t}(-\gamma_t \Delta t + \Delta \tilde{Z}_t) + \sigma_1 m_{1,t} \Delta W_{1,t}.$$  

(A.30)

(3) The industry’s customer base evolves according to

$$m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = 1 + \varsigma(-\gamma_t \Delta t + \Delta \tilde{Z}_t) + \left[ \alpha(1 - \tilde{\Theta}_1^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{(\epsilon - \eta)h}\omega^{-\epsilon h} - \rho \right] m_{1,t} \Delta t$$

$$+ \left[ \alpha(1 - \tilde{\Theta}_2^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{(\epsilon - \eta)h}\omega^{-\epsilon h} - \rho \right] (1 - m_{1,t}) \Delta t + \sigma_2 m_{1,t} \Delta W_{1,t} + \sigma (1 - m_{1,t}) \Delta W_{2,t}.$$  

(A.31)

(4) The aggregate state $\gamma_t$ evolves according to

$$\gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \bar{\gamma}) \Delta t - \pi(-\zeta \Delta t + \Delta \tilde{Z}_{\gamma,t}).$$

(A.32)

Recursive Formulation for the Deviation Value. The deviation value is obtained by assuming that firm $i$ optimally sets its profit margin conditional on firm $j$ following the collusive pricing rule $\tilde{\Theta}_2^C(m_{1,t}, \gamma_t)$. We exploit the linearity property and solve for firms’ deviation value as a function of their customer base shares. Denote by $\tilde{v}_i^D(m_{1,t}, \gamma_t; \tilde{\Theta}^C(\cdot))$ firm $i$’s deviation value.

Firm 1 solves the following normalized problem:

$$\tilde{v}_1^D(m_{1,t}, \gamma_t; \tilde{\Theta}^C(\cdot)) = \max_{\theta_{1,t}} \omega^{1-\epsilon} \theta_{1,t}(1 - \theta_{1,t})^{\eta - 1}(1 - \theta_{1,t})^{-\eta}m_{1,t} \Delta t$$

$$+ e^{-(r_f + \lambda) \Delta t} \left[ \frac{M_{t+\Delta t}}{M_t} \tilde{v}_1^D(m_{1,t+\Delta t}, \gamma_{t+\Delta t}; \tilde{\Theta}^C(\cdot)) + \xi \Delta t v_c^N(m_{1,t+\Delta t}, \gamma_{t+\Delta t}) \right].$$  

(A.33)

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \tilde{\Theta}_1^C = \left[ m_{1,t}(1 - \theta_{1,t})^{\eta - 1} + (1 - m_{1,t})(1 - \tilde{\Theta}_2^C(m_{1,t}, \gamma_t))^{\eta - 1} \right]^{\frac{1}{\eta - 1}}.$$  

(A.34)

(2) Firm 1’s customer base share evolves according to

$$m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1 - \theta_{1,t})^{\eta h}(1 - \theta_{1,t})^{(\epsilon - \eta)h}\omega^{-\epsilon h} - \rho \right] m_{1,t} \Delta t + \varsigma m_{1,t}(-\gamma_t \Delta t + \Delta \tilde{Z}_t) + \sigma_1 m_{1,t} \Delta W_{1,t}.$$  

(A.35)
We first illustrate the non-collusive equilibrium and then the collusive equilibrium.

Incentive-Compatibility Constraints and Optimal Collusive Profit Margins. The collusive equilibrium is a sub-game perfect Nash equilibrium if and only if the collusive profit margin function \( \hat{\Theta}^C(\cdot) \) satisfies the following incentive-compatibility constraints:

\[
\hat{v}_i^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \geq \hat{v}_i^D(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)),
\]

for all \( m_{1,t} \in [0,1], \gamma_t, \) and \( i = 1, 2. \)

There exist infinitely many sub-game perfect collusive equilibria. We focus on the collusive equilibrium with the collusive profit margin function that lies on the “Pareto-efficient frontier” (denoted by \( \Theta^C(\cdot) \)), which is obtained when all incentive-compatibility constraints are binding, i.e.,

\[
\hat{v}_i^C(m_{1,t}, \gamma_t; \Theta^C(\cdot)) = \hat{v}_i^D(m_{1,t}, \gamma_t; \Theta^C(\cdot)),
\]

for all \( m_{1,t} \in [0,1], \gamma_t, \) and \( i = 1, 2. \) We denote by \( v_i^C(m_{1,t}, \gamma_t) \) firm \( i \)'s value in the collusive equilibrium with collusive profit margin function \( \Theta^C(\cdot) \). Thus, by definition

\[
v_i^C(m_{1,t}, \gamma_t) = \hat{v}_i^C(m_{1,t}, \gamma_t; \Theta^C(\cdot)).
\]

H.2 The Extended Model

We first illustrate the non-collusive equilibrium and then the collusive equilibrium.

