

Friends or Foes?

The Interrelationship between Angel and Venture Capital Markets

— Online Appendix —

Angel market: equilibrium equity shares and entrepreneur's outside option.

According to the Nash product, α^* is implicitly defined by

$$\frac{dD_1^E(e_1^*)}{d\alpha} D_1^A(e_1^*) + (D_1^E(e_1^*) - U_1^E) \frac{dD_1^A(e_1^*)}{d\alpha} = 0. \quad (18)$$

Applying the Envelope Theorem we find that $dD_1^E(e_1^*)/d\alpha < 0$. We can then infer from Eq. (18) that $dD_1^A(e_1^*)/d\alpha > 0$ must hold for $\alpha = \alpha^*$. Using Eq. (18) we can implicitly differentiate α^* w.r.t. U_1^E :

$$\frac{d\alpha^*}{dU_1^E} = \frac{\frac{dD_1^A}{d\alpha}}{\frac{d}{d\alpha} \left[\frac{dD_1^E}{d\alpha} D_1^A + (D_1^E - U_1^E) \frac{dD_1^A}{d\alpha} \right]}. \quad (19)$$

Note that the denominator is strictly negative due to the second-order condition for α^* . Moreover, recall that $dD_1^A/d\alpha > 0$. Thus, $d\alpha^*/dU_1^E < 0$.

Angel market: optimal transfer payment.

Suppose the angel makes the transfer T to the entrepreneur in exchange for an additional equity stake $\hat{\alpha}(T)$. The angel's new equity share is then given by $\alpha(T) \equiv \alpha^* + \hat{\alpha}(T)$, with $\alpha'(T) > 0$, $\alpha(T) \geq 0 \forall T \geq 0$, and $\alpha(T) < 0 \forall T < 0$. Note that any post bargaining transfers aimed at adjusting the equity allocation, must improve joint efficiency to be implementable. The joint utility at the deal stage is

$$D_1^A + D_1^E = \rho_1(e_1) [g [U_1^A + U_2^E] + (1 - g)y_1] - k_1 - c(e_1), \quad (20)$$

where $e_1 \equiv e_1(\alpha(T))$. Thus, the marginal effect of a transfer T on joint utility is given by

$$\frac{d [D_1^A + D_1^E]}{dT} = \underbrace{[\rho_1'(e_1)g [U_2^A + U_2^E] - c'(e_1)]}_{\equiv X} \frac{de_1}{d\alpha} \frac{d\alpha(T)}{dT}. \quad (21)$$

Recall that $de_1/d\alpha < 0$, so that e_1 is maximized at $\alpha = 0$. Moreover, $X \geq 0$. Thus, $d[D_1^A + D_1^E]/dT > 0$ requires that $d\alpha(T)/dT < 0$, and therefore $T < 0$. However, because of his zero wealth, the entrepreneur cannot make a payment to the angel. Thus, $T^* = 0$.

Derivation of angel market equilibrium.

Using $q_1^A = x_1/M_1^A$ and $x_1 = \phi_1 [M_1^E M_1^A]^{0.5}$, we can write Eq. (6) as

$$\phi_1 D_1^A \left[\frac{M_1^E}{M_1^A} \right]^{0.5} = \sigma_1^A. \quad (22)$$

Using $\theta_1 = M_1^A/M_1^E$ we then get the equilibrium degree of competition for the angel market: $\theta_1^* = [\phi_1 D_1^A / \sigma_1^A]^2$. Next, note that we can write Eq. (22) as

$$M_1^A = M_1^E \left[\frac{\phi_1 \tilde{D}_1^A}{\sigma_1^A} \right]^2 = M_1^E \theta_1. \quad (23)$$

Solving Eq. (8) for M_1^E and using $q_1^E = \phi_1 [M_1^E M_1^A]^{0.5} / M_1^E = \phi_1 [M_1^A / M_1^E]^{0.5}$, we get the equilibrium stock of entrepreneurs in the early stage market:

$$M_1^{E*} = \frac{F(U_1^E)}{\delta_1 + q_1^E} = \frac{F(U_1^E)}{\delta_1 + \phi_1 [M_1^A / M_1^E]^{0.5}} = \frac{F(U_1^E)}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \quad (24)$$

Thus, the equilibrium stock of angels is given by

$$M_1^{A*} = M_1^{E*} \theta_1^* = \frac{F(U_1^E) \theta_1^*}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \quad (25)$$

Using $M_1^{E*} = M_1^{A*} / \theta_1^*$ we can then write x_1^* as

$$x_1^* = \phi_1 [M_1^{A*} M_1^{E*}]^{0.5} = \frac{\phi_1 M_1^{A*}}{\sqrt{\theta_1^*}} = F(U_1^E) \frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \quad (26)$$

Moreover, using Eq. (9) and $q_1^A = x_1/M_1^A$ we get $m_1^{A*} = q_1^A M_1^{A*} = x_1^*$.

Proof of Proposition 1.

