A Dynamic Theory of Multiple Borrowing

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Online Appendix

Microfounding Random Default with Moral Hazard

In this section we provide an alternative microfoundation of the random default through limited pledgeability and moral hazard. Instead of assuming the entrepreneur borrows against his default cost $\tilde{c}$ as we do in the main text of the paper, we endow the entrepreneur with a pledgeable stochastic cashflows from assets-in-place that are realized at $s = 2$. The pledgeable cashflows realize one of two values: zero or one. The good payoff occurs with probability p, which is chosen by the entrepreneur at quadratic effort cost p^2. Moral hazard arises because effort is chosen after cashflows are (partially) pledged to lenders. Since there are only two realizations of these cashflows and the entrepreneur has no other pledgeable wealth, it is without loss of generality that claims issued to lenders take the form of debt contracts: they are repaid in full when the good realization occurs and the entrepreneur defaults when cashflows are zero. With this in mind, assume the entrepreneur has issued a total face value of debt D in the first stage. At $s = 2$ the entrepreneur solves

$$p(D) \equiv \arg\max_{p} p[1 - D] - p^2$$

The entrepreneur expects the assets in place to pay out with probability p, and conditional on a positive payout the entrepreneur gets to keep $1 - D$ of the cashflows as the residual claim. The cost of choosing the probability of positive cashflows to be p is p^2. Thus $p(D)$ denotes the solution to the entrepreneur's choice of effort at stage $s = 2$ conditional on having a total outstanding face value of debt D. The solution of entrepreneur's problem yields $p(D) = \frac{1-D}{2}$, and entrepreneur's expected payoff from the residual claim is $\frac{(1-D)^2}{4}$ The value function in (2) is modified to

$$V(D) = \max_{D'} z\tilde{p}(D')(D'-D) - (1-q)\frac{(1-D)^2}{4} + qV(D')$$

and all of our results go through analogously.

Proof of Proposition 1

Borrower's problem can be formulated recursively as:

$$V(D) = \max_{D'} z\tilde{p}(D')(D'-D) + (1-q)\left[\frac{(1-D')^2}{2} - \frac{1}{2}\right] + qV(D')$$

where

$$\tilde{p}(D') \equiv \mathbb{E}\left[1 - D^{agg} | D'\right]$$

We guess that borrower's policy function $g(\cdot)$ and lenders' loan pricing function $\tilde{p}(\cdot)$ both take a linear form and are each characterized by a single endogenous variable, b and ℓ, respectively:

$$\tilde{p}(D) = \ell(1-D)$$

$$1 - g(D) = b(1-D)$$

To solve for borrower's policy function, we proceed to take first order condition and use the envelope condition for borrower's problem. The first order condition is:

$$-z\ell(g(D) - D) + z\ell(1 - g(D)) - (1-q)(1-g(D)) + qV'(g(D)) = 0$$

and the envelope condition is:

$$V'(D) = -z\ell(1 - g(D))$$

Plugging the envelope condition into the first-order condition and after simplifying, we can express the Euler condition as a quadratic function of b:

$$q z \ell b^2 + (1 - q - 2z\ell)\, b + z\ell = 0$$

We thus solve for the endogenous parameter b that governs the borrower's policy function as:[12]

$$b = \frac{(2z\ell - (1-q)) - \sqrt{(1 - q - 2z\ell)^2 - 4qz^2\ell^2}}{2qz\ell} \tag{9}$$

To solve for the lender's loan pricing function, note

$$\begin{aligned}
\tilde{p}(D) &= \mathbb{E}\left[1 - D^{agg} | D\right] \\
&= (1-q)\left[(1-D) + q\,(1 - g(D)) + q^2\,(1 - g(g(D))) + \cdots\right] \\
&= (1-q)\left[(1-D) + qb\,(1-D) + q^2 b^2\,(1-D) + \cdots\right] \\
&= \frac{1-q}{1-qb}\,(1-D)
\end{aligned}$$

Hence

$$\ell = \frac{1-q}{1-qb} \tag{10}$$

Equation (9) characterizes b as a decreasing function of ℓ. On the other hand, equation (10) characterizes ℓ as an increasing function of b. The two equations therefore yields a unique solution (b^*, ℓ^*) for each $q \in [0,1)$ and $z \in (1, \infty)$. In particular, we have

$$b^* = \frac{z}{\left(z - \frac{1}{2}\right) + \sqrt{\left(z - \frac{1}{2}\right)^2 - qz\,(z-1)}}$$

$$\ell^* = \frac{(1-q)\,(z-1)}{\sqrt{\left(z - \frac{1}{2}\right)^2 - qz\,(z-1)} - \frac{1}{2}}$$

To solve for borrower welfare, $V(0)$, we can plug these expressions back into the value function, which we guess and verify has quadratic form $V(D) = aD^2 + bD + c$. Observe that $V(1) = -\frac{1}{2}$

[12] There are two roots to the quadratic equation, one of which leads to explosive debt accumulation. We choose the other, stable root.

and $V'(1) = 0$, which imply the functional form must satisfy $V(D) = -\frac{1}{2} + a(1-D)^2$ for some constant a. We can find the constant a by plugging back into the value function our equilibrium expressions

$$V(D) = z\tilde{p}(D')(D'-D) + (1-q)\left[\frac{(1-D')^2}{2} - \frac{1}{2}\right] + qV(D')$$

$$-\frac{1}{2} + a(1-D)^2 = \left[z\ell b(1-b) + (1-q)b^2/2 + qab^2\right](1-D)^2 - (1-q)\sum_{i=0}^{\infty} q^i \frac{1}{2}$$

which, solving for a and simplifying substantially, gives

$$V(D) = -\frac{1}{2} + \left(\frac{z^2(1-q)}{2z(1-q) - 1 + 2\sqrt{\left(z-\frac{1}{2}\right)^2 - qz(z-1)}}\right)(1-D)^2$$

$\square$

Lemma 1. *The best response functions have the following properties:*

$$\frac{\partial L(b;z,q)}{\partial b} \geq 0; \quad \frac{\partial L(b;z,q)}{\partial z} = 0;$$
$$\frac{\partial B(\ell;z,q)}{\partial z} \leq 0; \quad \frac{\partial B(\ell;z,q)}{\partial \ell} \leq 0;$$

where the inequalities are strict for $q \in (0,1)$ and $z > 1$.

