Online Appendix

for

"Agency Conflicts and Short- versus Long-Termism in Corporate Policies"

Sebastian Gryglewicz* Simon Mayer† Erwan Morellec‡

August 13, 2019

*Erasmus University Rotterdam. Email: gryglewicz@ese.eur.nl

†Erasmus University Rotterdam. E-mail: mayer@ese.eur.nl.

‡EPF Lausanne, Swiss Finance Institute, and CEPR. Phone: +41 21 693 0116. Fax: +41 21 693 0060. E-mail: erwan.morellec@epfl.ch.

A Asymmetric performance pay with convex cost

In this section, we demonstrate that asymmetric performance-pay may also arise in our baseline model with strictly convex adjustement cost of investment. This is the case when the bound $\ell_{\max}$ becomes relevant for the principal's maximization problem. In general, optimal effort levels are given by:

$$
\begin{aligned}
s = s(w) &= \frac{\alpha + p''(w)\rho\sigma_X\sigma_K\lambda_s\left(\lambda_\ell\ell(w) - w\right)}{\lambda_s\alpha - p''(w)(\lambda_s\sigma_X)^2} \;\wedge\; s_{\max} \\
\ell = \ell(w) &= \frac{\mu\left(p(w) - p'(w)w\right) + p''(w)\rho\sigma_X\sigma_K\lambda_\ell\lambda_s s(w) - p''(w)w\lambda_\ell\sigma_K^2}{\lambda_\ell\mu - p''(w)(\lambda_\ell\sigma_K)^2} \;\wedge\; \ell_{\max}.
\end{aligned}
$$

The following Lemma demonstrates that asymmetric performance-pay arises when $\ell = \ell_{\max}$.

Lemma 1. *Let $w \in (0, \overline{w}]$ such that in optimum $\ell(w) = \ell = \ell_{\max}$ and $s(w) = s \in [0, s_{\max}]$. Assume that parameters satisfy $-\rho\sigma_K\lambda_\ell\ell_{\max} < \sigma_X\lambda_s s_{\max}$ for $\rho \in (-1,1)$. Then*

$$\beta^\ell \equiv \beta^\ell(w) = \max\left\{\lambda_\ell\ell_{\max}, w - \rho\frac{\sigma_X}{\sigma_K}\lambda_s s\right\} \text{ and } \beta^s \equiv \beta^s(w) = \lambda_s s.$$

In particular, the short-run IC-condition is always tight under the conditions stated.

Proof. Given the optimal choice $\ell(w) = \ell_{\max}$, $s(w) = s$, the tuple $(\beta^s(w), \beta^\ell(w))$ must satisfy

$$
\begin{gathered}
(\beta^s(w), \beta^\ell(w)) = \arg\min_{\beta^s, \beta^\ell}\left[(\beta^s\sigma_X)^2 + \sigma_K^2(\beta^\ell - w)^2 + 2\rho\sigma_X\sigma_K\beta^s(\beta^\ell - w)\right] \\
\text{subject to } \beta^\ell \geq \lambda_\ell\ell_{\max} \text{ and } \beta^s \geq \lambda_s s,
\end{gathered}
$$

where the last inequality is tight, unless $s = s_{\max}$. Using standard arguments, one obtains:

$$
\begin{aligned}
&\beta^\ell \equiv \beta^\ell(w) = \max\left\{\lambda_\ell\ell_{\max}, w - \rho\frac{\sigma_X}{\sigma_K}\beta^s\right\}; \\
&\beta^s \equiv \beta^s(w) = \max\left\{\lambda_s s_{\max}, \rho\frac{\sigma_K}{\sigma_X}(w - \beta^\ell)\right\} \text{ if } s = s_{\max} \text{ and } \beta^s = \lambda_s s \text{ otherwise.}
\end{aligned}
$$

The claim is trivial if $s < s_{\max}$ or $\rho = 0$.

Let us suppose $s = s_{\max}$, $\rho \neq 0$ and $\beta^s > \lambda_s s$. Hence, $\beta^s = \rho\frac{\sigma_K}{\sigma_X}(w - \beta^\ell)$. If now $\beta^\ell > \lambda_\ell\ell$, then $\beta^\ell = w - \rho\sigma_X/\sigma_K\beta^s$. This implies $\rho\sigma_K/\sigma_X(w - \beta^\ell) = \rho^2\beta^s < \beta^s$ and hence $\beta^s = \lambda_s s_{\max}$, a contradiction.