Recall that the equilibrium of the angel market is determined by the deal values D_1^E and D_1^A , and therefore by the late stage utilities U_2^E and U_2^A , as well as by the entrepreneur's outside option U_1^E (through α^*). We will show in Proof of Proposition 4 that U_2^E and U_2^A do not

depend on ϕ_1 , δ_1 , σ_1^E , σ_1^A , and k_1 . Next we need to derive a condition which defines U_1^E . The equilibrium condition (5) can be written as

$$U_1^E [r + \delta_1] = -\sigma_1^E + q_1^E [D_1^E - U_1^E]. \quad (27)$$

Using $q_1^E = \phi_1 [M_1^{A*}/M_1^{E*}]^{0.5} = \phi_1 \sqrt{\theta_1^*} = \phi_1^2 D_1^A / \sigma_1^A$ we get the following condition which defines U_1^E :

$$U_1^E [r + \delta_1] - \frac{\phi_1^2}{\sigma_1^A} D_1^A [D_1^E - U_1^E] + \sigma_1^E = 0. \quad (28)$$

Now consider the equilibrium degree of competition θ_1^* . Differentiating θ_1^* w.r.t. δ_1 yields

$$\frac{d\theta_1^*}{d\delta_1} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{dD_1^A}{d\delta_1} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dU_1^E} \frac{dU_1^E}{d\delta_1}. \quad (29)$$

Next we define $\Gamma \equiv D_1^A [D_1^E - U_1^E]$. We then get

$$\frac{dU_1^E}{d\delta_1} = - \frac{U_1^E}{r + \delta_1 - \frac{\phi_1^2}{\sigma_1^A} \left[\frac{d\Gamma}{d\alpha} \frac{d\alpha^*}{dU_1^E} + \frac{\partial \Gamma}{\partial U_1^E} \right]}. \quad (30)$$

Note that $d\Gamma/d\alpha = 0$ due to the first-order condition for α^* . Moreover, $\partial \Gamma / \partial U_1^E = -D_1^A$. Consequently,

$$\frac{dU_1^E}{d\delta_1} = - \frac{U_1^E}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} < 0. \quad (31)$$

This in turn implies that $d\theta_1^*/d\delta_1 > 0$. Likewise,

$$\frac{d\theta_1^*}{d\sigma_1^E} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dU_1^E} \frac{dU_1^E}{d\sigma_1^E}, \quad (32)$$

with $dD_1^A/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and

$$\frac{dU_1^E}{d\sigma_1^E} = - \frac{1}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} < 0. \quad (33)$$

Thus, $d\theta_1^*/d\sigma_1^E > 0$. Moreover, note that $dD_1^A/dk_1 < 0$. Consequently, $d\theta_1^*/dk_1 < 0$. For the remaining comparative statics it is useful to express the condition for U_1^E in terms of θ_1^* :

$$U_1^E [r + \delta_1] - \phi_1 \sqrt{\theta_1^*} [D_1^E - U_1^E] + \sigma_1^E = 0, \quad (34)$$

so that

$$\frac{dU_1^E}{d\theta_1^*} = \frac{\phi_1 \frac{1}{2\sqrt{\theta_1^*}} [D_1^E - U_1^E]}{r + \delta_1 + \phi_1 \sqrt{\theta_1^*}} > 0. \quad (35)$$

Moreover, using the definition of θ_1^* we define

$$G \equiv \theta_1^* - \left[\frac{\phi_1}{\sigma_1^A} D_1^A \right]^2 = 0 \quad (36)$$

where $D_1^A = D_1^A(\alpha^*(U_1^E(\theta_1^*)))$. We get

$$\frac{d\theta_1^*}{d\phi_1} = \frac{2 \frac{\phi_1}{[\sigma_1^A]^2} [D_1^A]^2}{1 - 2 \left[\frac{\phi_1}{\sigma_1^A} \right]^2 D_1^A \frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dU_1^E} \frac{dU_1^E}{d\theta_1^*}}. \quad (37)$$

Recall that $dD_1^A/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and $dU_1^E/d\theta_1^* > 0$. Thus, the denominator is positive, which implies that $d\theta_1^*/d\phi_1 > 0$. Likewise, using Eq. (36), we get

$$\frac{d\theta_1^*}{d\sigma_1^A} = - \frac{2 \frac{\phi_1^2}{[\sigma_1^A]^3} [D_1^A]^2}{1 - 2 \left[\frac{\phi_1}{\sigma_1^A} \right]^2 D_1^A \frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dU_1^E} \frac{dU_1^E}{d\theta_1^*}}. \quad (38)$$

Again, the denominator is positive, which implies that $d\theta_1^*/d\sigma_1^A < 0$.

Next, note that $dm_1^{E*}/dU_1^E = F'(U_1^E) > 0$, and recall that $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E < 0$. Moreover, using Eq. (28) we find

$$\frac{dU_1^E}{d\phi_1} = \frac{2 \frac{\phi_1}{\sigma_1^A} D_1^A [D_1^E - U_1^E]}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} > 0 \quad (39)$$

$$\frac{dU_1^E}{d\sigma_1^A} = - \frac{\frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A [D_1^E - U_1^E]}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} < 0. \quad (40)$$

Likewise, using $\Gamma = D_1^A [D_1^E - U_1^E]$,

$$\frac{dU_1^E}{dk_1} = \frac{\frac{\phi_1^2}{\sigma_1^A} \frac{d\Gamma}{dk_1}}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A}, \quad (41)$$

with

$$\frac{d\Gamma}{dk_1} = \underbrace{\frac{d\Gamma}{d\alpha}}_{=0} \frac{d\alpha^*}{dk_1} + \frac{\partial\Gamma}{\partial k_1} = -[D_1^E - U_1^E] < 0. \quad (42)$$

Thus, $dU_1^E/dk_1 < 0$. All this implies that m_1^{E*} is increasing in ϕ_1 , and decreasing in δ_1 , σ_1^E , σ_1^A , and k_1 .

