Online Appendix: Volatility, Intermediaries, and Exchange Rates

Xiang Fang* Yang Liu*

1 Equilibrium conditions characterization

In this appendix, we characterize all the equilibrium conditions of our model.

Domestic and foreign households’ consumption aggregation:

\[ C^{t-1}_{t} = (1 - \alpha)C^{t-1}_{x,t} + \alpha C^{t-1}_{y,t}, \]  
(1)

\[ C^{*t-1}_{t} = \alpha C^{*}_{x,t} + (1 - \alpha)C^{*}_{y,t}. \]  
(2)

Domestic and foreign households’ optimality conditions for relative consumption and relative price:

\[ \frac{C_{y,t}}{C_{x,t}} = \left( \frac{P_{x,t}}{P_{y,t}} \right) \frac{1 - \alpha}{\alpha}. \]  
(3)

\[ \frac{C^{*}_{y,t}}{C^{*}_{x,t}} = \left( \frac{P_{x,t}}{P_{y,t}} \right) \frac{1 - \alpha}{\alpha}. \]  
(4)

Domestic and foreign households’ period budget constraint:

\[ P_{x,t}C_{x,t} + P_{y,t}C_{y,t} = C_{t}, \]  
(5)

\[ P_{x,t}C^{*}_{x,t} + P_{y,t}C^{*}_{y,t} = Q_{t}C^{*}_{t}. \]  
(6)

*HKU Business School, University of Hong Kong, Hong Kong, China. E-mail addresses: yangliu5@hku.hk (Y. Liu), xiangf@hku.hk (X. Fang).
Domestic and foreign households’ Euler equations:

\[ E_t M_{t+1} R_{f,t} = 1, \]

\[ E_t M^*_t R^*_{f,t} = 1. \]  

(7)  

(8)

Return on the domestic and foreign risky asset (denominated in the local consumption basket):

\[ R_{s,t+1} = \frac{P_{x,t+1} X_{t+1} + P_{s,t+1}}{P_{s,t}}, \]

\[ R^*_{s,t+1} = \frac{P_{y,t+1} Y_{t+1} + P^*_s}{P^*_s}. \]  

(9)  

(10)

Return on the international bond:

\[ R_{i,t+1} = R_{b,t} \frac{1 + Q_{t+1}}{1 + Q_t}. \]  

(11)

The endogenous state variable, the domestic share of total wealth, is defined as:

\[ \omega_t = \frac{P_{s,t-1} R_{s,t-1} S_{x,t-1} + P^*_s R^*_{s,t-1} Q_{t-1} S_{y,t-1} + D_{I,t-1} R_{I,t}}{P_{s,t-1} R_{s,t} + P^*_s R^*_{s,t} Q_{t}}. \]

(12)

Domestic consolidated budget constraint for households and intermediaries:

\[ C_{x,t} + P_{s,t} S_{x,t} + P^*_s Q_{t} S_{y,t} + D_{I,t} = P_{s,t-1} R_{s,t} S_{x,t-1} + P^*_s R^*_{s,t-1} Q_{t-1} S_{y,t-1} + D_{I,t-1} R_{I,t}. \]  

(13)

Intermediaries’ value functions:

\[ \theta_t (P_{s,t} S_{x,t} + P^*_s Q_{t} S_{y,t} + D_{I,t}) = \frac{1}{1 - \kappa_t} N_t, \]

\[ \theta^*_t \left( \frac{P_{s,t}}{Q_{t}} (1 - S_{x,t}) + P^*_s (1 - S_{y,t}) + D^*_I \right) = \frac{1}{1 - \kappa^*_t} N^*_t. \]  

(14)

Market clearing conditions:

\[ C_{x,t} + C^*_{x,t} = X_t, \]

\[ C_{y,t} + C^*_{y,t} = Y_t, \]

\[ D_{I,t} + D^*_I Q_{t} = 0. \]  

(15)  

(16)  

(17)

Exogenous domestic and foreign intermediaries’ net worth:

\[ N_t = \eta X_t, \]

\[ N^*_t = \eta Y_t. \]  

(18)  

(19)
Exogenous processes:

\[ \Delta \log X_{t+1} = \tau (\log Y_t - \log X_t) + \bar{\sigma} \sigma_{x,t} u_{x,t+1}, \]  
(20)

\[ \Delta \log Y_{t+1} = -\tau (\log Y_t - \log X_t) + \bar{\sigma} \sigma_{y,t} u_{x,t+1}, \]  
(21)

\[ \log \sigma_{x,t+1} = \rho \sigma_{x,t} + \sigma_{\omega} \omega_{x,t+1}, \]  
(22)

\[ \log \sigma_{y,t+1} = \rho \sigma_{x,t} + \sigma_{\omega} \omega_{y,t+1}, \]  
(23)

\[ \log \theta_t = \log \theta_0 + \theta_1 \log \sigma_{x,t}, \]  
(24)

\[ \log \theta_t^* = \log \theta_0 + \theta_1 \log \sigma_{y,t}. \]  
(25)