H.2.1 The Non-collusive Equilibrium

Firm 1 solves the following normalized problem:

\[
\hat{v}_1^N(m_{1,t}, \gamma_t; \theta_{2,t}) = \max_{\theta_{1,t}} \omega^{1-\epsilon} \theta_{1,t}(1-\theta_{1,t})^{\eta-1}(1-\theta_t)\epsilon^{-\eta} m_{1,t} \Delta t + e^{-(r_f + \lambda) t} \mathbb{E}_t [\frac{M_{t+\Delta t}}{M_t} v_1^N(m_{1,t+\Delta t}, \gamma_t+\Delta t)],
\]

subject to the following constraints. (1) The industry’s profit margin is given by

\[
1 - \theta_t = [m_{1,t}(1-\theta_{1,t})^{\eta-1} + (1-m_{1,t})(1-\theta_{2,t})^{\eta-1}] \frac{1}{\eta-1}.
\]

(2) Firm 1’s customer base share evolves according to

\[
m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1-\theta_{1,t})^{\eta h}(1-\theta_t)^{\epsilon-\eta} \omega^{e \theta} - \rho \right] m_{1,t} \Delta t + \varsigma m_{1,t}(\gamma_t \Delta t + \Delta \hat{Z}_t) + \sigma_M m_{1,t} \Delta W_{1,t}.
\]
We first state the collusion decision, which is determined by comparing the firm’s value with collusion and that without

\[
\text{Collusion Decisions.}
\]

\[\text{conditions that determine the optimal collusive profit margins.}\]

Finally, we present the incentive-compatibility constraints and the

collusion. Then, we present the recursive formulation for the two values. Next, we present the firm’s value when

\[
\text{The Non-Collusive (Nash) Equilibrium. Denote the equilibrium profit margin functions by } \theta^N_i(1, t; \gamma_t) \text{ and the}
\]

\[\text{off-equilibrium profit margin functions by } \tilde{\theta}^N_i(1, t; \gamma_t; \theta_i, t). \text{ Exploiting the symmetry between firm 1 and firm 2, we}
\]

\[\text{can obtain firm 2’s off-equilibrium value and policy functions as } \tilde{v}^N_1(1, t; \gamma_t; \theta_i, t), \text{ and}
\]

\[\tilde{\theta}^N_1(1, t; \gamma_t; \theta_i, t). \quad (A.46)\]

\[\tilde{v}^N_2(1, t; \gamma_t; \theta_i, t). \quad (A.47)\]

Given firm \(j = 1, 2\)’s profit margin \(\theta_j, t\), firm \(i\) optimally sets the profit margin \(\theta_i, t\). The non-collusive (Nash) equilibrium is derived from the fixed point—each firm’s profit margin is optimal given the other firm’s optimal profit margin:

\[\theta^N_1(1, t; \gamma_t) = \tilde{\theta}^N_1(1, t; \gamma_t; \theta_2, t), \quad (A.48)\]

\[\theta^N_2(1, t; \gamma_t) = \tilde{\theta}^N_2(1, t; \gamma_t; \theta_1, t). \quad (A.49)\]

The equilibrium value functions are given by

\[v^N_1(1, t; \gamma_t) = \tilde{v}^N_1(1, t; \gamma_t; \theta_2, t), \quad (A.50)\]

\[v^N_2(1, t; \gamma_t) = \tilde{v}^N_2(1, t; \gamma_t; \theta_1, t). \quad (A.51)\]