Next, recall that $m_1^{A*} = x_1^*$ is given by

$$m_1^{A*} = x_1^* = F(U_1^E) \underbrace{\frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}}_{\equiv T}. \quad (43)$$

It is straightforward to show that $dT/d(\phi_1 \sqrt{\theta_1^*}) > 0$. Because $dU_1^E/d\phi_1 > 0$ and $d\theta_1^*/d\phi_1 > 0$, we then have $dm_1^{A*}/d\phi_1 = dx_1^*/d\phi_1 > 0$. Likewise, we know that $dU_1^E/d\sigma_1^A$, $dU_1^E/dk_1 < 0$ and $d\theta_1^*/d\sigma_1^A$, $d\theta_1^*/dk_1 < 0$. Thus, $dm_1^{A*}/d\sigma_1^A = dx_1^*/d\sigma_1^A < 0$ and $dm_1^{A*}/dk_1 = dx_1^*/dk_1 < 0$. Moreover, we have shown that $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E < 0$, while $d\theta_1^*/d\delta_1$, $d\theta_1^*/d\sigma_1^E > 0$. Thus, the effects of δ_1 and σ_1^E on $m_1^{A*} = x_1^*$ are ambiguous.

Now consider the equilibrium valuation V_1^* . Note that V_1^* is decreasing in the angel's equilibrium equity share α^* , which is defined by Eq. (18). Recall that $d\alpha^*/dU_1^E < 0$, $dU_1^E/d\phi_1 > 0$ and $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^E$, $dU_1^E/d\sigma_1^A < 0$. Consequently, $d\alpha^*/d\phi_1 < 0$ and $d\alpha^*/d\delta_1$, $d\alpha^*/d\sigma_1^E$, $d\alpha^*/d\sigma_1^A > 0$. All this implies that V_1^* is increasing in ϕ_1 , and decreasing in δ_1 , σ_1^E and σ_1^A . Furthermore, note that k_1 affects D_1^A and U_1^A . Using Eq. (18) we get

$$\frac{d\alpha^*}{dk_1} = -\frac{\frac{dD_1^E}{d\alpha} \frac{\partial D_1^A}{\partial k_1} - \frac{dU_1^E}{dk_1} \frac{dD_1^A}{d\alpha} + (D_1^E - U_1^E) \frac{d^2 D_1^A}{d\alpha dk_1}}{\frac{d}{d\alpha} \left[\frac{dD_1^E}{d\alpha} D_1^A + (D_1^E - U_1^E) \frac{dD_1^A}{d\alpha} \right]}, \quad (44)$$

where the denominator is strictly negative due to the second-order condition for α^* . Thus, to prove that $d\alpha^*/dk_1 > 0$, we need to show that the numerator is positive. We know that $dD_1^E/d\alpha < 0$, $dD_1^A/d\alpha > 0$, and $dU_1^E/dk_1 < 0$. Moreover, $\partial D_1^A/\partial k_1 = -1$ and $d^2 D_1^A/(d\alpha dk_1) = 0$. Thus, the numerator is strictly positive, so that $d\alpha^*/dk_1 > 0$. This in turn implies that the effect of k_1 on $V_1^* = k_1/\alpha^*$ is ambiguous.

Finally consider the equilibrium success probability $\rho_1(e_1^*)$, with $\rho_1'(e_1^*) > 0$. Using Eq. (3) we get

$$\frac{de_1^*}{d\alpha} = \frac{\rho_1'(e_1)(1-q)y_1}{\frac{d}{de_1} [\rho_1'(e_1) [qU_2^A + (1-q)(1-\alpha)y_1] - c'(e_1)]}, \quad (45)$$

where the denominator is strictly negative due to the second-order condition for e_1^* . Thus, $de_1^*/d\alpha < 0$. Our comparative statics results for α^* then imply that $d\rho_1(e_1^*)/d\phi_1 > 0$ and $d\rho_1(e_1^*)/d\delta_1, d\rho_1(e_1^*)/d\sigma_1^E, d\rho_1(e_1^*)/d\sigma_1^A, d\rho_1(e_1^*)/dk_1 < 0$. \square

Early Stage Investment and Valuation.

Consider first our base model with endogenous effort. Differentiating V_1^* w.r.t. k_1 yields

$$\frac{dV_1^*}{dk_1} = \frac{d}{dk_1} \left(\frac{k_1}{\alpha^*} \right) = \frac{\alpha^* - k_1 \frac{d\alpha^*}{dk_1}}{[\alpha^*]^2}. \quad (46)$$

Note that $dV_1^*/dk_1 > 0$ when $k_1 \rightarrow 0$. Thus, the equilibrium valuation V_1^* is decreasing in k_1 when k_1 is sufficiently small.