Proof. The results with respect to lender's best response immediately follow from equation (10).

We now with with equation (9) to derive the results respect to borrower's best response function. Let $x \equiv 2z\ell$, we have

$$\begin{aligned} b &= \frac{(x-(1-q)) - \sqrt{(1-q-x)^2 - qx^2}}{qx} \\ &= \frac{1}{q} - \frac{1-q+\left((1-q)(x-1)^2 - q(1-q)\right)^{\frac{1}{2}}}{qx} \end{aligned}$$

Let $\Delta \equiv \left((1-q)(x-1)^2 - q(1-q) \right)$ and take derivative with respect to x, we have

$$\begin{aligned}
\frac{\partial b}{\partial x} &= -\frac{qx(1-q)(x-1) - q\left((1-q)\Delta^{\frac{1}{2}} + (1-q)(x-1)^2 - q(1-q) \right)}{(qx)^2 \Delta^{\frac{1}{2}}} \\
&= -\frac{q(1-q)}{(qx)^2 \Delta^{\frac{1}{2}}} \left(x^2 - x - \Delta^{\frac{1}{2}} - (x-1)^2 + q \right) \\
&= -\frac{q(1-q)}{(qx)^2 \Delta^{\frac{1}{2}}} \left(x + q - 1 - \Delta^{\frac{1}{2}} \right)
\end{aligned}$$

Since $\Delta^{\frac{1}{2}} = \sqrt{(x+q-1)^2 - qx^2} \leq x + q - 1$, we have that

$$\frac{\partial b}{\partial x} \leq 0$$

and the inequality is strict for $q \in (0,1)$. Given the definition of x, we have that $\frac{\partial b(\ell;z,q)}{\partial z} \leq 0$ and $\frac{\partial b(\ell;z,q)}{\partial \ell} \leq 0$. $\square$

Proof of Proposition2

The equilibrium sequence of debt issuance is straightforward to compute. Recall that the borrower promises to repay the $i+1$ lender an amount $d_{i+1} = g(D_i) - D_i$,where D_i is the cumulative repayment already pledged to previous lenders. Also recall that this pledge of d_{i+1} generates current proceeds of $k_{i+1} = \tilde{p}(g(D_i))(g(D_i) - D_i)$. In equilibrium we can write these expressions as simply

$$d_{i+1} = (1 - b^*)(1 - D_i)$$

$$k_{i+1} = \ell^* b^* (1 - b^*)(1 - D_i)^2$$

These can also be expressed recursively as

$$d_{i+1} = b^* \times d_i$$

$$k_{i+1} = (b^*)^2 \times k_i$$

The proposition follows from the fact that $b^* \leq 1$ with equality only when $q = 0$ (full commitment).

Proof of Proposition 3

We begin with an outline of the proof. The proof will first show how to solve the finite-lender game by backwards induction, generating a recursive formulation for borrower and lender strategies. Next, we show that the fixed point of this recursion generates the strategies of the infinite-lender equilibrium defined in the main text. Finally, to demonstrate convergence, we show that this recursive formulation of strategies is characterized by a contraction mapping. This implies that, considering the strategies at a given lender, as the number of potential subsequent lenderse goes to infinity, the equilibrium strategies at this lender converge uniquely to the fixed point and thus to the strategies of the infinite-lender equilibrium.

Backward Induction in the Finite-Lender Game

Consider a finite version of the game with N lenders. For this proof, we abuse notation and index periods counting backwards from the end. Thus the last lender is indexed 1, and the first lender is indexed N. Therefore, after lender i there are at most $i-1$ more lenders for the borrower to visit. Let D_i denote the amount of cumulative debt the borrower accumulates from meeting lenders N through $i+1$, thus D_0 denotes the total amount of debt the borrower will accumulate if it gets to meet all N lenders. The probability of default from the last lender's perspective will be $\tilde{p}_1(D_0) = 1 - D_0$ if the lender accepts a propsal that brings the borrower's cumulative debt to D_0. This defines the unique strategy of the final lender to accept only weakly profitable loans. Assume the borrower has any arbitrary face value of debt D_1 upon meeting the final lender. The borrower solves

$$V_1(D_1) \equiv \max_{D_0} z(D_0 - D_1)(1 - D_0) + \left(\frac{D_0^2}{2} - D_0\right)$$

and the solution is

$$1 - D_0 = (1 - D_1) \underbrace{\frac{z}{2z - 1}}_{\equiv B_1}$$

with borrower's maximized value function being

$$V_1 (D_1) = (1 - D_1)^2 \underbrace{\left[\frac{2z^3 - z^2}{2 (2z - 1)^2} \right]}_{\equiv W_1} - \frac{1}{2}$$

We define

$$\begin{aligned}
B_1 &\equiv \frac{z}{2z - 1} \\
L_1 &\equiv 1 \\
W_1 &\equiv \frac{2z^3 - z^2}{2 (2z - 1)^2}
\end{aligned}$$

Thus the unique subgame perfect equilibrium strategies conditional on arriving to the last lender with some amount of debt D are:

$$\begin{aligned}
1 - g (D) &= B_1 (1 - D) \\
\tilde{p}_1 (D) &= L_1 (1 - D)
\end{aligned}$$

and any borrower considering leaving the second-to-last lender with a total face value of debt D realizes that continuation utility if it reaches the last lender is given by

$$V_1 (D) = W_1 (1 - D)^2 - \frac{1}{2}.$$

Thus we know that at lender $i = 1$ players use strategies linear in $1 - D$. Now we show by induction that all lenders use such linear strategies. Assume for some n that players at all stages $i < n$ use linear strategies and that the maximized value function at lender i is proportional to $(1 - D_{i+1})^2$. We will show that players at stage n also use linear strategies and that the maximized value function at lender n is proportional to $(1 - D_{n+1})^2$, and thus by induction prove that these claims do indeed hold for all $n \in \mathbb{N}$.