After finding the equilibrium value and solving the policy functions above, we can verify that the following conditions

\[v^N_1(1, t; \gamma_t) = v^N_2(1, t; \gamma_t), \quad (A.52)\]

\[\theta^N_1(1, t; \gamma_t) = \theta^N_2(1, t; \gamma_t). \quad (A.53)\]

H.2.2 The Collusive Equilibrium

We first state the collusion decision, which is determined by comparing the firm’s value with collusion and that without

\[\text{collusion. Then, we present the recursive formulation for the two values. Next, we present the firm’s value when}

\[\text{the firm deviates from the collusive equilibrium. Finally, we present the incentive-compatibility constraints and the}

\[\text{conditions that determine the optimal collusive profit margins.}\]

Collusion Decisions. Denote by \(\tilde{v}^C_i(1, t; \gamma_t; \tilde{\Theta}^C(-))\) firm \(i\)’s value in the collusive equilibrium, given collusive profit margin function \(\tilde{\Theta}^C(-)\). Denote by \(\tilde{v}^C_i(1, t; \gamma_t; \tilde{\Theta}^C(-))\) firm \(i\)’s value when the two firms choose to maintain collusion, given collusive profit margin function \(\tilde{\Theta}^C(-)\). Denote by \(\tilde{v}^N_i(1, t; \gamma_t; \tilde{\Theta}^C(-))\) firm \(i\)’s value when the two firms choose not to maintain collusion, given the collusive profit margin function \(\tilde{\Theta}^C(-)\).
The two firms collude with each other at time $t$ if

$$\pi_1^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) - \nu dt \geq \pi_1^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot))$$  \hspace{1cm} (A.54)

and $$\pi_2^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) - \nu dt \geq \pi_2^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)).$$  \hspace{1cm} (A.55)

In this case, the firm’s value in the collusive equilibrium is given by $\hat{\pi}_i^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \pi_i^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot))$ for $i = 1, 2$.

The two firms do not collude with each other at time $t$ if

$$\pi_1^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) - \nu dt < \pi_1^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot))$$  \hspace{1cm} (A.56)

or $$\pi_2^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) - \nu dt < \pi_2^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)).$$  \hspace{1cm} (A.57)

In this case, the firm’s value in the collusive equilibrium is given by $\hat{\pi}_i^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \pi_i^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot))$ for $i = 1, 2$.

**The Firm’s Value with Collusion.** Firm 1 solves the following normalized problem:

$$\pi_1^C(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \omega^{1-\hat{\theta}_1^C(m_{1,t}, \gamma_t)}(1 - \hat{\theta}_1^C(m_{1,t}, \gamma_t))^{n-1}(1 - \hat{\theta}_2^C(m_{1,t}, \gamma_t))^{\gamma - \eta}m_{1,t} \Delta t$$

$$+ e^{-(\rho + \lambda)M_t}E_t \left[ \frac{M_{t+\Delta t}^{\tau}}{M_t^{\tau}} \pi_1^C(m_{1,t+\Delta t}, \gamma_t + \Delta Z_t; \hat{\Theta}^C(\cdot)) \right],$$

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \hat{\theta}_1^C = \left[ m_{1,t}(1 - \hat{\theta}_1^C(m_{1,t}, \gamma_t))^{n-1} + (1 - m_{1,t})(1 - \hat{\theta}_2^C(m_{1,t}, \gamma_t))^{n-1} \right]^{\frac{1}{n-1}}.$$  \hspace{1cm} (A.59)

(2) Firm 1’s customer base share evolves according to

$$m_{1,t+\Delta t} = m_{1,t} + \left[ \alpha(1 - \hat{\theta}_1^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{\gamma h} - \rho \right] m_{1,t} \Delta t + \varsigma m_{1,t}(-\gamma_t \Delta t + \Delta Z_t) + \sigma_M m_{1,t} \Delta W_{1,t}.$$  \hspace{1cm} (A.60)

(3) The industry’s customer base evolves according to

$$m_{1,t+\Delta t} = 1 + \varsigma(-\gamma_t \Delta t + \Delta Z_t) + \left[ \alpha(1 - \hat{\theta}_1^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{\gamma h} - \rho \right] m_{1,t} \Delta t$$

$$+ \left[ \alpha(1 - \hat{\theta}_2^C(m_{1,t}, \gamma_t))^{\eta h}(1 - \theta_t)^{\gamma h} - \rho \right] (1 - m_{1,t}) \Delta t + \sigma_M m_{1,t} \Delta W_{1,t} + \sigma_M (1 - m_{1,t}) \Delta W_{2,t}.$$  \hspace{1cm} (A.61)