Next, suppose the entrepreneur's effort e_1 is exogenous, and define $\rho_1 \equiv \rho_1(e_1)$. The early stage deal values are then given by

$$D_1^E = \rho_1 [gU_2^E + (1-g)(1-\alpha)y_1] - c \quad (47)$$

$$D_1^A = \rho_1 [gU_2^A + (1-g)\alpha y_1] - k_1, \quad (48)$$

where c is the entrepreneurs disutility of providing effort e_1 . The optimal equity share for the angel, α^* , then satisfies the symmetric Nash bargaining solution, which accounts for the outside option of each party (U_1^E for the entrepreneur, and 0 for the angel because of free entry). Let \tilde{D}_1^E and \tilde{D}_1^A denote the deal values reflecting the Nash bargaining solution, which are given by

$$\tilde{D}_1^E = \frac{1}{2} [\rho_1 [g(U_2^E + U_2^A) + (1-g)y_1] - k_1 - c + U_1^E] \quad (49)$$

$$\tilde{D}_1^A = \frac{1}{2} [\rho_1 [g(U_2^E + U_2^A) + (1-g)y_1] - k_1 - c - U_1^E]. \quad (50)$$

The equilibrium equity share for the angel, α^* , then satisfies $D_1^E(\alpha^*) = \tilde{D}_1^E$ and $D_1^A(\alpha^*) = \tilde{D}_1^A$. Recall that $U_2^A = U_2^E$ in equilibrium. Thus,

$$\alpha^* = \frac{1}{2} + \frac{k_1 - c - U_1^E}{2\rho_1(1-g)y_1}. \quad (51)$$

The equilibrium early stage valuation is $V_1^* = k_1/\alpha^*$. We get

$$\frac{dV_1^*}{dk_1} = \frac{\overbrace{\alpha^* - k_1}^{\equiv N} \frac{d\alpha^*}{dk_1}}{[\alpha^*]^2}. \quad (52)$$

The denominator is always non-negative. Moreover, note that $N \geq 0$ for $k_1 \rightarrow 0$, which implies that $dV_1^*/dk_1 \geq 0$ for $k_1 \rightarrow 0$. To show that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$, it is thus sufficient to verify that $dN/dk_1 > 0$:

$$\frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left(\frac{d\alpha^*}{dk_1} + k_1 \frac{d^2\alpha^*}{dk_1^2} \right) = -k_1 \frac{d^2\alpha^*}{dk_1^2}. \quad (53)$$

We need to find the sign of $d^2\alpha^*/dk_1^2$. We start by taking the first derivative of α^* w.r.t. k_1 :

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2\rho_1(1-g)y_1} \left[1 - \frac{dU_1^E}{dk_1} \right]. \quad (54)$$

It is easy to see that $\tilde{D}_1^E - U_1^E = \tilde{D}_1^A$. Thus, the condition defining U_1^E simplifies to

$$U_1^E [r + \delta_1] - \frac{\phi_1^2}{\sigma_1^A} \left[\tilde{D}_1^A \right]^2 + \sigma_1^E = 0. \quad (55)$$

Thus,

$$\frac{dU_1^E}{dk_1} = -\frac{a_1 \tilde{D}_1^A}{r + \delta_1 + a_1 \tilde{D}_1^A}, \quad (56)$$

where $a_1 = \phi_1^2/\sigma_1^A$. Consequently,

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2\rho_1(1-g)y_1} \left[1 + \frac{1}{(r + \delta_1) \left[a_1 \tilde{D}_1^A \right]^{-1} + 1} \right]. \quad (57)$$

We then get

$$\frac{d^2\alpha^*}{dk_1^2} = \frac{1}{2\rho_1(1-g)y_1} \frac{-\frac{1}{2}a_1 (r + \delta_1) \left[a_1 \tilde{D}_1^A \right]^{-2} \left[1 + \frac{dU_1^E}{dk_1} \right]}{\left[(r + \delta_1) \left[a_1 \tilde{D}_1^A \right]^{-1} + 1 \right]^2}. \quad (58)$$

Note that

$$1 + \frac{dU_1^E}{dk_1} = 1 - \frac{a_1 \tilde{D}_1^A}{r + \delta_1 + a_1 \tilde{D}_1^A} = \frac{r + \delta_1}{r + \delta_1 + a_1 \tilde{D}_1^A} > 0. \quad (59)$$

Thus, $d^2\alpha^*/dk_1^2 < 0$. This implies that $dN/dk_1 > 0$, and therefore $dV_1^*/dk_1 > 0$.

Proof of Proposition 2.

In equilibrium, $U_2^E = U_2^A$. Moreover, we will show in Proof of Proposition 3 that $dU_2^E/d\phi_2 > 0$ and $dU_2^E/d\delta_2, dU_2^E/d\sigma_2, dU_2^E/d\sigma_2^V, dU_2^E/dk_2 < 0$. Consider the equilibrium degree of competition θ_1^* . With $U_2^E = U_2^A$ note that

$$\frac{d\theta_1^*}{dU_2^E} = 2 \frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{dU_2^A}. \quad (60)$$

For a given α we find that

$$\frac{dD_1^A}{dU_2^E} = \rho_1'(e_1^*) \frac{de_1^*}{dU_2^E} [gU_2^E + (1-g)\alpha y_1] + \rho_1(e_1^*)g > 0. \quad (61)$$

Moreover, applying the Envelope Theorem we get $dD_1^E/dU_2^E = g\rho_1(e_1^*) > 0$. Thus, the bargaining frontier shifts outwards, so that $dD_1^E/dU_2^E > 0$ and $dD_1^A/dU_2^E > 0$ at the equilibrium equity share α^* . This implies that $d\theta_1^*/dU_2^E > 0$, and consequently, $d\theta_1^*/d\phi_2 > 0$ and $d\theta_1^*/d\delta_2, d\theta_1^*/d\sigma_2, d\theta_1^*/d\sigma_2^V, d\theta_1^*/dk_2 < 0$.