Now consider the subgame where the borrower meets lender n with cumulative debt D_n obtained from previous lenders. Since all future lenders and borrowers use linear strategies, we can compute lender $n's$ expected probability of repayment:

$$
\begin{aligned}
\tilde{p}_n (D_{n-1}) &= (1-q)(1-D_{n-1}) + q\tilde{p}_{n-1}(D_{n-2}) \\
&= (1-q)(1-D_{n-1}) + qL_{n-1}(1-D_{n-2}) \\
&= (1-q)(1-D_{n-1}) + qL_{n-1}B_{n-1}(1-D_{n-1}) \\
&= [(1-q) + qB_{n-1}L_{n-1}](1-D_{n-1})
\end{aligned}
$$

where the first to second line follows from the assumption that lender $n-1$ is using a linear strategy $\tilde{p}_{n-1}(D) = L_{n-1}(1-D)$, and moving from the second to the third line relies on the assumption that the borrower at $n-1$ is using a linear strategy $1 - D_{n-2} = B_{n-1}(1-D_{n-1})$. Thus we know that lender n follows the strategy given by

$$
L_n = 1 - q + qB_{n-1}L_{n-1}
$$

Next, under our inductive hypothesis we can write the borrower's problem visiting lender n as:

$$
\begin{aligned}
V_n(D_n) &= \max_{D_{n-1}} z\tilde{p}_n(D_{n-1})(D_{n-1} - D_n) + (1-q)\left(\frac{D_{n-1}^2}{2} - D_{n-1}\right) + qV_{n-1}(D_{n-1}) \\
&= \max_{D_{n-1}} \Bigg\{ z(D_{n-1} - D_n)(1-D_{n-1})L_n + (1-q)\left(\frac{D_{n-1}^2}{2} - D_{n-1}\right) \\
&\quad +q\left(W_{n-1}(1-D_{n-1})^2 - \frac{1}{2}\right)\Bigg\}
\end{aligned}
$$

Taking the first order condition and solving or D_{n-1} verifies that the borrower's strategy does indeed have the hypothesized linear form:

$$1 - D_{n-1} = (1 - D_n) \underbrace{\frac{zL_n}{2zL_n - (1 - q + 2qW_{n-1})}}_{\equiv B_n}$$

and maximized value function

$$V_n(D_n) = (1 - D_n)^2 \underbrace{\left(z(1 - B_n) B_n L_n + (1 - q) \frac{B_n^2}{2} + qB_n^2 W_{n-1} \right)}_{\equiv W_n} - \frac{1}{2}$$

Thus the inductive proof is completed and all strategies satisfy the proposed form.

Recursive Formulation of Strategies

It is clear from above that the vector (B_n, L_n, W_n) is generated by a system of 3 difference equations:

$$\begin{aligned} L_n &= (1 - q) + qB_{n-1}L_{n-1} \\ B_n &= \frac{zL_n}{2zL_n - (1 - q + 2qW_{n-1})} \\ W_n &= z(1 - B_n) B_n L_n + (1 - q) \frac{B_n^2}{2} + qB_n^2 W_{n-1} \end{aligned}$$

rearranging so each of (L_n, B_n, W_n) is a function of only the lagged variables:

$$\begin{aligned} L_n &= 1 - q + qB_{n-1}L_{n-1} \\ B_n &= \frac{z(1 - q + qB_{n-1}L_{n-1})}{2z(1 - q + qB_{n-1}L_{n-1}) - (1 - q + 2qW_{n-1})} \\ W_n &= \frac{z^2(1 - q + qB_{n-1}L_{n-1})^2}{4z(1 - q + qB_{n-1}L_{n-1}) - 2(1 - q + 2qW_{n-1})} \end{aligned}$$

where the last equation can be simplified to

$$W_n = \frac{z}{2} B_n L_n.$$

Convergence to Infinite-Lender Strategies

Defining a new variable $x_n \equiv B_n L_n$, the set of difference equations above can be rewritten as

$$L_n = (1-q) + qx_{n-1}$$

$$B_n = \frac{z\left((1-q)+qx_{n-1}\right)}{2z\left((1-q)+qx_{n-1}\right)-\left(1-q+zqx_{n-1}\right)}$$

$$W_n = \frac{z}{2}x_n$$

Hence the sequence $\{x_n\}$ is defined by $x_1 \equiv B_1 L_1 = \frac{z}{2z-1}$ and a continuous function $f(\cdot)$ such that $x_n = f(x_{n-1})$, where

$$f(x) = \frac{z\left((1-q)+qx\right)^2}{(1-q)(2z-1)+zqx}$$

First note that $f(x) > 0$ if $x > 0$, and since $x_1 > 0$, we have $x_n > 0$ for all n. It is also easily verified that $f(\cdot)$ has a unique fixed point x^*, which corresponds to the unique fixed point (B^*, L^*, W^*) of the system of difference equations above. Moreover, it is a matter of algebra to show that the fixed point coincides with our closed form solution of the infinite lender game, implying that $b^* = B^*$ and $\ell^* = L^*$.

We next show that $f(\cdot)$ defines a contraction mapping which, given the continuity of $f(\cdot)$, shows that $x_n \to x^*$. This in turn implies $(B_n \to b^*, L_n \to \ell^*)$ as $n \to \infty$. In words, this means that as the number of potential future lenders in the game after lender n grows towards infinity, the unique subgame perfect strategies the players at stage n converge to the stationary strategies employed by all players in the infinite-lender game.

To show $f(\cdot)$ defines a contraction mapping, we show $||f'(x)|| < 1$. Taking derivatives of f with respect to x, we have (after much simplification)

$$\begin{aligned} f'(x) &= q - \frac{q(z-1)^2(1-q)^2}{(2z+q-2zq+zqx-1)^2} \\ &= q\left(1-\left(\frac{(z-1)(1-q)}{(2z-1)(1-q)+zqx}\right)^2\right) \end{aligned}$$

Since $x > 0$, $1 > q \geq 0$, $z > 1$, we have

$$0 < \frac{(z-1)(1-q)}{(2z-1)(1-q)+zqx} < 1$$

$$0 < \frac{(z-1)(1-q)}{(z-1)(1-q)+z(1-q)+zqx} < 1$$

Hence

$$0 < f'(x) < 1$$

which shows that $f(\cdot)$ is a contraction mapping.