(4) The aggregate state $\gamma_t$ evolves according to

$$\gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \bar{\gamma}) \Delta t - \pi(-\gamma_t + \Delta Z_{\gamma_t}).$$  \hspace{1cm} (A.62)

**The Firm’s Value without Collusion.** Let $\hat{\pi}_i^N(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot))$ be firm $i(=1,2)$’s value when its competitor $j$’s profit margin is set to any (off-equilibrium) value $\theta_{j,t}$, given the collusive profit margin function $\hat{\Theta}^C(\cdot)$. 

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Firm 1 solves the following normalized problem:

$$\hat{\nu}^N_1(m_{1,t}, \gamma_t; \theta_{2,t}, \Theta^C(\cdot)) = \max_{\hat{\theta}_{1,t}} \omega^{1-\epsilon} \theta_{1,t}(1-\theta_{1,t})^{\eta-1}(1-\theta_{1,t})^{\eta-1}m_{1,t}\Delta t$$

\[+ e^{-(r_f+\lambda)\Delta t}E_t \left[ \frac{M_{t+\Delta t}}{M_t} \hat{\nu}^C_1(m_{1,t+\Delta t}, \gamma_{t+\Delta t}; \Theta^C(\cdot)) \right], \quad (A.63)\]

subject to the following constraints. (1) The industry’s profit margin is given by

$$1 - \theta_t = \left[ m_{1,t}(1-\theta_{1,t})^{\eta-1} + (1-m_{1,t})(1-\theta_{2,t})^{\eta-1} \right]^{\frac{1}{\eta}}. \quad (A.64)$$

(2) Firm 1’s customer base share evolves according to

$$m_{1,t+\Delta t} \frac{M_{t+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1-\theta_{1,t})^{\eta} (1-\theta_{t})^{(e-\eta)h}\omega^{-\nu} - \rho \right] m_{1,t}\Delta t + \varsigma_{m_{1,t}}(-\gamma_t\Delta t + \Delta Z_t) + \sigma_M m_{1,t}\Delta W_{1,t}. \quad (A.65)$$

(3) The industry’s customer base evolves according to

$$\frac{M_{t+\Delta t}}{M_t} = 1 + c(-\gamma_t\Delta t + \Delta Z_t) + \left[ \alpha(1-\theta_{1,t})^{\eta} (1-\theta_{t})^{(e-\eta)h}\omega^{-\nu} - \rho \right] m_{1,t}\Delta t$$

\[+ \left[ \alpha(1-\theta_{2,t})^{\eta} (1-\theta_{t})^{(e-\eta)h}\omega^{-\nu} - \rho \right] (1-m_{1,t})\Delta t + \sigma_M m_{1,t}\Delta W_{1,t} + \sigma_M (1-m_{1,t})\Delta W_{2,t}. \quad (A.66)\]

(4) The aggregate state $\gamma_t$ evolves according to

$$\gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \tau)\Delta t - \pi(-\zeta\Delta t + \Delta Z_{\gamma,t}). \quad (A.67)$$

Denote the equilibrium profit margin functions by $\bar{\nu}^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot))$ and the off-equilibrium profit margin functions by $\hat{\nu}^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot))$. Exploiting the symmetry between firm 1 and firm 2, we can obtain firm 2’s off-equilibrium value and policy functions as

$$\hat{\nu}^N_1(m_{1,t}, \gamma_t; \theta_{2,t}, \Theta^C(\cdot)) = \hat{\nu}^N_2(1-m_{1,t}, \gamma_t; \theta_{1,t}, \Theta^C(\cdot)), \quad (A.68)$$

$$\hat{\nu}^N_2(m_{1,t}, \gamma_t; \theta_{2,t}, \Theta^C(\cdot)) = \hat{\nu}^N_2(1-m_{1,t}, \gamma_t; \theta_{1,t}, \Theta^C(\cdot)). \quad (A.69)$$