Now consider the equilibrium inflow of entrepreneurs $m_1^{E*} = F(U_1^E)$, with $F'(U_1^E) > 0$. Using Eq. (28) we get

$$\frac{dU_1^E}{dU_2^E} = \frac{\frac{\phi_1^2}{\sigma_1^A} \frac{d\Gamma}{dU_2^E}}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A}, \quad (62)$$

where $\Gamma = D_1^A [D_1^E - U_1^E]$. Note that

$$\frac{d\Gamma}{dU_2^E} = \frac{d\Gamma}{de_1} \frac{de_1^*}{U_2^E} + \frac{d\Gamma}{d\alpha} \frac{d\alpha^*}{U_2^E} + \frac{\partial \Gamma}{\partial U_2^E}, \quad (63)$$

where $de_1^*/dU_2^E > 0$ and $d\Gamma/d\alpha = 0$ (see Eq. (18)). Moreover,

$$\frac{d\Gamma}{de_1} = \frac{dD_1^A}{de_1} [D_1^E - U_1^E] + D_1^A \underbrace{\frac{D_1^E}{de_1}}_{=0} = \rho_1'(e_1^*) \underbrace{[gU_2^A + (1-g)\alpha y_1]}_{>0} \underbrace{[D_1^E - U_1^E]}_{>0} > 0 \quad (64)$$

$$\frac{\partial \Gamma}{\partial U_2^E} = \underbrace{\frac{\partial D_1^A}{\partial U_2^E}}_{>0} \underbrace{[D_1^E - U_1^E]}_{>0} + D_1^A \underbrace{\frac{\partial D_1^E}{\partial U_2^E}}_{>0} > 0 \quad (65)$$

This implies that $dU_1^E/dU_2^E > 0$, and therefore, $dF(U_1^E)/dU_2^E > 0$. Our comparative statics results for U_2^E (see Proof of Proposition 3) then imply that $dm_1^{E*}/d\phi_2 > 0$ and $dm_1^{E*}/d\delta_2$, $dm_1^{E*}/d\sigma_2$, $dm_1^{E*}/d\sigma_2^V$, $dm_1^{E*}/dk_2 < 0$.

Next consider the equilibrium inflow of angels, m_1^{A*} , which is defined by

$$m_1^{A*} = x_1^* = \underbrace{F(U_1^E)}_{=m_1^{E*}} \underbrace{\frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}}_{\equiv T}. \quad (66)$$

One can show that $dT/d\sqrt{\theta_1^*} > 0$. Our comparative statics results for m_1^{E*} and θ_1^* then imply that $dm_1^{A*}/d\phi_2 > 0$ and $dm_1^{A*}/d\delta_2$, $dm_1^{A*}/d\sigma_2$, $dm_1^{A*}/d\sigma_2^V$, $dm_1^{A*}/dk_2 < 0$.

Now consider the equilibrium equity share α^* for angels. Recall that $dD_1^E/dU_2^E > 0$ and $dD_1^A/dU_2^E > 0$ at the equilibrium equity share α^* . Moreover, using the Envelope Theorem it is straightforward to show that $dD_1^A/dU_2^E > dD_1^E/dU_2^E$. The Nash bargaining solution then implies that $d\alpha^*/dU_2^E < 0$. Thus, $d\alpha^*/d\phi_2 < 0$ and $d\alpha^*/d\delta_2$, $d\alpha^*/d\sigma_2$, $d\alpha^*/d\sigma_2^V$, $d\alpha^*/dk_2 > 0$. For the equilibrium valuation $V_1^* = k_1/\alpha^*$ we can then infer that $dV_1^*/d\phi_2 > 0$ and $dV_1^*/d\delta_2$, $dV_1^*/d\sigma_2$, $dV_1^*/d\sigma_2^V$, $dV_1^*/dk_2 < 0$.

Finally consider the equilibrium success rate $\rho_1(e_1^*)$, with $\rho_1'(e_1^*) > 0$. Using Eq. (3) it is straightforward to show that $\partial e_1^*/\partial U_2^E > 0$ and $\partial e_1^*/\partial \alpha < 0$. Using our comparative statics results for U_2^E and α^* we can then infer that $de_1^*/d\phi_2 > 0$ and $de_1^*/d\delta_2$, $de_1^*/d\sigma_2$, $de_1^*/d\sigma_2^V$, $de_1^*/dk_2 < 0$. Consequently, $d\rho_1(e_1^*)/d\phi_2 > 0$ and $d\rho_1(e_1^*)/d\delta_2$, $d\rho_1(e_1^*)/d\sigma_2$, $d\rho_1(e_1^*)/d\sigma_2^V$, $d\rho_1(e_1^*)/dk_2 < 0$. \square

VC market: derivation of deal values and equity shares.