□

Proof of Proposition 4

Before proving the comparative static propositions, it will be useful to derive some expressions for equilibrium objects of interest in terms of parameters z and q. Let K^{agg} denote the aggregate investment and D^{agg} denote the aggregate debt that have been attained when the lending market game ends. In equilibrium, ex-ante, these are random variables with respect to the number of lenders the borrower will be able to visit.

Lemma 2. $\mathbb{E}[K^{agg}] = \mathbb{E}[p(D^{agg})D^{agg}]$

Proof. Let N denote the random number of lenders the borrower gets to visit before losing access to the lending market game. The random aggregate face value of debt and aggregate investment can be expressed as:

$$D^{agg} = \sum_{j=1}^{\infty} d_j \mathbf{1}(N \geq j)$$

$$K^{agg} = \sum_{j=1}^{\infty} k_j \mathbf{1}(N \geq j)$$

where d_j is the amount of debt given by the j-th lender . Similarly denote k_j to be the amount of investment capital provided by the j-th lender. Pick any $j > 0$, the zero-profit condition for his loan and investment size is:

$$\mathbb{E}\left[p\left(D^{agg}\right)|N \geq j\right] d_j \mathbf{1}\left(N \geq j\right) = k_j \mathbf{1}\left(N \geq j\right)$$

Taking expectation over N on both sides and applying the law of iterated expectation, we get:

$$\mathbb{E}\left[p\left(D^{agg}\right) d_j \mathbf{1}\left(N \geq j\right)\right] = \mathbb{E}\left[k_j \mathbf{1}\left(N \geq j\right)\right]$$

We next sum the previous equation over all lenders. By the linearity of the expectations operator, we can bring the sum inside:

$$\mathbb{E}\left[p\left(D^{agg}\right) \sum_{j=1}^{\infty} d_j \mathbf{1}\left(N \geq j\right)\right] = \mathbb{E}\left[\sum_{j=1}^{\infty} k_j \mathbf{1}\left(N \geq j\right)\right]$$

Substituting in the defintions of D^{agg} and I^{agg}:

$$\mathbb{E}\left[p\left(D^{agg}\right) D^{agg}\right] = \mathbb{E}\left[K^{agg}\right]$$

□

Lemma 3. *We can express the expected debt and investment as functions of b:*

$$\mathbb{E}\left[D^{agg}\right] = \frac{1-b}{1-qb}$$

$$\mathbb{E}\left[K^{agg}\right] = \frac{b\left(1-b\right)\left(1-q\right)}{\left(1-bq\right)\left(1-b^2q\right)}$$

Proof. Denote the expected aggregate debt upon leaving a given lender with cumulative debt D as $\mathbb{E}\left[D^{agg}|D\right]$. From lender's zero-profit condition, we have

$$\mathbb{E}\left[D^{agg}|D\right] = 1 - \tilde{p}\left(D\right)$$

The ex-ante expected aggregate debt $\mathbb{E}\left[D^{agg}\right]$ is simply the expected aggregate debt upon leaving a lender with zero outstanding debt, times $\frac{1}{q}$ (since the borrower meets the first lender with certainty, not probability q). Thus we have

$$\begin{aligned}
\mathbb{E}\left[D^{agg}\right] &= \frac{1}{q}\mathbb{E}\left[D^{agg}|0\right] \\
&= \frac{1}{q}\left(1-\frac{1-q}{1-qb}\right) \\
&= \frac{1-b}{1-qb}
\end{aligned}$$

To get the expression for expected investment:

$$\begin{aligned}
\mathbb{E}\left[K^{agg}\right] &= \tilde{p}\left(g\left(0\right)\right)g\left(0\right)+q\tilde{p}\left(g^{2}\left(0\right)\right)\left[g^{2}\left(0\right)-g\left(0\right)\right]+q^{2}\tilde{p}\left(g^{3}\left(0\right)\right)\left[g^{3}\left(0\right)-g^{2}\left(0\right)\right]+... \\
&= \ell\left[\left(1-g\left(0\right)\right)g\left(0\right)+q\left(1-g^{2}\left(0\right)\right)\left[\left(1-g\left(0\right)\right)-\left(1-g^{2}\left(0\right)\right)\right]+...\right] \\
&= \ell\left[b\left(1-b\right)+qb^{2}\left[b-b^{2}\right]+q^{2}b^{3}\left[b^{2}-b^{3}\right]+...\right] \\
&= \ell b\left(1-b\right)\left[1+qb^{2}+q^{2}b^{4}+...\right] \\
&= \ell b\frac{1-b}{1-qb^{2}} \\
&= \frac{b\left(1-b\right)\left(1-q\right)}{\left(1-bq\right)\left(1-qb^{2}\right)}
\end{aligned}$$

□

Lemma 4. *Let* $x\equiv\sqrt{4\left(1-q\right)\left(z^{2}-z\right)+1}$. *The analytic solution of expected debt, investment, and welfare can be expressed as the following functions of parameters* q *and* z*:*

$$\mathbb{E}\left[D^{agg}\right]=\frac{2z-1-x}{2qz}$$

$$\mathbb{E}\left[K^{agg}\right]=\frac{\left(z-1\right)\left(x+1-2z\left(1-q\right)\right)}{2zq\left(2z-1\right)}$$

$$V\left(0\right)=\frac{1-2z\left(1-q\right)-2q+x}{4q}$$

Proof. These expressions can be obtained by substituting the analytic solution of b^* from lemma 1 into the expressions in lemma 2. $\square$

Now continuing on, we can express equilibrium b^* as

$$(b^*) = \frac{2z - 1 - x}{2q(z-1)}$$

Also note that

$$\frac{\partial x}{\partial z} = x^{-1}(1-q)\,2\,(2z-1) > 0$$
$$\frac{\partial x}{\partial q} = -2x^{-1}\left(z^2 - z\right) < 0$$

We now proceed to prove Proposition 4 claim by claim.

Claim 1. $\mathbb{E}[D^{agg}]$ *is increasing in* q *and* $\partial D_i / \partial q < 0$ for all i.