Domestic and foreign intermediaries’ Euler equations:

\[ E_t M_{t+1} R_{s,t+1} = 1 + \kappa_i \theta_t, \]  
(26)

\[ E_t M_{t+1}^* R_{s,t+1}^* \frac{Q_{t+1}}{Q_t} = 1 + \kappa_i \theta_t^*, \]  
(27)

\[ E_t M_{t+1} R_{I,t+1} = 1 + \kappa_i \theta_t, \]  
(28)

\[ E_t M_{t+1}^* R_{I,t+1}^* \frac{Q_{t+1}}{Q_t} = 1 + \kappa_i^* \theta_t^*, \]  
(29)

\[ E_t M_{t+1}^* R_{s,t+1}^* \frac{Q_{t+1}}{Q_t} = 1 + \kappa_i^* \theta_t^*, \]  
(30)

\[ E_t M_{t+1}^* R_{I,t+1}^* \frac{Q_{t+1}}{Q_t} = 1 + \kappa_i^* \theta_t^*. \]  
(31)

2 Model solution and estimation method

In this section, we describe in details the solution and the estimation method.

2.1 Solution method

The model is solved numerically using a global projection method (Fernández-Villaverde, Rubio-Ramírez, and Schorfheide, 2016). The model has the following four state variables: the domestic share of total wealth \( \omega_t \), the relative size of endowment \( \log Y_t - \log X_t \), and the volatility of the domestic and foreign country \( \sigma_{x,t} \) and \( \sigma_{y,t} \). We use Smolyak polynomials on sparse grids as
the basis functions to approximate the policy and pricing functions \( \{C_{x,t}, S_{x,t}, S_{y,t}, P_{s,t}, P_{s,t}^*\} \).

Denote the state variables as \( \mathcal{X}_t = [\omega_t, \log Y_t - \log X_t, \sigma_{x,t}, \sigma_{y,t}] \). Use rescale function \( \Phi : \mathbb{R}^4 \to [-1, 1]^4 \) to rescale the state variables between -1 and 1. For example,

\[
\Phi(\omega_t) = -1 + 2 \frac{\omega_t - \omega_{\text{min}}}{\omega_{\text{max}} - \omega_{\text{min}}}
\]

Each policy and pricing function is approximated as \( \hat{f}(\mathcal{X}_t; b) = \sum_{n=1}^{N_p} b_n \Psi_n(\Phi(\mathcal{X}_t)) \), where \( b \) is the approximation parameter and \( \Psi_n \) are the polynomial functions. Given the state variables and the approximated \( \{C_{x,t}, S_{x,t}, S_{y,t}, P_{s,t}, P_{s,t}^*\} \), other endogenous variables can be solved accordingly by the equilibrium conditions listed in Section 1. We iterate on the five Euler equations, Eq. (26) to (30), until convergence to find the correct approximations for \( \{C_{x,t}, S_{x,t}, S_{y,t}, P_{s,t}, P_{s,t}^*\} \). The expectations are computed using monomial integration methods. We compute the state variables at each monomial node in the next period. The law of motion of \( \omega_{t+1} \) is unknown, and we approximate it with Smolyak polynomials. Given the state variables, we solve the other endogenous variables and compute the errors of the five Euler equations. The algorithm can be briefly structured as follows.

Given the approximation parameters \( b \), at each grid point,

1. Approximate the five policy and pricing functions, \( \{C_{x,t}, S_{x,t}, S_{y,t}, P_{s,t}, P_{s,t}^*\} \).
2. Solve the other endogenous variables using the equilibrium conditions.
3. Solve the endogenous state variable \( \omega' \) and other endogenous variables for the next period.
4. Compute the Euler equation approximation errors.

Iterate on parameter \( b \) to minimize the approximation errors over the grids.

We rely on a Fortran-based numerical optimizer. A crucial element is the initial guess of the solution. We compute the average portfolio holdings with the perturbation method shown in Section 4 and use it as the initial guess. We gradually increase the grids and polynomials to facilitate the convergence and to increase the accuracy.