Let CV_i denote the value generated by the coalition $i = EAV, EV, EA, AV, E, A, V$. Using the Shapley value we get the following general deal values from the tripartite bargaining game:

$$D_2^E = \frac{1}{3} [CV_{EAV} - CV_{AV}] + \frac{1}{6} [CV_{EA} - CV_A] + \frac{1}{6} [CV_{EV} - CV_V] + \frac{1}{3} CV_E \quad (67)$$

$$D_2^A = \frac{1}{3} [CV_{EAV} - CV_{EV}] + \frac{1}{6} [CV_{EA} - CV_E] + \frac{1}{6} [CV_{AV} - CV_V] + \frac{1}{3} CV_A \quad (68)$$

$$D_2^V = \frac{1}{3} [CV_{EAV} - CV_{EA}] + \frac{1}{6} [CV_{EV} - CV_E] + \frac{1}{6} [CV_{AV} - CV_A] + \frac{1}{3} CV_V \quad (69)$$

We note that $CV_{EAV} = \pi$ and $CV_{AV} = CV_{EV} = CV_E = CV_A = CV_V = 0$. Moreover, by assumption we have $U_2^E + U_2^A > y_1$, so that $CV_{EA} = U_2^E + U_2^A$. Thus,

$$D_2^E = \frac{1}{3}\pi + \frac{1}{6} [U_2^E + U_2^A] \quad (70)$$

$$D_2^A = \frac{1}{3}\pi + \frac{1}{6} [U_2^E + U_2^A] \quad (71)$$

$$D_2^V = \frac{1}{3}\pi - \frac{1}{3} [U_2^E + U_2^A] \quad (72)$$

The deal values then allow us to derive the equilibrium equity shares β^{E*} , β^{A*} , and β^{V*} . The equilibrium equity share for entrepreneurs, β^{E*} , ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*}y_2 = D_2^E$. Solving this for β^{E*} yields

$$\beta^{E*} = \frac{D_2^E}{y_2} = \frac{1}{6y_2} [2\pi + U_2^E + U_2^A]. \quad (73)$$

Likewise we get

$$\beta^{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2} [2\pi + U_2^E + U_2^A] \quad (74)$$

$$\beta^{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{3y_2} [3k_2 + \pi - (U_2^E + U_2^A)]. \quad (75)$$

Derivation of VC market equilibrium.

The first part of the derivation follows along the lines of the derivation of the angel market equilibrium: Using Eq. (13) we get $\theta_2^* = [\phi_2 D_2^V / \sigma_2^V]^2$. Moreover, using Eq. (14) and the relationship $M_2^{V*} = M_2^{E*} \theta_2^*$ we find

$$M_2^{V*} = g\rho_1(e_1^*)x_1^* \frac{\theta_2^*}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}. \quad (76)$$

Using $M_2^{E*} = M_2^{V*} / \theta_2^*$ and the definition of M_2^{V*} , we can write x_2^* as

$$x_2^* = \phi_2 [M_2^{V*} M_2^{E*}]^{0.5} = \frac{\phi_2 M_2^{V*}}{\sqrt{\theta_2^*}} = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}, \quad (77)$$

where $m_2^{E*} = g\rho_1(e_1^*)x_1^*$. Furthermore, using Eq. (15) and $q_2^V = x_2 / M_2^V$ we find that $m_2^{V*} = q_2^V M_2^{V*} = x_2^*$.

Finally, using the equilibrium equity share β^{V*} for VCs we can write V_2^* as follows:

$$V_2^* = \frac{k_2}{\beta^{V*}} = \frac{k_2 y_2}{k_2 + D_2^V} = \left(\frac{3k_2}{3k_2 + \pi - (U_2^E + U_2^A)} \right) y_2. \quad (78)$$

Proof of Proposition 3.

First we need to derive a condition which defines U_2^E . We can write Eq. (12) as

$$U_2^E [r + \delta_2] = -\sigma_2 + q_2^E [D_2^E - U_2^E]. \quad (79)$$

Note that $D_2^E - U_2^E = \pi/3 - 2U_2^E/3 = D_2^V$. Using $q_2^E = \phi_2 [M_2^{V*} / M_2^{E*}]^{0.5} = \phi_2 \sqrt{\theta_2^*} = \phi_2^2 D_2^V / \sigma_2^V$, we get the following condition which defines U_2^E :

$$U_2^E [r + \delta_2] - \frac{\phi_2^2}{\sigma_2^V} [D_2^V]^2 + \sigma_2 = 0. \quad (80)$$

Consider the equilibrium degree of competition θ_2^* . Recall that $U_2^A = U_2^E$ in equilibrium; thus,

$$\frac{d\theta_2^*}{dU_2^A} = \frac{d\theta_2^*}{dU_2^E} = 2 \frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} \frac{dD_2^V}{dU_2^E} = -\frac{4}{3} \frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} < 0. \quad (81)$$