Proof. We first express $\mathbb{E}[D^{agg}]$ as a function of z, q, and x:

$$\mathbb{E}[D^{agg}] = \frac{1-b}{1-qb}$$

$$\begin{aligned}
\mathbb{E}[D^{agg}] &= \frac{1 - \frac{2z-1-x}{2q(z-1)}}{1 - \frac{2z-1-x}{2(z-1)}} \\
&= \frac{2q(z-1) - 2z + 1 + x}{2q(z-1) - 2qz + q + qx} \\
&= \frac{(2qz - 2q - 2z + 1 + x)(x+1)}{q(x-1)(x+1)} \\
&= \frac{(2z-1-x)(1-q)(z-1)}{2zq(1-q)(\alpha-1)} \\
&= \frac{2z-1-x}{2zq}
\end{aligned}$$

Differentiating with respect to q, we get

$$\frac{d\mathbb{E}[D^{agg}]}{dq} = \frac{-2zq\frac{dx}{dq} - (2z-1-x)\,2z}{(2qz)^2}$$

which implies

$$\begin{aligned} sign\left(\frac{\partial \mathbb{E}\left[D^{agg}\right]}{\partial q}\right) &= sign\left(2q\left(z^2-z\right)-\left(2z-1-x\right)x\right) \\ &= sign\left(2q\left(z^2-z\right)+4\left(1-q\right)\left(z^2-z\right)+1-\left(2z-1\right)x\right) \\ &= sign\left(\left(z^2-z\right)\left(4-2q\right)+1-\left(2z-1\right)x\right) \end{aligned}$$

Let $RHS \equiv (z^2-z)(4-2q)+1-(2z-1)x$. The remaining proof consists of three steps: 1) show $\frac{dRHS}{dz} \geq 0$ for all $z \geq 1$, $q \in [0,1]$; 2) show $\frac{dRHS}{dq} \geq 0$ for all $z \geq 1$, $q \in [0,1]$, with equality holding only when $z=1$ or $q=0$; 3) RHS evaluated at $z=1, q=0$ is zero, concluding that $RHS > 0$ for $z > 1, q > 0$.

Step 1: show $\frac{dRHS}{dz} > 0$ for all $z \geq 1$, $q \in [0,1]$. Differentiating RHS with respect to z, we have

$$\begin{aligned} \frac{dRHS}{dz} &= (2z-1)(4-2q)-2x-(2z-1)\frac{\partial x}{\partial z} \\ &= (2z-1)\left(4-2q-x^{-1}(1-q)(4z-2)\right)-2x \\ &> (2z-1)\left(4-2q-x^{-1}(1-q)(4z-2)\right) \\ &\geq (2z-1)\left(4-\max_{q\in[0,1]}(2q)-\max_{q\in[0,1]}x^{-1}(1-q)(4z-2)\right) \\ &= (2z-1)(4-2-2) \\ &= 0 \end{aligned}$$

Step 2: show $\frac{dRHS}{dq} > 0$ for all $z \geq 1$, $q \in [0,1]$, with equality holding only when $z=1$ or $q=0$. Differentiating RHS with respect to q, we have

$$\begin{aligned} \frac{dRHS}{dq} &= -2\left(z^2-z\right)-(2z-1)\frac{\partial x}{\partial q} \\ &= \left(z^2-z\right)\left(2x^{-1}(2z-1)-2\right) \\ &= \underbrace{2x^{-1}}_{>0}\underbrace{\left(z^2-z\right)}_{\geq 0}(2z-1-x) \end{aligned}$$

The last term is non-negative and is zero only when $q = 0$. To see this, note

$$\begin{aligned} x &= \sqrt{4(1-q)(z^2-z)+1} \\ \text{(equal only if } q=0) &\leq \sqrt{4z^2-4z+1} \\ &= 2z-1 \end{aligned}$$

Hence we have $\frac{dRHS}{dq} > 0$.

Step 3: conclude the proof. Note

$$RHS|_{q=0,z=1} = 0$$

Hence we have, for any $z > 1$ and $q > 0$, $RHS > 0$. Thus $\frac{\partial \mathbb{E}[D^{agg}]}{\partial q} > 0$.

To show the second part, that $dD_i/dq < 0$ for all i, note that $D_i = 1 - b^i$ and thus it suffices to show $db/dq > 0$. To show this, we rely on the fact that b and ℓ form the fixed point of the best response,

$$\ell = L(b; z, q) \qquad \text{and} \qquad b = B(\ell; z, q).$$

Totally differentiating the best response, we get

$$\frac{d\ell}{dq} = L_b \frac{db}{dq} + L_q, \qquad \frac{db}{dq} = B_l \frac{d\ell}{dq} + B_q.$$

Substitute out $d\ell/dq$, we get

$$\frac{db}{dq} = \frac{B_l L_q + B_q}{1 - B_l L_b}$$

We know $L_b > 0$, $B_l < 0$, $L_q < 0$, and $B_q > 0$; hence $db/dq > 0$, as desired. □

Claim 2. $\mathbb{E}[K^{agg}]$ is decreasing in q.

Proof. Given the result in lemma (4), we first show that investment is decreasing in q if and only if the exante welfare for the borrower is decreasing in q. To see this, note

$$\begin{aligned}
\mathbb{E}\left[K^{agg}\right] &= \frac{(z-1)\left(x+1-2z\left(1-q\right)\right)}{2zq\left(2z-1\right)} \\
&= \frac{2\left(z-1\right)}{z\left(2z-1\right)}\cdot\left[\frac{\left(x+1-2z\left(1-q\right)-2q\right)}{4q}+\frac{1}{2}\right] \\
&= \frac{2\left(z-1\right)}{z\left(2z-1\right)}\cdot\left[V\left(0\right)+\frac{1}{2}\right]
\end{aligned}$$

To show $V(0)$ is decreasing in q, first note

$$V\left(0\right)=\frac{1-2z\left(1-q\right)-2q+x}{4q}$$

$$\begin{aligned}
\frac{dV\left(0\right)}{dq} &= \frac{4q\left(2z-2+x_q\right)-4\left(1+2z\left(q-1\right)-2q+x\right)}{16q^2} \\
&= \frac{1}{4q^2}\left[q\left(2z-2+x_q\right)-1+2z\left(1-q\right)+2q-x\right] \\
&= \frac{1}{4q^2}\left[-2z\left[\left(z-1\right)\left(\frac{q}{x}\right)-1\right]-\left(1+x\right)\right]
\end{aligned}$$

Define $\Omega\equiv-2z\left[\left(z-1\right)\left(\frac{q}{x}\right)-1\right]-\left(1+x\right)$, we then have $sign\left(\frac{dV(0)}{dq}\right)=sign\left(\Omega\right)$.