Given $j = 1, 2$’s profit margin $\theta_{j,t}$, firm $i$ optimally sets the profit margin $\theta_{i,t}$. The non-collusive (Nash) equilibrium is derived from the fixed point—each firm’s profit margin is optimal given the other firm’s optimal profit margin:

$$\bar{\theta}^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot)) = \bar{\theta}^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot)), \quad (A.70)$$

$$\bar{\theta}^N_2(m_{1,t}, \gamma_t; \Theta^C(\cdot)) = \bar{\theta}^N_2(m_{1,t}, \gamma_t; \Theta^C(\cdot)). \quad (A.71)$$

The equilibrium value functions are given by

$$\pi^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot)) = \pi^N_1(m_{1,t}, \gamma_t; \Theta^C(\cdot)), \quad (A.72)$$

$$\pi^N_2(m_{1,t}, \gamma_t; \Theta^C(\cdot)) = \pi^N_2(m_{1,t}, \gamma_t; \Theta^C(\cdot)). \quad (A.73)$$

After finding the equilibrium value and solving the policy functions above, we can verify that the following conditions
are satisfied due to symmetry:

\[
\pi^N_1(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \pi^N_2(1 - m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)), \tag{A.74}
\]

\[
\pi^N_1(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \pi^N_2(1 - m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)). \tag{A.75}
\]

**The Firm’s Deviation Value.** Denote by \( \hat{\psi}^D_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) firm \( i \)'s deviation value, given the collusive profit margin function \( \hat{\Theta}^C(\cdot) \).

Because at time \( t \), firms may or may not collude as we discussed above, we have to separately solve for the deviation value in the two cases. In the first case, denote by \( \hat{\psi}^{D,C}_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) the deviation value when the two firms collude at time \( t \). That is, \( \hat{\psi}^{D,C}_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) is obtained by assuming that firm \( i \) optimally sets its profit margin conditional on firm \( j \) following the collusive profit margin function \( \hat{\Theta}^C_j(m_{1,t}, \gamma_t) \). In the second case, denote by \( \hat{\psi}^{D,N}_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) the deviation value when the two firms do not collude at time \( t \).

The deviation value \( \hat{\psi}^D(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) is determined by the following:

\[
\hat{\psi}^D_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \hat{\psi}^{D,C}_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \quad \text{if the two firms collude with each other at time } t, \ i.e.,
\]

\[
\hat{\psi}^D(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \hat{\psi}^{D,N}_i(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \quad \text{if the two firms do not collude with each other at time } t, \ i.e.,
\]

I. **The first case:**

\( \hat{\psi}^{D,C}_1(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) \) is obtained from solving the following problem for firm 1:

\[
\hat{\psi}^{D,C}_1(m_{1,t}, \gamma_t; \hat{\Theta}^C(\cdot)) = \max_{\theta_t} \omega^{1 - \epsilon} \theta_1(1 - \theta_1)^{\gamma - 1}(1 - \theta_1)^{\epsilon - \gamma} m_{1,t} \Delta t + e^{-(r_f + \delta) \Delta t} \mathbb{E}_t \left[ M_{t+\Delta t} \left[ (1 - \xi \Delta t) \hat{\psi}^D(m_{1,t+\Delta t}, \gamma_t; \hat{\Theta}^C(\cdot)) + \xi \Delta t \pi^N(m_{1,t+\Delta t}, \gamma_t; \hat{\Theta}^C(\cdot)) \right] \right], \tag{A.80}
\]

subject to the following constraints. (1) The industry’s profit margin is given by

\[
1 - \theta_t = \left[ m_{1,t}(1 - \theta_1)^{\gamma - 1} + (1 - m_{1,t})(1 - \theta_2(m_{1,t}, \gamma_t))^{\gamma - 1} \right]^\frac{1}{\gamma - 1}. \tag{A.81}
\]

(2) Firm 1’s customer base share evolves according to

\[
m_{1,t+\Delta t} = m_{1,t} + \left[ \alpha(1 - \theta_1)^{\gamma(h(1 - \theta_1)\omega^{\epsilon} - \rho)} m_{1,t} \Delta t + \zeta m_{1,t}(-\gamma \Delta t + \Delta \tilde{Z}_t) \right] + \sigma_m m_{1,t} \Delta W_{1,t}. \tag{A.82}
\]