### 2.2 Estimation method

The equilibrium model is estimated by simulated method of moments (SMM). The estimation methods are detailed in Singleton (2009) and Fernández-Villaverde, Rubio-Ramírez, and
Denote the moment from the data sample by \( \hat{m}_T(Y) \) and model-implied moments \( \hat{m}_T(Y; \theta_0) \) under the model with parameter \( \theta_0 \). Define the discrepancy as

\[
G_T(\theta|Y) = \hat{m}_T(Y) - \hat{m}_T(Y; \theta_0).
\]

Our estimator \( \hat{\theta}_{smm} \) minimizes the criterion function of the weighted discrepancy

\[
\hat{\theta}_{smm} = \arg\min_{\theta} G_T(\theta|Y)' W G_T(\theta|Y)
\]

Suppose that there is a unique \( \theta_0 \) that \( G_T(\theta|Y) \to 0 \) almost surely, then the estimator is consistent and asymptotically normal:

\[
\sqrt{T} (\hat{\theta}_{smm} - \theta_0) \rightarrow N(0, (DWD')^{-1} DW \Omega WD' (DWD')^{-1}).
\]

where \( D = \nabla \theta G_T(\theta_0|Y) \) and \( \Omega = (1 + \frac{1}{\lambda})(\lim_{T \to \infty} T \text{Var}[\hat{m}_T(Y; \theta_0)]) \). \( \lambda \) is the ratio between number of observations in the simulated data to that in the real data. We weight the squared differences by the inverse of the square of the sample moments. The off-diagonal elements of the weighting matrix are zero:

\[
W_{i,j} = \begin{cases} 
1/\hat{m}_T(Y_i)^2 & i = j \\
0 & i \neq j 
\end{cases}
\]

In this case, the weight matrix adjusts for the difference in units. When computing the model-implied moments, we simulated the model for 10,000 periods.

3 Data sources

In this appendix, we report the details of the data that are used in our paper, including the measurement, the data source, and the sample coverage.

Consumption, output, import and export data. Source: Quarterly National Accounts of OECD Database, NIPA Table for US. Sample period: 1973-2018 quarterly.

Share of home assets. Measurement: \( \frac{\text{MarketCap} - \text{ExternalLiabilities}}{\text{MarketCap}} \). Source: Market cap data are from the World Bank. External liabilities in stocks are from the database compiled by Lane and
Milesi-Ferretti (2007); the most updated version is extended to 2015. Sample period: 1985-2015 annual.


Banks’ leverage ratio. Measurement: \( \frac{\text{TotalAssets} - \text{TotalLiabilities}}{\text{TotalAssets}} \). Source: Financial Account Database of the US, reported by the Federal Reserve Board. Sample period: 1973-2018 quarterly.


Exchange rate. Source: Datastream and GFD. Sample period: 1980-2019 daily. The daily data are used to calculate the realized volatility.


4 Comparison of solutions using global and local methods

We solve the model with the estimated parameters using a local perturbation method and compare the solution with the one obtained using the global method. In our model, the portfolio holdings of the domestic risky asset and foreign risky asset are indeterminate at the steady state. Therefore, we follow the method in Devereux and Sutherland (2011) to solve the model via perturbation, which combines the second-order approximation of the Euler equations and the first-order approximation of other equilibrium conditions.

In the model, we impose the steady state of the international bond position to be zero due to the symmetry of the two countries. The steady-state domestic intermediaries’ holding of domestic risky asset \( S_x \) is solved using the second-order approximation of the intermediaries’ Euler equations, and the steady-state domestic intermediaries’ holding of the foreign risky asset is \( 1 - S_x \). Due to symmetry, the foreign intermediaries hold \( 1 - S_x \) share of the domestic risky asset and \( S_x \). 
Table 1: Comparison of the solutions between the global and local perturbation method
The table reports the model-implied moments that are used for estimation when the model is solved with the global projection method (the benchmark) and the local perturbation method.

<table>
<thead>
<tr>
<th>Moments</th>
<th>Benchmark</th>
<th>Perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td>sd(Δc)</td>
<td>2.01</td>
<td>2.00</td>
</tr>
<tr>
<td>P_yC_y/C</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>S_x</td>
<td>0.89</td>
<td>0.88</td>
</tr>
<tr>
<td>sd(NX/GDP)</td>
<td>1.98</td>
<td>1.98</td>
</tr>
<tr>
<td>sd(log(σ_x))</td>
<td>0.21</td>
<td>0.22</td>
</tr>
<tr>
<td>r_f</td>
<td>0.73</td>
<td>0.76</td>
</tr>
<tr>
<td>r_s - r_f</td>
<td>0.99</td>
<td>0.93</td>
</tr>
<tr>
<td>φ</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>corr(Δq, Δc − Δc^*)</td>
<td>-0.05</td>
<td>-0.19</td>
</tr>
<tr>
<td>β_fq</td>
<td>1.63</td>
<td>1.70</td>
</tr>
<tr>
<td>sd(Δq)</td>
<td>5.27</td>
<td>6.39</td>
</tr>
<tr>
<td>r_cip</td>
<td>-0.25</td>
<td>-0.20</td>
</tr>
<tr>
<td>sd(r_cip)</td>
<td>0.23</td>
<td>0.19</td>
</tr>
</tbody>
</table>

share of the foreign risky asset. In the perturbation solution, the portfolio holding S_x is a constant.

The average Euler equation errors using perturbation are approximately $10^{-3}$, which are two orders of magnitude larger than the global method.

References