To compute the derivative of Ω with respect to q:

$$\begin{aligned}
\frac{d\Omega}{dq} &= -2z\left(z-1\right)\frac{x-qx_q}{x^2}-x_q \\
&= -\frac{1}{x^3}4z^2\left(z-1\right)^2q\leq 0
\end{aligned}$$

Therefore Ω is declining in q for all $z>1$. This means to check that $\frac{dV(0)}{dq}<0$ it is sufficient to check that $\left.\Omega\right|_{q=0}\leq 0$.

$$\begin{aligned}
\Omega\left(q=0\right) &= \left(2z-1\right)-x \\
&= 0
\end{aligned}$$

Hence welfare is decreasing in q. $\square$

Claim 3. Default probability is increasing in q.

Proof. Note $\mathbb{E}\left[Pr\left(Default\right)\right] = \mathbb{E}\left[D^{agg}\right]$ and the claim follows directly from claim 1. $\square$

Claim 3. $\mathbb{E}\left[D^{agg}\right]/\mathbb{E}\left[K^{agg}\right]$ *is increasing in q.*

Proof. Follows directly from claims 1 and 2. $\square$

Claim 4. $\lim_{q\to 1} V\left(0\right) = 0$ *and* $\lim_{q\to 1}\mathbb{E}\left[D^{agg}\right] = \frac{z-1}{z}$.

Proof. We can write the ex-ante welfare as

$$V\left(0\right) = \frac{1 + 2z\left(q-1\right) - 2q + x}{4q}$$

Using the fact that $\lim_{q\to 1} x = 1$ and taking limit, the first result is immediate.

To show $\mathbb{E}\left[D^{agg}\right] \to \frac{z-1}{z}$, note $\mathbb{E}\left[D^{agg}\right] = \frac{1-b}{1-qb}$. Using the expression that $b = \frac{z}{\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)}}$, we can write

$$1 - b = \frac{\sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)} - 1/2}{\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)}}$$

$$1 - qb = \frac{z\left(1-q\right) + \sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)} - 1/2}{\left(z-\frac{1}{2}\right)+\sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)}}$$

Hence

$$\begin{aligned}
\mathbb{E}\left[D^{agg}\right] &= \frac{1-b}{1-qb} \\
&= \frac{\sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)} - 1/2}{z\left(1-q\right) + \sqrt{\left(z-\frac{1}{2}\right)^2 - qz\left(z-1\right)} - 1/2}
\end{aligned}$$

Using L'Hopital's rule, we get

$$\begin{aligned}
\lim_{q\to 1}\mathbb{E}\left[D^{agg}\right] &= \frac{\lim_{q\to 1}\left[\frac{1}{2}z\left(z-1\right)\left(\left(z-\frac{1}{2}\right)^2-qz\left(z-1\right)\right)^{-1/2}\right]}{\lim_{q\to 1}\left[z+\frac{1}{2}z\left(z-1\right)\left(\left(z-\frac{1}{2}\right)^2-qz\left(z-1\right)\right)^{-1/2}\right]}\\
&= \frac{z\left(z-1\right)}{z+z\left(z-1\right)}\\
&= \frac{z-1}{z},
\end{aligned}$$

as desired. □

Proof of Proposition 5

From lemma (3) we have

$$\mathbb{E}\left[D^{agg}\right] = \frac{1-b}{1-qb}$$

$$\mathbb{E}\left[D^{agg}\right]/\mathbb{E}\left[K^{agg}\right] = \frac{1-b^2q}{b\left(1-q\right)}$$

both of which are decreasing in b. It is thus sufficient to show that in equilibrium b^* is increasing in z. To see this, consider the equilibrium condition

$$b^* = b\left(\ell\left(b^*, z\right), z\right)$$

differentiating with respect to z gives

$$\frac{db^*}{dz} = \frac{\ell_z b_\ell + b_z}{1-b_\ell \ell_b}.$$

We know from Lemma 1 that $\ell_z = 0$, $b_z < 0$,$b_\ell < 0$, and $\ell_b > 0$. Plugging these signs in above gives immediately that $\frac{db^*}{dz} < 0$, establishing the results to prove part 1 of the proposition.

From Lemma (4) we have

$$\mathbb{E}\left[K^{agg}\right]=\frac{(z-1)\left(x+1-2z\left(1-q\right)\right)}{2zq\left(2z-1\right)}$$

Differentiating with respect to z, we get

$$\frac{d\mathbb{E}\left[K^{agg}\right]}{dz}=\frac{q\left(2z-2z^2x-10z^2+8z^3\right)-\left(x-6z+4z^2x+12z^2-8z^3-4zx+1\right)}{2z^2q\left(2z-1\right)^2x}$$

From this expression, one can verify that

$$\frac{\partial^2\mathbb{E}\left[K^{agg}\right]}{\partial q\partial z}<0,$$

and that for any given $q\in(0,1)$, there exists an unique $\bar{z}\left(q\right)\in(1,\infty)$ such that

$$\frac{d\mathbb{E}\left[K^{agg}\right]}{dz}\begin{cases}<0 & \text{for } 1\leq z<\bar{z}\left(q\right)\\ >0 & \text{for } z>\bar{z}\left(q\right)\end{cases}$$

and

$$\frac{d\bar{z}\left(q\right)}{dq}<0.$$

To show welfare is increasing in z we can start with the expression for welfare and differentiate with respect to z

$$V\left(0\right)\propto 1-2z\left(1-q\right)-2q+\sqrt{\left(2z-1\right)^2\left(1-q\right)+q}$$

$$\begin{aligned}\frac{\partial V\left(0\right)}{\partial z}&\propto 2\left(1-q\right)\left[\frac{2z-1}{\sqrt{\left(2z-1\right)^2\left(1-q\right)+q}}-1\right]\\&\propto\left[\frac{\left(2z-1\right)}{\sqrt{\left(2z-1\right)^2-q\left(\left(2z-1\right)^2-1\right)}}-1\right]\end{aligned}$$

so we have

$$\begin{aligned}
\frac{\partial V(0)}{\partial z} > 0 &\iff (2z-1) > \sqrt{(2z-1)^2 - q\left((2z-1)^2 - 1\right)} \\
&\iff (2z-1)^2 > (2z-1)^2 - q\left((2z-1)^2 - 1\right) \\
&\iff (2z-1)^2 > 1
\end{aligned}$$

which is true because $z > 1$. Thus welfare is increasing in z.