Note that δ_2 only affects U_2^E in the definition of θ_2^* . Implicitly differentiating U_2^E w.r.t. δ_2 yields

$$\frac{dU_2^E}{d\delta_2} = -\frac{U_2^E}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0, \quad (82)$$

which implies that $d\theta_2^*/d\delta_2 > 0$. Likewise, σ_2 only affects U_2^E in the definition of θ_2^* . We get

$$\frac{dU_2^E}{d\sigma_2} = -\frac{1}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \quad (83)$$

Thus, $d\theta_2^*/d\sigma_2 > 0$. Next, differentiating U_2^E w.r.t. ϕ_2 yields

$$\frac{d\theta_2^*}{d\phi_2} = 2\frac{\phi_2 D_2^V}{[\sigma_2^V]^2} \left[D_2^V + \phi_2 \frac{dD_1^V}{dU_2^E} \frac{dU_2^E}{d\phi_2} \right] = 2\frac{\phi_2 D_2^V}{[\sigma_2^V]^2} \left[D_2^V - \frac{2}{3}\phi_2 \frac{dU_2^E}{d\phi_2} \right], \quad (84)$$

with

$$\frac{dU_2^E}{d\phi_2} = \frac{2\frac{\phi_2}{\sigma_2^V} [D_2^V]^2}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} > 0. \quad (85)$$

Therefore,

$$\frac{d\theta_2^*}{d\phi_2} = 2\frac{\phi_2 D_2^V}{[\sigma_2^V]^2} \frac{(r + \delta_2)D_2^V}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} > 0. \quad (86)$$

Likewise,

$$\frac{d\theta_2^*}{d\sigma_2^V} = 2\frac{\phi_2^2 D_2^V}{\sigma_2^V} \frac{1}{[\sigma_2^V]^2} \left[-\frac{2}{3}\frac{dU_2^E}{d\sigma_2^V} \sigma_2^V - D_2^V \right], \quad \text{with} \quad \frac{dU_2^E}{d\sigma_2^V} = -\frac{\frac{\phi_2^2 D_2^V D_2^V}{[\sigma_2^V]^2}}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \quad (87)$$

Consequently,

$$\frac{d\theta_2^*}{d\sigma_2^V} = -2\frac{\phi_2^2 D_2^V}{\sigma_2^V} \frac{1}{[\sigma_2^V]^2} \frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V} [D_2^V]^2 + (r + \delta_2)D_2^V}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \quad (88)$$

Moreover, we get

$$\frac{d\theta_2^*}{dk_2} = 2\frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} \frac{dD_2^V}{dk_2} = 2\frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} \left[-\frac{1}{3} - \frac{2}{3}\frac{dU_2^E}{dk_2} \right], \quad \text{with} \quad \frac{dU_2^E}{dk_2} = -\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \quad (89)$$

Thus,

$$\frac{d\theta_2^*}{dk_2} = -\frac{2}{3}\frac{\phi_2^2 D_2^V}{[\sigma_2^V]^2} \frac{r + \delta_2}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} < 0. \quad (90)$$

Next, recall that $m_2^{V^*} = x_2^*$ is given by

$$m_2^{V^*} = x_2^* = \underbrace{g\rho_1(e_1^*)x_1^*}_{=m_2^{E^*}} \underbrace{\frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}}_{\equiv T}. \quad (91)$$

We have shown in Proof of Proposition 2 that $dx_1^*/d\phi_2 > 0$ and $dx_1^*/d\delta_2, dx_1^*/d\sigma_2, dx_1^*/d\sigma_2^V, dx_1^*/dk_2 < 0$. Likewise, we have shown that $d\rho_1(e_1^*)/d\phi_2 > 0$ and $d\rho_1(e_1^*)/d\delta_2, d\rho_1(e_1^*)/d\sigma_2, d\rho_1(e_1^*)/d\sigma_2^V, d\rho_1(e_1^*)/dk_2 < 0$. Moreover, it is straightforward to verify that $dT/d(\phi_2\sqrt{\theta_2^*}) > 0$. Using our comparative statics results for θ_2^* , we can infer that $dT/d\phi_2, dT/d\delta_2, dT/d\sigma_2 > 0$, and $dT/d\sigma_2^V, dT/dk_2 < 0$. All this implies that $dm_2^{V^*}/d\phi_2 > 0$ and $dm_2^{V^*}/d\sigma_2^V, dm_2^{V^*}/dk_2 < 0$, while the effects of δ_2 and σ_2 on $m_2^{V^*}$ are ambiguous.

Now consider the equilibrium late stage valuation V_2^* :

$$V_2^* = \left(\frac{3k_2}{3k_2 + \pi - 2U_2^E} \right) y_2. \quad (92)$$

Recall that $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2, dU_2^E/d\sigma_2^V, dU_2^E/d\delta_2 < 0$. Thus, $dV_2^*/d\phi_2 > 0$ and $dV_2^*/d\sigma_2, dV_2^*/d\sigma_2^V, dV_2^*/d\delta_2 < 0$. Furthermore, recall that V_2^* can also be written as $V_2^* = k_2 y_2 / (k_2 + D_2^V)$. Taking the first derivative of V_2^* w.r.t. k_2 yields

$$\frac{dV_2^*}{dk_2} = \frac{k_2 + D_2^V - k_2 \left[1 - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} \right]}{[k_2 + D_2^V]^2} y_2 = \frac{\overbrace{\frac{1}{3}k_2 + D_2^V + \frac{2}{3}k_2 \frac{dU_2^E}{dk_2}}^{\equiv N}}{[k_2 + D_2^V]^2} y_2. \quad (93)$$