Next, suppose $\rho < 0$ and $\beta^\ell = \lambda_\ell\ell_{\max}$. Hence, $w > \lambda_\ell\ell_{\max}$. Since $\beta^\ell = \lambda_\ell\ell_{\max}$ it follows that $\lambda_\ell\ell_{\max} > w - \rho\sigma_X/\sigma_K\beta^s$ and - using $\beta^s = \rho\frac{\sigma_K}{\sigma_X}(w - \beta^\ell)$ - one obtains $\lambda_\ell\ell_{\max} > w - \rho^2(w - \lambda_\ell\ell_{\max})$. Hence, $\lambda_\ell\ell_{\max} > w$, a contradiction.

Finally, assume $s = s_{\max}$, $\rho < 0$ and $\beta^\ell = \lambda_\ell\ell_{\max}$. Hence, $\lambda_\ell\ell_{\max} > w$ and $\rho\sigma_K/\sigma_X(w - \lambda_\ell) > \lambda_s s_{\max}$, which implies $w - \lambda_\ell\ell_{\max} < \lambda_s s_{\max}\sigma_X/(\sigma_K\rho)$. Therefore, $-\rho\sigma_K\lambda_\ell\ell_{\max} > \sigma_X\lambda_s s_{\max}$, which contradicts the hypothesis. □

By means of the previous Lemma it is obvious, that asymmetric performance pay always arises when $\ell_{\max}$ is sufficiently low.

Next, we state Lemma 2, which shows that asymmetric performance pay occurs generally for large values of w and the set on which it occurs is convex. That is, there is asymmetric performance pay exactly above some threshold $w' < \overline{w}$, i.e., on the set $(w', \overline{w}]$.

Lemma 2. *Assume* $-\rho\sigma_K\lambda_\ell\ell_{\max} < \sigma_X\lambda_s s_{\max}$. *If there exists* $w' \geq \lambda_\ell\ell_{\max} + \max\{\rho, 0\}\sigma_X/\sigma_K s_{\max}$ *with* $\ell(w') = \ell_{\max}$, *then* $\ell(w) = \ell_{\max}$ *and* $\beta^\ell = w - \rho\sigma_X/\sigma_K s(w)$ *for all* $w \geq w'$.

Proof. Let us start at the point w' and plug-in optimal incentives

$$\max\left\{\lambda_\ell\ell_{\max}, w' - \rho\frac{\sigma_X}{\sigma_K}\lambda_s s\right\} = w' - \rho\frac{\sigma_X}{\sigma_K}\lambda_s s$$

into the HJB equation , so as to obtain the squared volatility $\Sigma(w') = (\lambda_s s(w)\sigma_X s(w))^2(1-\rho^2)$, which does not depend on ℓ anymore. Therefore, a necessary and sufficient condition for $\ell(w') = \ell_{\max}$ being optimal reads

$$p(w) - wp'(w) \geq \lambda_\ell\ell_{\max}$$

Owing to the concavity, the benefits of long-run investment, i.e., $p(w) - wp'(w)$ increase in w, while there is no agency-cost associated with long-run incentives when $\ell = \ell_{\max}$. Thus, $\ell(w) = \ell_{\max}$ is optimal for $w \geq w'$. □

Corollary 1. *Asymmetric performance-pay arises for* λ_ℓ *sufficiently low.*

Proof. Clearly, the limit $\lambda_\ell \to 0$ leads to $\ell(w) \to \ell_{\max}$ for all w, while $\lim_{\lambda_\ell\to 0}\overline{w} > 0$ owing to $\sigma_X, \sigma_K > 0$. The claim follows, as $\beta^\ell = \ell_{\max}\lambda_\ell$. □

B Model solution with private cost

In this section, we solve the model, when the cost of investment is private. For brevity, we only discuss the solution under the assumption of interior first-best investment levels, i.e., $k^{FB} < k_{\max}$ for $k = s, \ell$, and zero correlation.