$\square$

Proof of Proposition 6

Proof: Define $i(z;q) \equiv \frac{\mathbb{E}[K^{agg}(z);q]}{\int \mathbb{E}[K^{agg}(z);q]dF(z)}$, which measures the expected investment of a firm with productivity z relative to the average investment level in the entire economy. Observe that by construction $\int i(z)\,dF(z) = 1$. We know from Proposition 5 that $\frac{\partial^2 \mathbb{E}[K^{agg};q]}{\partial q \partial z} < 0$; hence, for any $q_2 > q_1$, $i(z;q_1)$ and $i(z;q_2)$ cross exactly once, meaning there exists a cutoff z^{cross} such that $i(z;q_2) \gtreqless i(z;q_1)$ for $z \lesseqgtr z^{cross}$. We can therefore write:

$$\begin{aligned}
\tilde{Z}(q_2) - \tilde{Z}(q_1) &= \int_1^{z^{cross}} \underbrace{\left(i_2(z) - i_1(z)\right)}_{>0} z dF(z) + \int_{z^{cross}}^{\infty} \underbrace{\left(i_2(z) - i_1(z)\right)}_{<0} z dF(z) \\
&< \int_1^{z^{cross}} \underbrace{\left(i_2(z) - i_1(z)\right)}_{>0} z^{cross} dF(z) + \int_{z^{cross}}^{\infty} \underbrace{\left(i_2(z) - i_1(z)\right)}_{<0} z^{cross} dF(z) \\
&= \int_1^{\infty} i_2(z)\,dF(z) - \int_1^{\infty} i_1(z)\,dF(z) \\
&= 0.
\end{aligned}$$

Thus we can conclude that $\tilde{Z}$ is decreasing in q.

Proof of Proposition 7

Let $EK^b(z,q)$ and $ED^b(z,q)$ denote the expected investment and debt in the baseline model, and let $EK^p(z,\theta,q)$ and $ED^p(z,\theta,q)$ denote the expected investment and debt with pledgeability θ. Observe that $EK^p(z,\theta,q) = EK^b\left(\frac{z-\theta}{1-\theta}, q\right)\frac{1}{1-\theta}$ and $ED^p(z,\theta,q) = ED^b\left(\frac{z-\theta}{1-\theta}, q\right)$. The proposition follows directly from these observations.

□

Proof of Proposition 9

Let S be the sum of lender and borrower surplus. We know $\tilde{p}(D+d)\,d - k = \beta S$. We now write S as a function of D and d only, substituting out k:

$$\begin{aligned} S &= (z-1)\,k + W(D+d) - W(D) + \tilde{p}(D+d)\,d \\ &= (z-1)\left(\tilde{p}(D+d)\,d - \beta S\right) + W(D+d) - W(D) + \tilde{p}(D+d)\,d \\ &= \frac{z\tilde{p}(D+d)\,d + W(D+d) - W(D)}{1+(z-1)\beta}. \end{aligned}$$

At the time of contracting, the two parties choose d to maximize joint surplus and then choose k according to the Nash bargaining equation to divide surplus. Hence borrower's value function $V(\cdot)$ can be written recursively as

$$V(D) = \underbrace{W(D)}_{\text{outside option}} \underbrace{+\,(1-\beta)\max_{d}\frac{z\tilde{p}(D+d)\,d + W(D+d) - W(D)}{1+(z-1)\beta}}_{\text{borrower captures } (1-\beta) \text{ fraction of joint surplus}}$$

Substitute using $V(D) = \frac{W(D)+(1-q)\mathbb{E}[\min\{D,\tilde{c}\}]}{q}$, we have

$$\frac{W(D) + (1-q)\,\mathbb{E}\left[\min\{D,\tilde{c}\}\right]}{q} = W(D) + (1-\beta)\max_{d}\frac{z\tilde{p}(D+d)\,d + W(D+d) - W(D)}{1+(z-1)\beta}$$

and, after simplifying and substituting $\hat{q} \equiv \frac{q(1-\beta)}{1-\beta+z\beta(1-q)}$,

$$\begin{aligned} W(D) &= -(1-\hat{q})\left(D - D^2/2\right) + \hat{q}\times\max_{d}\left\{z\tilde{p}(D')(D'-D) + W(D')\right\} \\ &= -(1-\hat{q})\left(D - D^2/2\right) + \hat{q}\times\max_{D}\left\{z\ell\left(1-g(D)\right)(D'-D) + W(D')\right\} \end{aligned}$$

Envelope Condition:

$$W'(D) = -(1-\hat{q})(1-D) - \hat{q}z\tilde{p}(g(D))$$
$$= -(1-\hat{q})(1-D) - \hat{q}z\ell(1-g(D))$$

$$W'(D') = -(1-\hat{q})(1-g(D)) - \hat{q}z\ell\left(1-g^2(D)\right)$$

FOC

$$0 = -\hat{q}z\ell(g(D)-D) + \hat{q}z\ell(1-g(D)) + \hat{q}W'(g(D))$$

Plugging in envelope into FOC and simplifying gives

$$\hat{q}z\ell b^2 + (1-q-2z\ell)b + z\ell = 0$$

which is the same quadratic as in the proof of Lemma 1. Hence, the best response of b given ℓ, z, and $\hat{q}$ has the same form as in Lemma 1, but with $\hat{q}$ replacing q.