(3) The industry’s customer base evolves according to

\[
M_{t+\Delta t} = M_t + (\alpha(1 - \theta_1)^{\gamma(h(1 - \theta_1)\omega^{\epsilon} - \rho)} m_{1,t} \Delta t + \zeta m_{1,t}(-\gamma \Delta t + \Delta \tilde{Z}_t) + \sigma_m m_{1,t} \Delta W_{1,t} + \sigma_m (1 - m_{1,t}) \Delta W_{2,t}. \tag{A.83}
\]
The aggregate state $\gamma_t$ evolves according to
\[
\gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \tau)\Delta t - \pi(-\zeta\Delta t + \Delta Z_{\gamma,t}).
\] (A.84)

**II. The second case:**
$\tilde{v}_i^{D,N}(m_{1,t}, \gamma_t; C(\cdot))$ is obtained from solving the following problem for firm 1:
\[
\tilde{v}_i^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)) = \omega^{1-\eta} \hat{\theta}_1^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot))(1 - \hat{\theta}_2^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta-1}(1 - \hat{\theta}_1^{D,N})^\eta m_{1,t}\Delta t
\]
\[+ e^{-(r_f+\lambda)\Delta t} E_t \left[ \frac{M_{1+\Delta t}}{M_t} \tilde{v}_i^{D}(m_{1,t+\Delta t}, \gamma_t+\Delta t; \hat{C}(\cdot)) \right],
\] (A.85)
subject to the following constraints. (1) The industry’s profit margin is given by
\[
1 - \hat{\theta}_1^{D,N} = \left[ m_{1,t}(1 - \hat{\theta}_1^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta-1} + (1 - m_{1,t})(1 - \hat{\theta}_2^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta-1} \right]^{\frac{1}{\eta-1}}.
\] (A.86)

(2) Firm 1’s customer base share evolves according to
\[
m_{1,t+\Delta t} \frac{M_{1+\Delta t}}{M_t} = m_{1,t} + \left[ \alpha(1 - \hat{\theta}_1^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta h}(1 - \theta_t)(\epsilon-\eta)\omega - \rho \right] m_{1,t}\Delta t + \varsigma m_{1,t}(-\gamma_t\Delta t + \Delta Z_t) + \sigma_M m_{1,t} \Delta W_{1,t}.
\] (A.87)

(3) The industry’s customer base evolves according to
\[
\frac{M_{1+\Delta t}}{M_t} = 1 + \varsigma(-\gamma_t\Delta t + \Delta Z_t) + \left[ \alpha(1 - \hat{\theta}_1^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta h}(1 - \theta_t)(\epsilon-\eta)\omega - \rho \right] m_{1,t}\Delta t
\]
\[+ \left[ \alpha(1 - \hat{\theta}_2^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot)))^{\eta h}(1 - \theta_t)(\epsilon-\eta)\omega - \rho \right] (1 - m_{1,t})\Delta t + \sigma_M m_{1,t} \Delta W_{1,t} + \sigma_M (1 - m_{1,t})\Delta W_{2,t}.
\] (A.88)

(4) The aggregate state $\gamma_t$ evolves according to
\[
\gamma_{t+\Delta t} = \gamma_t - \varphi(\gamma_t - \tau)\Delta t - \pi(-\zeta\Delta t + \Delta Z_{\gamma,t}).
\] (A.89)

Note that $\hat{\theta}_i^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot))$ are the non-collusive profit margins conditional on firm 1 taking the deviation strategy in the collusion state (and hence obtaining continuation value $\hat{v}_i^D(m_{1,t+\Delta t}, \gamma_t+\Delta t; \hat{C}(\cdot))$). $\hat{\theta}_1^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot))$ are different from $\hat{\theta}_2^{D,N}(m_{1,t}, \gamma_t; \hat{C}(\cdot))$. They are the non-collusive profit margins conditional on both firms following the collusion strategy in the collusion state (and hence obtaining continuation value $\hat{v}_i^C(m_{1,t+\Delta t}, \gamma_t+\Delta t; \hat{C}(\cdot))$).