The agent's continuation value $\{W\}$ reads for $t < \tau$:

$$W_t = \mathbb{E}_t\left[\int_t^\tau e^{-\gamma(u-t)}\big(dC_u - K_u\mathcal{C}(s_u, \ell_u)du\big)\right],$$

while the principal's continuation value under the optimal contract is given by

$$P(W, K) \equiv \mathbb{E}_t\left[\int_t^\tau e^{-r(u-t)}(dX_u - dC_u) + e^{-r(\tau-t)}RK_\tau\Big|W_t = W, K_t = K\right]. \tag{1}$$

By the martingale representation theorem, $\{W\}$ solves the SDE:

$$dW_t + dC_t = \gamma W_t dt + K_t\mathcal{C}(s_t, \ell_t)dt + \beta_t^s K_t\sigma_X dZ_t^X + \beta_t^\ell K_t\sigma_K dZ_t^K$$

for progressively measurable processes $\{\beta^s\}, \{\beta^\ell\}$. The incentive conditions are derived as:

$$\beta_t^s \geq \mathcal{C}_s(s_t, \ell_t) \iff \beta_t^s \geq \lambda_s s_t$$
$$\beta_t^\ell \geq \mathcal{C}_\ell(s_t, \ell_t) \iff \beta_t^\ell \geq \lambda_\ell\ell_t,$$

where the respective inequality is strict for interior levels.

The value function scales in captial, i.e., $P(W,K) = Kp(w)$ for $w = W/K$, and $p(w)$ solves the following HJB equation :

$$(r+\delta)p(w) = \max_{s,\ell,\beta^s,\beta^\ell} \Bigg\{ \alpha s + p'(w)w(\gamma+\delta-\mu\ell) + p'(w)\mathcal{C}(s,\ell) + \mu\ell p(w) \\ + \frac{p''(w)}{2}\Big[(\beta^s\sigma_X)^2 + \sigma_K^2(\beta^\ell - w)^2 + 2\rho\sigma_X\sigma_K\beta^s(\beta^\ell - w)\Big]\Bigg\},$$

which is solved subject to $p(0) - R = p'(\overline{w}) - 1 = p''(\overline{w}) = 0$ and the incentive compatibility conditions.

The optimal investment levels s, ℓ follow from the FOC of maximization:

$$s = s(w) = \frac{\alpha}{-p'(w)\lambda_s\alpha - p''(w)(\lambda_s\sigma_X)^2} \wedge s_{\max} \text{ if } -p'(w)\lambda_s\alpha - p''(w)(\lambda_s\sigma_X)^2 > 0$$
$$\ell = \ell(w) = \frac{\mu(p(w) - p'(w)w) - p''(w)w\lambda_\ell\sigma_K^2}{-p'(w)\lambda_\ell\mu - p''(w)(\lambda_\ell\sigma_K)^2} \wedge \ell_{\max} \text{ if } -p'(w)\lambda_\ell\mu - p''(w)(\lambda_\ell\sigma_K)^2 > 0,$$

and

$$s = s(w) = s_{\max} \text{ if } -p'(w)\lambda_s\alpha - p''(w)(\lambda_s\sigma_X)^2 \le 0$$
$$\ell = \ell(w) = \ell_{\max} \text{ if } -p'(w)\lambda_\ell\mu - p''(w)(\lambda_\ell\sigma_K)^2 \le 0.$$

Note that the direct marginal cost of investment is given by $-p'(w)\lambda_s s\alpha$ (resp. $-p'(w)\lambda_\ell\ell\mu$), which is unambiguously negative for $w \in [0, w^*]$, where w^* solves $p'(w^*) = 0$. Hence, incentivizing investment is beneficial since it induces a positive drift component in the agent's continuation value w, which moves w on average away from the liquidation boundary (and thereby relaxes the non-negativity constraint of wages dC).

Departing from there, we can state and prove the following Proposition.

Proposition 1 (Short- and Long-termism)**.** *The optimal investment levels s, ℓ satisfy:*

i) *$s(\overline{w}) = s^{FB}$ and $\ell(w) < \ell^{FB}$ in a neighbourhood of $\overline{w}$*

ii) *If $\sigma_X > 0$, then there exist values $w^L < w^H < \overline{w}$ with $\ell(w) > \ell^{FB}$ for $w \in (w^L, w^H)$, provided $\sigma_K > 0$ is sufficiently low and $\ell_{max} > \ell^{FB}$*

iii) *If $\sigma_K > 0$, then there exist values $w^L < w^H < \overline{w}$ with $s(w) > s^{FB}$ for $w \in (w^L, w^H)$, provided $\sigma_X > 0$ is sufficiently low and $s_{\max} > s^{FB}$.*