$$b = \frac{(2z\ell-(1-\hat{q})) - \sqrt{(2z\ell-(1-\hat{q}))^2 - 4\hat{q}z^2\ell^2}}{2\hat{q}z\ell}$$

$$\ell = \frac{1-q}{1-qb}$$

Defining $\gamma \equiv \frac{1-\beta}{1-\beta+z\beta(1-q)}$ and plugging in $\ell(b)$ into $b(\ell)$, rearranging, and squaring and taking the correct root, we have

$$0 = \left[b^2\gamma q + 1\right] z(1-q) + b(1-\gamma q)(1-qb) - 2bz(1-q)$$

$$0 = b^2\left[(\gamma z - 1) - \gamma q(z-1)\right] + b\left[(1-\gamma q) - 2z(1-q)\right] + \left[z(1-q)\right]$$

$$b^* = \frac{2z\left(1-q\right)-\left(1-\gamma q\right)-\sqrt{\left(\left(1-\gamma q\right)-2z\left(1-q\right)\right)^2-4q\left(\left(\gamma z-1\right)-\gamma q\left(z-1\right)\right)z\left(1-q\right)}}{2q\left(\left(\gamma z-1\right)-\gamma q\left(z-1\right)\right)}$$

an algabraeic rearrangment also gives

$$b^* = \frac{2z\left(1-q\right)}{\left(2z\left(1-q\right)-\left(1-\gamma q\right)\right)+\sqrt{\left(\left(1-\gamma q\right)-2z\left(1-q\right)\right)^2-4q\left(\left(\gamma z-1\right)-\gamma q\left(z-1\right)\right)z\left(1-q\right)}}$$

which reduces to the expression for b^* derived above when $\beta = 0$.

$\square$

Proof of Proposition 10

Proof of Claim 1. The first-order condition in equation (1) that characterizes the single-lender equilibrium is

$$z \times \left[p\left(D^*\right)+p'\left(D^*\right)D^*\right] = p\left(D^*\right)$$

If an interest rate cap were set to be $1+\bar{r}^{SL} \equiv \frac{1}{p(D^*)}$, the borrower could propose to pledge D^* and raise $p\left(D^*\right)D^*$ from the very first lender. No future lender would be willing to provide additional investment to the borrower because doing so would require an interest rate higher than $1+\bar{r}^{SL}$ to break even, but such a rate is prohibited by the interest rate cap. Hence the full commitment allocation can be achieved under $\bar{r}^{SL}$. Substituting for $p\left(D\right) = 1-D$, the full-commitment debt issuance is $D^* = \frac{z-1}{2z-1}$. The optimal interest cap is thus

$$\begin{aligned} 1+\bar{r}^{SL} &= \frac{1}{1-D^*} \\ &= 1-\frac{1}{z} \end{aligned}$$

Claim 2 follows directly from the fact that there is an one-to-one relationship between risky debt issuance and the probability of repayment.

Proof of Claim 3. When $\bar{r} < \bar{r}^{SL}$, the interest rate cap is inefficiently low and the borrower can pledge less debt facevalue than he would have done under full commitment. The unique equilibirum under the interest rate cap would involve the borrower pledging $D = 1 - \frac{1}{1+\bar{r}}$ debt and raising $(1-D)\,D$ investment from the very first lender. In the extreme case where $\bar{r} = 0$, the borrower would be unable to raise any investment from the lenders, achieving an even lower level of welfare than under the unregulated equilibrium.

Proof of Claim 4. When $\bar{r} > \bar{r}^{Sl}$, the full commitment allocation is unattainable as with probability q the borrower will meet the second lender and pledge a strictly positive amount of debt for any level of outstanding debt below one.

Next we show that for $\bar{r} < \infty$ the cap unambiguously improves expected investment and welfare while lowering expected debt and interest rate relative to the unregulated equilibrium. Using the techniques in the proof for proposition 3, for any game with finite lenders K we can find a sequence of aggregate debt $\left\{D_1^K, ..., D_K^K\right\}$ where D_i^K corresponds to the aggregate debt level had the borrower reach lender i in a game with total lender K, where lender indices start backwards with the last lender being lender 1. Using a simple perturbation argument, we know that for K such that $D_1^K < \bar{D} \leq D_1^{K+1}$, the infinite lender game with debt cap $\bar{D}$ would have a unique SPE where the borrower reaches the debt cap when borrowing from $(K+1)$-th lender. Furthermore, using the same recursive definition of lender and borrower strategies in equilibrium as we adopted in proposition 3, it is clear that borrower's ex-ante expected investment, aggregate debt level, and welfare with debt cap $\bar{D}$ is in between the corresponding equilibrium quantities for the finite lender games with K and $K+1$ lenders.

Proof of Proposition 11

In equilibrium, interest rates are set to exactly compensate lenders for their expected risk of default and to cover their net tax liability. If lenders know they will be compensated for dilution associated with future borrowing, they are willing to price loans as if they expected no future borrowing. Thus we know that an equilibrium loan satisfies

$$k_n + \tau_n = p\,(D_n)\,d_n, \tag{11}$$

and the taxes and transfers are chosen so that

$$\tau_n = [p(D_{n-1}) - p(D_n)]\, D_{n-1}, \tag{12}$$

$$\tau_n^i = [p(D_{n-1}) - p(D_n)]\, d_i. \tag{13}$$

We first show $\tau_n = K_{n-1}\frac{d_n}{D_n} - k_n\frac{D_{n-1}}{D_n}$. Note that equations (11) and (12) jointly and inductively imply that $K_n = p(D_n)\, D_n$. Hence

$$\begin{aligned}
\tau_n &= [p(D_{n-1}) - p(D_n)]\, D_{n-1} \\
&= K_{n-1} - p(D_n)\, D_{n-1} \\
&= K_{n-1} - \left(\frac{k_n + \tau_n}{d_n}\right) D_{n-1} \\
&= K_{n-1}\frac{d_n}{D_n} - k_n\frac{D_{n-1}}{D_n},
\end{aligned}$$

where the third equality follows by substituting for $p(D_n)$ using equation (11).

That $\tau_n^i = \frac{d_i}{D_{n-1}} \times \tau_n$ follows immediately from equation (13).

$\square$