**Incentive-Compatibility Constraints and Optimal Collusive Profit Margins.** The collusive equilibrium is a sub-game perfect Nash equilibrium if and only if the collusive profit margins $\hat{\Theta}(\cdot)$ satisfy the following incentive-compatibility constraints:
\[
\hat{v}_i^C(m_{1,t}, \gamma_t; \hat{C}(\cdot)) \geq \hat{v}_i^D(m_{1,t}, \gamma_t; \hat{C}(\cdot)),
\] (A.90)
for all $m_{1,t} \in [0, 1]$, $\gamma_t$, and $i = 1, 2$.

There exist infinitely many sub-game perfect collusive equilibria. We focus on the collusive equilibrium with the collusive profit-margin scheme that lies on the “Pareto-efficient frontier” (denoted by $\hat{C}(\cdot)$), which is obtained when all incentive-compatibility constraints are binding, i.e.,
\[
\hat{v}_i^C(m_{1,t}, \gamma_t; \hat{C}(\cdot)) = \hat{v}_i^D(m_{1,t}, \gamma_t; \hat{C}(\cdot)),
\] (A.91)
for all \( m_{1,t} \in [0, 1], \gamma_t, \) and \( j = 1, 2. \) We denote by \( v_i^C(m_{1,t}, \gamma_t) \) firm \( i \)'s value in the collusive equilibrium with the collusive profit margin function \( \Theta^C(\cdot) \). Thus, by definition
\[
v_i^C(m_{1,t}, \gamma_t) = \tilde{v}_i^C(m_{1,t}, \gamma_t; \Theta^C(\cdot)). \tag{A.92}
\]

### H.3 Discretization and Simulation

We discretize the idiosyncratic cash-flow shocks \( dW_{i,t} \) based on \( n_w \) grids spanning the range from \(-3\sigma_M \) to \( 3\sigma_M \) using the method of Tauchen (1986). We approximate the persistent process of \( \gamma_t \) using \( n_\gamma \) discrete states based on the method of Rouwenhorst (1995). The time-line is discretized into intervals of length \( \Delta t \). We choose a large \( n_\gamma \) to ensure the continuous process is accurately approximated.

We use collocation methods to solve each firm’s problem. Let \( S_m \times S_\gamma \) be the grid of collocation nodes for a firm’s equilibrium value, and \( S_m \times S_\gamma \times S_\theta \) be the grid of collocation nodes for a firm’s off-equilibrium value. We have \( S_m = \{m_1, m_2, \ldots, m_{n_m}\} \), \( S_\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{n_\gamma}\} \), and \( S_\theta = \{\theta_1, \theta_2, \ldots, \theta_{n_\theta}\} \).

We approximate the firm’s value function \( v(\cdot) \) on the grid of collocation nodes using a linear spline with coefficients corresponding to each grid point. We first take a guess of the spline’s coefficients, then we iterate to obtain a vector that solves the system of Bellman equations.

When conducting simulations, we assume that the value of \( \lambda_t \) remains the same until the industry is hit by an idiosyncratic Poisson shock with a small rate \( \chi \). Conditional on receiving the Poisson shock, a new characteristic is drawn randomly from the set \( \{\lambda_1, \cdots, \lambda_N\} \) each with equal probability, where \( 0 \leq \lambda_1 < \cdots < \lambda_N \). The rate \( \chi \) is chosen to match the high yearly autocorrelation of \( \lambda \) in the data, 0.977.

### H.4 Implementation

The numerical algorithms are implemented in C++. The program is run on the servers of MIT’s Economics Department, supply.mit.edu and demand.mit.edu, which are hosted on Dell PowerEdge R910 (64 cores, Intel(R) Xeon(R) CPU E7-4870, 2.4GHz) and Dell PowerEdge R920 (48 cores, Intel(R) 4 Xeon E7-8857 v2 CPUs). We use OpenMP for parallelization when iterating value functions and simulating the model.

**Selection of Grids** We set \( n_\gamma = 34, n_m = 21, \Delta t = 1/12, n_\theta = 11. \) The grid of customer base share \( S_m \) is discretized into 21 nodes from \( 10^{-7} \) to \( 1 - 10^{-7} \) with equal spacing. We do not set \( S_m \) from 0 to 1 to avoid the indeterminacy of the optimal profit margin with \( m = 0. \) The time interval \( \Delta t \) is set to 1/12 (i.e., one month). With \( \Delta t = 1/12, 2,000 \) iterations allow us to achieve convergence in value functions. The profit margin grid is discretized into 11 nodes from 0 to \( 1/\epsilon \) with equal spacing.