Proof. i) Utilizing the boundary conditions $p'(\overline{w}) - 1 = p''(\overline{w}) = 0$ yields $s(\overline{w}) = 1/\lambda_s = s^{FB}$. Owing to agency-induced termination, $\mathcal{P}(\tau < \infty) = 1$, we have that $p(w) - wp'(w) < p^{FB}$. Again invoking the boundary conditions yields:

$$\ell(\overline{w}) = \frac{p(w) - p'(w)w}{\lambda_\ell} < \frac{p^{FB}}{\lambda_\ell} = \ell^{FB}$$

and by continuity the relationship holds in an appropriate left neighbourhood of $\overline{w}$.

ii) By Berge's maximum theorem, the solution $\{p_{\sigma_K}, \overline{w}_{\sigma_K}\}_{\sigma_K}$ is continuous in $\sigma_K > 0$ and converges to a well behaved solution with payout threshold $\overline{w} > 0$ when $\sigma_K \to 0$, because of $\sigma_X > 0$. Then, by continuity, there exist values $w' \in (0, \overline{w})$ and σ_K sufficiently small, so that effective (marginal) cost become negative, for $w = w'$:

$$-p'(w)\lambda_\ell \mu - p''(w)(\lambda_\ell \sigma_K)^2 = -p'(w)\lambda_\ell \mu + o(\sigma_K^2) \leq 0,$$

in which case clearly $\ell(w') = \ell_{\max} > \ell^{FB}$, thereby concluding the proof.

iii) By Berge's maximum theorem, the solution $\{p_{\sigma_X}, \overline{w}_{\sigma_X}\}_{\sigma_X}$ is continuous in $\sigma_X > 0$ and converges to a well behaved solution with payout threshold $\overline{w} > 0$ when $\sigma_X \to 0$, because of $\sigma_K > 0$. Then, by continuity, there exist values $w' \in (0, \overline{w})$ and σ_X sufficiently small, so that effective (marginal) cost become negative, for $w = w'$:

$$-p'(w)\lambda_s \alpha - p''(w)(\lambda_s \sigma_X)^2 = -p'(w)\lambda_s \alpha + o(\sigma_X^2) \leq 0,$$

in which case clearly $s(w') = s_{\max} > s^{FB}$, thereby concluding the proof.

□

The proof relied on exploiting the direct cost effect. While we are also able to prove short- and long-termism in the case of private investment cost, the key differences to our results presented in the main-text are as follows.

First, the statement is "if" and not "if and only if". While a dual moral hazard problem implies short-termism (resp. long-termism) when short-run (resp. long-run) risk is sufficiently low, it could also be that short-termism (resp. long-termism) arises in a model with $\sigma_K = 0$ (resp. $\sigma_X = 0$). This is due to the direct cost effect, which renders it beneficial to incur investment cost when w is low.

Second, short-termism can arise even without correlation between permanent and transitory shocks.

C Agent's Limited Wealth

Let us now consider what happens when the agent possesses zero wealth. For simplicity, we focus in the following on the case of quadratic investment cost, zero correlation and, without loss of generality, $\delta = 0$. Given prescribed investment levels (s_t, ℓ_t), if the agent were to increase short-term investment by some small amount $\varepsilon > 0$, she would require additional funds $\varepsilon \mathcal{C}_s(s_t, \ell_t) = \lambda_s \alpha \varepsilon$. Due to the lack of private wealth, the only possibility is to curb long-term investment by $\varepsilon \frac{\mathcal{C}_s(s_t, \ell_t)}{\mathcal{C}_\ell(s_t, \ell_t)} = \varepsilon \frac{\lambda_s \alpha s}{\lambda_\ell \mu \ell}$ and therefore (mis)-allocate this amount from the long-term towards short-term investment. The above reallocation boosts the cash-flow rate by $K_t \alpha \varepsilon$, while lowering the growth rate of assets by $K_t \varepsilon \frac{\lambda_s \alpha s}{\lambda_\ell \mu \ell} \mu$, so that incentive compatibility requires $\beta_t^s \geq \beta_t^\ell \frac{\lambda_s s_t}{\lambda_\ell \ell_t}$. To preclude symmetric redirecting from investment funds from the short- towards the long-term, we get the reverse inequality. Combining these conditions implies:

$$\frac{\beta_t^s}{\lambda_s s_t} = \frac{\beta_t^\ell}{\lambda_\ell \ell_t}. \tag{2}$$

The standard incentive conditions are additionally required to discourage the agent to divert from investment funds for her own consumption:

$$\beta_t^s \geq \lambda_s s_t \text{ and } \beta_t^\ell \geq \lambda_\ell \ell_t. \tag{3}$$

By standard arguments, the HJB equation describing the principal's problem reads then:

$$rp(w) = \max_{s,\ell,\beta^s,\beta^\ell} \left\{ \alpha s - \mathcal{C}(s,\ell) + p'(w)w(\gamma - \mu\ell) + \mu\ell p(w) + \frac{p''(w)}{2}\Big[(\beta^s\sigma_X)^2 + \sigma_K^2(\beta^\ell - w)^2\Big] \right\},$$

subject to the incentive constraints (2) and (3) and the usual boundary conditions.

To see why our results on short- and long-termism are practically unaffected by the assumption of limited wealth, let us substitute Eq. (2) into the HJB equation and eliminate β^s and analyze the optimality conditions for the controls. Because

$$\frac{\partial p(w)}{\partial s} \propto \alpha - \lambda_s \alpha s + \underbrace{p''(w)\left(\frac{\beta^\ell \lambda_s}{\lambda_\ell \ell}\sigma_X\right)^2 s}_{\text{Agency cost of investment } (<0)} \;,$$

it is clear that $s(w) < s^{FB}$ for all $w \in [0, \overline{w})$ owing to the agency cost associated with short-term investment, which confirms the result of Proposition 3.

Next, note that

$$\frac{\partial p(w)}{\partial \ell} \propto \underbrace{\mu\big(p(w) - wp'(w) - \lambda_\ell \ell\big)}_{\text{Investment Benefit-Cost;}\in o(\mu)} \underbrace{-\frac{p''(w)}{\ell}\left(\frac{\beta^\ell \lambda_s s}{\lambda_\ell \ell}\sigma_X\right)^2}_{\text{Additional Incentives;}>0} + \underbrace{\mathbf{1}_{\{\beta^\ell = \lambda_\ell \ell\}}\sigma_K^2 p''(w)(\beta^\ell - w)}_{\text{Effective Agency Cost of Investment}}. \tag{4}$$

Interestingly, by increasing long-term investment and owing to the convexity of the cost function, the principal makes misallocations of funds from the long- towards the short-term more costly for the agent and therefore provides *effectively* additional incentives for the manager to implement the prescribed investment allocation. The remaining terms in Eq. (4) are standard with the sole caveat that $\beta^\ell \geq \ell\lambda_\ell$ need not be tight, in which case long-term investment ℓ can be boosted without incurring additional agency cost.

To continue, observe that for μ sufficiently low, the first-term becomes negligible. If the incentive compatibility condition with respect to long-term incentives is tight (such that $\beta^\ell = \lambda_\ell \ell$), then for $\gamma - r$ sufficiently low, we find $w < \overline{w}$ with $w > \lambda_\ell \ell^{FB}$, in which case $\frac{\partial p(w)}{\partial \ell} > 0$ for $\ell \leq \ell^{FB}$ and thus $\ell(w) > \ell^{FB}$.[1] If $\beta^\ell > \lambda_\ell \ell$, then the right-hand side of (4) is strictly positive for a low growth rate μ. In either case, we are able to recover our result from Proposition 5.[2]

Moreover, one can solve for the optimal level of long-term incentives, which are now given

[1] This claim relies on the premise that $\lim_{\gamma \downarrow r} \overline{w} = \infty$ and that $\overline{w}$ increases in σ_X, σ_K and λ_s. These claims can be proven utilizing the proof technique used to establish the analogous claims in the baseline model.

[2] It is obvious that when either $\sigma_X = 0$ or $\sigma_K = 0$, we have only one relevant incentive constraint: either $\beta^s \geq \lambda_s s$ or $\beta^\ell \geq \lambda_\ell \ell$. From there it is straightforward to verify that no long-termism can occur. The claim is slightly more involved for $\sigma_X = 0$. Then, the principal set $\beta^\ell = \max\{w, \lambda_\ell \ell\}$. Arguments similar to ones used in the proof of Proposition 5 can then be used to show that $\lambda_\ell \ell(w) > w$ for all w, so that the problem becomes standard and agency cost of investment induce underinvestment.

by:

$$\beta^\ell = \max\left\{\lambda_\ell \ell, \frac{w}{1+\pi^2}\right\} \text{ for } \pi = \frac{\lambda_s s \sigma_X}{\lambda_\ell \ell \sigma_K}$$

As a consequence, the incentive constraint need not be binding for high levels of w, which leads to asymmetric performance-pay as in Proposition 8.