**Solving the Non-Collusive Equilibrium.** Given the value functions from the previous iteration, we use the golden section search method to find the equilibrium profit margins. The computational complexity of this algorithm is on the order of \( \log(n) \), so it is much faster and more accurate than a simple grid search.

Searching for the equilibrium profit margin is very challenging because we have to solve a fixed-point problem (Eq. (A.48)- Eq. (A.51)) that involves the simultaneous profit-margin-setting decisions of both firms. Our solution technique proceeds in three iterative steps.

First, given \( v_1^N(m_1, \gamma) \), we solve for the off-equilibrium value \( \hat{v}_1^N(m_1, \gamma; \theta_2) \) and the off-equilibrium policy function \( \hat{\theta}_1^N(m_1, \gamma; \theta_2) \). Exploiting symmetry, we obtain \( \hat{v}_2^N(m_1, \gamma; \theta_1) \) and \( \hat{\theta}_2^N(m_1, \gamma; \theta_1) \). Second, for each \( (m_1, \gamma) \in S_m \times S_\gamma \), we use a nonlinear solver Knitro to solve Eq. (A.48)- Eq. (A.49) and obtain the equilibrium profit margins \( \theta_1^N(m_1, \gamma), \theta_2^N(m_1, \gamma) \). Third, we solve Eq. (A.50)- Eq. (A.51) and obtain equilibrium value functions \( v_1^N(m_1, \gamma) \) and \( v_2^N(m_1, \gamma) \).
Solving the Collusive Equilibrium. To solve the collusive equilibrium, we have to simultaneously solve for the endogenous collusion boundaries and the endogenous collusive profit margins within these boundaries. We implement a nested iteration method. First, we guess the collusion boundaries and solve for the highest collusive profit margins within these boundaries using the algorithm below. Then we check whether the implied collusion boundaries are consistent with our guessed boundaries. If not, we update our guess and resolve the highest collusive profit margins.

We modify the golden section search method to find the highest collusive profit margins $\theta^C(m_1, \gamma)$ by iteration. Within each iteration, we solve for firms’ collusion value and deviation value using standard recursive methods given $\hat{\theta}^C_i(m_1, \gamma)$.

There are two key differences between our method and a standard golden section search method. First, to increase efficiency, we guess and update the collusive profit margin function $\hat{\theta}^C_i(m_1, \gamma)$ simultaneously for all $(m_1, \gamma) \in S_m \times S_\gamma$, instead of doing it one by one for each state. A natural problem introduced by the simultaneous updating is that there might be overshooting. For example, if for some particular state $(m_1^*, \gamma^*)$, we made a collusive profit margin $\hat{\theta}^C_i(m_1^*, \gamma^*)$ too high in the previous iteration, the collusive profit margin for some other states $(m_1, \gamma) \neq (m_1^*, \gamma^*)$ might be affected in the current iteration and may never achieve a binding incentive-compatibility constraint. Eventually, this may lead to non-convergence.

We solve this problem by gradually updating the collusive profit margins. In particular, in each iteration, we first compute the updated collusive profit margin function $\hat{\theta}^C_i^\prime(m_1, \gamma)$ implied by the golden section search method. Then, instead of directly changing the upper search bound or lower search bound to $\hat{\theta}^C_i^\prime(m_1, \gamma)$, we change it to $(1 - \text{adj}) \times \hat{\theta}^C_i(m_1, \gamma) + \text{adj} \times \hat{\theta}^C_i^\prime(m_1, \gamma)$, a weighted average of the current collusive profit margin $\hat{\theta}^C_i(m_1, \gamma)$ and the updated collusive profit margin $\hat{\theta}^C_i^\prime(m_1, \gamma)$. If $\text{adj} = 0.5$, our algorithm is essentially the same as a bisection search algorithm. A lower $\text{adj}$ is more suitable for solving the problem in which different states have a higher degree of interdependence. In our extended model, we set a relatively low $\text{adj} = 0.15$ to ensure convergence.
References