While not shown explicitly here, continuing this line of arguments, we could also recover our results of Propositions 6 and 7. In particular, the same forces drive short- and long-termism as in our baseline model. In conclusion, the assumption that the manager has unlimited wealth is indeed without loss of generality while simplifying the exposition.

D Incentives contingent on stock price and earnings

Fix throughout the optimal controls $\{s, \ell\}$ and focus on the baseline case, in which effort costs are quadratic and effort is interior, i.e., $(s_t, \ell_t) \in (0, s_{\max}) \times (0, \ell_{\max})$.

D.1 Step I

To start with, let us recall that the HJB equation implies the relationship:

$$rP(W_t, K_t)dt = \mathbb{E}[dX_t - K_t\mathcal{C}(s_t, \ell_t)dt + dP(W_t, K_t)]$$

where the expectation is taken under the probability measure $\mathcal{P}$. As shown in Appendix A.2.3, the above is equivalent to:

$$(r - \mu\ell_t)p(w_t)dt = [\alpha s_t - \mathcal{C}(s_t, \ell_t)]dt + \tilde{\mathbb{E}}[dp(w_t)],$$

where the expectation $\tilde{\mathbb{E}}$ is taken under the equivalent probability measure $\tilde{\mathcal{P}}$, with its Radon-Nikodym derivative defined by Eq. (A3). Under this probability measure, $\{w\}$ follows:

$$dw_t + dc_t = (\gamma - \mu\ell_t)w_t dt + \beta_t^s \sigma_X d\widetilde{Z}_t^X + (\beta_t^\ell - w_t)\sigma_K d\widetilde{Z}_t^K,$$

where $\{\tilde{Z}^X\}$ and $\{\tilde{Z}^K\}$ are standard Brownian Motions under $\mathcal{P}$ with correlation ρ.

Defining the stock-return from holding a stake within the firm over $[t, t+dt)$:

$$dR_t := \frac{dX_t - K_t\mathcal{C}(s_t, \ell_t)dt + dP(W_t, K_t)}{P(W_t, K_t)},$$

we can use the previous relationship to obtain:

$$dR_t = rdt + \frac{1 + p'(w_t)\lambda_s s_t}{p(w_t)}\sigma_X d\tilde{Z}_t^X + \frac{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)}{p(w_t)}\sigma_K d\tilde{Z}_t^K$$

Next, we can readily calculate:

$$\begin{aligned}\frac{dP_t}{P_t} &= \frac{dP(W_t,K_t)}{P(W_t,K_t)} = dR_t - \frac{dX_t - K_t\mathcal{C}(s_t,\ell_t)}{P(W_t,K_t)} \\ &= rdt - [\alpha s_t - \mathcal{C}(s_t,\ell_t)]dt + \frac{p'(w_t)\lambda_s s_t}{p(w_t)}\sigma_X d\tilde{Z}_t^X + \frac{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)}{p(w_t}\sigma_K d\tilde{Z}_t^K \\ &=: \mu_t^P dt + \Sigma_t^X d\tilde{Z}_t^X + \Sigma_t^K d\tilde{Z}_t^K.\end{aligned}$$

with

$$\begin{aligned}\mu_t^P &:= r - [\alpha s_t - \mathcal{C}(s_t,\ell_t)] \\ \Sigma_t^X &:= \frac{p'(w_t)\lambda_s s_t}{p(w_t)}\sigma_X \\ \Sigma_t^K &:= \frac{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)}{p(w_t)}\sigma_K,\end{aligned}$$

or equivalently:

$$dP_t = \mu_t^P p(w_t) K_t dt + \left[p'(w_t)\lambda_s s_t\right] K_t \sigma_X d\tilde{Z}_t^X + \left[p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)\right] K_t \sigma_K d\tilde{Z}_t^K.$$

If one were to prefer to look at the expressions under the physical measure $\mathcal{P}$ rather than the auxiliary measure $\tilde{\mathcal{P}}$, one can derive:

$$\frac{dP_t}{P_t} = \mu_t^P dt + \Sigma_t^X dZ_t^X + \Sigma_t^K dZ_t^K.$$

In the following subsection, we verify this relationship by direct calculation. In case the reader is not interested in this, there is no loss in skipping the following subsection and directly proceeding to Step II.

D.1.1 Calculation of dP_t/P_t under physical measure

First calculate:

$$\begin{aligned}dw_t &= d\left(\frac{W_t}{K_t}\right) = \frac{dW_t}{K_t} - \frac{W_t}{K_t^2}dK_t + \frac{W_t}{K_t^3} < dK_t, dK_t > - \frac{1}{K_t^2} < dW_t, dK_t > \\ &= \left[(\gamma - \mu\ell_t) + (w_t - \lambda_\ell \ell_t)\sigma_K^2 - \lambda_s s_t \sigma_X \sigma_K \rho\right]dt + \lambda_s s_t \sigma_X dZ_t^X + (\lambda_\ell \ell_t - w_t)\sigma_K dZ_t^K,\end{aligned}$$

where $< \cdot, \cdot >$ denotes the quadratic variation (e.g. $< dZ_t^X, dZ_t^X >= dt$). From there it follows that:

$$
\begin{aligned}
\frac{dP_t}{P_t} &= \frac{dP(W_t, K_t)}{P(W_t, K_t)} = \frac{d(K_t p(w_t))}{K_t p(w_t)} \\
&= \frac{dK_t}{K_t} + \frac{p'(w_t)dw_t + 0.5p''(w_t) < dw_t, dw_t >}{p(w_t)} + \frac{p'(w_t) < dK_t, dw_t >}{K_t p(w_t)} \\
&= rdt - (\alpha s_t - \mathcal{C}(s_t, \ell_t))dt + \left(\frac{(w_t - \lambda_\ell \ell_t)\sigma_K^2 - \lambda_s s_t \sigma_X \sigma_K \rho}{p(w_t)} - \frac{(w_t - \lambda_\ell \ell_t)\sigma_K^2 - \lambda_s s_t \sigma_X \sigma_K \rho}{p(w_t)} \right) dt \\
&\quad + \Sigma_t^X dZ_t^X + \Sigma_t^K dZ_t^K \\
&= \mu_t^P dt + \Sigma_t^X dZ_t^X + \Sigma_t^K dZ_t^K,
\end{aligned}
$$

where the third equality utilizes the HJB equation, evaluated under the optimal controls, $\{s, \ell\}$.

D.2 Step II

Finally, we can demonstrate how the optimal contract can be implemented by exposing the agent to unexpected price and earnings changes, where:

$$
\beta_t^E := \frac{dW_t}{dE_t} \text{ and } \beta_t^P := \frac{dW_t}{dP_t}, \tag{5}
$$

where earnings follow:

$$
dE_t = [\alpha s_t - \mathcal{C}(s_t, \ell_t)]K_t dt + K_t \sigma_X dZ_t^X
$$

We set β_t^P such that it matches the exposure to long-run shocks dK_t:

$$
\frac{dW_t}{dZ_t^K} = \lambda_\ell \ell_t \sigma_K K_t = \beta_t^P \left[p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t) \right] K_t \sigma_K = \frac{dW_t}{dP_t} \frac{dP_t}{dZ_t^K},
$$

so that:

$$
\beta_t^P = \frac{\lambda_\ell \ell_t}{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)} \tag{6}
$$

Since price changes are dependent on earning changes, β_t^P already exposes the agent to dX_t. We set now β_t^E so as to match:

$$
\frac{dW_t}{dZ_t^X} = \lambda_s s_t \sigma_X K_t = \beta_t^E \sigma_X K_t + \beta_t^P [p'(w_t)\lambda_s s_t] \sigma_X K_t = \frac{dW_t}{dE_t} \frac{dE_t}{dZ_t^X} + \frac{dW_t}{dP_t} \frac{dP_t}{dZ_t^X},
$$

which can be solved for:

$$
\begin{aligned}
\beta_t^E &= \lambda_s s_t - \beta_t^P [p'(w_t)\lambda_s s_t] \\
&= \lambda_s s_t \left[1 - \frac{p'(w_t)\lambda_\ell \ell_t}{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)} \right] \\
&= \lambda_s s_t \left[\frac{p(w_t) - p'(w_t) w_t}{p(w_t) + p'(w_t)(\lambda_\ell \ell_t - w_t)} \right].
\end{aligned} \tag{7}
$$