Note that the denominator is always positive. Moreover, we have $N > 0$ for $k_2 \rightarrow 0$. Thus, $dV_2^*/dk_2 > 0$ for $k_2 \rightarrow 0$. To verify that $dV_2^*/dk_2 > 0$ for all $k_2 > 0$, it is sufficient to show that $dN/dk_2 > 0$:

$$\frac{dN}{dk_2} = \frac{1}{3} - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} + \frac{2}{3} \left[\frac{dU_2^E}{dk_2} + k_2 \frac{d^2U_2^E}{dk_2^2} \right] = \frac{2}{3} k_2 \frac{d^2U_2^E}{dk_2^2}. \quad (94)$$

It remains to identify the sign of $d^2U_2^E/dk_2^2$. Using $a_2 \equiv \phi_2^2/\sigma_2^V$ we can write dU_2^E/dk_2 as

$$\frac{dU_2^E}{dk_2} = -\frac{\frac{2}{3}a_2D_2^V}{r + \delta_2 + \frac{4}{3}a_2D_2^V} = -\frac{\frac{2}{3}}{(r + \delta_2) [a_2D_2^V]^{-1} + \frac{4}{3}}. \quad (95)$$

Thus,

$$\frac{d^2U_2^E}{dk_2^2} = \frac{\frac{2}{9}a_2(r + \delta_2)[a_2D_2^V]^{-2}\left[1 + 2\frac{dU_2^E}{dk_2}\right]}{\left[(r + \delta_2)[a_2D_2^V]^{-1} + \frac{4}{3}\right]^2}. \quad (96)$$

Note that

$$1 + 2\frac{dU_2^E}{dk_2} = 1 - \frac{\frac{4}{3}a_2D_2^V}{r + \delta_2 + \frac{4}{3}a_2D_2^V} = \frac{r + \delta_2}{r + \delta_2 + \frac{4}{3}a_2D_2^V} > 0. \quad (97)$$

Hence, $d^2U_2^E/dk_2^2 > 0$, so that $dN/dk_2 > 0$. Consequently, $dV_2^*/dk_2 > 0$. \square

Proof of Proposition 4.

We can see from Eq. (80) that U_2^E (and therefore U_2^A) does not depend on the early stage parameters ϕ_1 , δ_1 , σ_1^E , σ_1^A , and k_1 . This also implies that D_2^V , and therefore θ_2^* and V_2^* , do not depend on these parameters.

Now consider the equilibrium inflow of start-ups $m_2^{E*} = g\rho_1(e_1^*)x_1^*$. Recall from Proposition 1 that $dx_1^*/d\phi_1 > 0$ and $dx_1^*/d\sigma_1^A$, $dx_1^*/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous. Moreover, we know from Proposition 1 that $d\rho_1(e_1^*)/d\phi_1 > 0$ and $d\rho_1(e_1^*)/d\delta_1$, $d\rho_1(e_1^*)/d\sigma_1^E$, $d\rho_1(e_1^*)/d\sigma_1^A$, $d\rho_1(e_1^*)/dk_1 < 0$. This implies that $dm_2^{E*}/d\phi_1 > 0$ and $dm_2^{E*}/d\sigma_1^A$, $dm_2^{E*}/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous.

Finally consider the equilibrium inflow of VCs m_2^{V*} , as defined by

$$m_2^{V*} = x_2^* = m_2^{E*} \frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}. \quad (98)$$

Recall that θ_2^* does not depend on the early stage parameters. Our comparative statics results for m_2^{E*} then imply that $dm_2^{V*}/d\phi_1 > 0$ and $dm_2^{V*}/d\sigma_1^A$, $dm_2^{V*}/dk_1 < 0$, while the effects of δ_1 and σ_1^E are ambiguous. \square

Angel protection: derivation of deal values and equity shares.

The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda\pi$, and $CV_{AV} = CV_E = CV_A = CV_V = 0$. Using the general deal values (67), (68), and (69), we get

$$D_2^E = \frac{1}{6} [2 + \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] \quad (99)$$

$$D_2^A = \frac{1}{3} [1 - \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] \quad (100)$$

$$D_2^V = \frac{1}{6} [2 + \lambda] \pi - \frac{1}{3} [U_2^E + U_2^A] \quad (101)$$

The new equilibrium equity share for entrepreneurs, β^{E*} , ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*} y_2 = D_2^E$. Solving this for β^{E*} yields

$$\beta^{E*} = \frac{D_2^E}{y_2} = \frac{1}{6y_2} [(2 + \lambda) \pi + U_2^E + U_2^A]. \quad (102)$$

Likewise we get

$$\beta^{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2} [2(1 - \lambda) \pi + U_2^E + U_2^A] \quad (103)$$

$$\beta^{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{6y_2} [6k_2 + (2 + \lambda) \pi - 2(U_2^E + U_2^A)]. \quad (104)$$

Proof of Proposition 5.

We first show that $dU_2^A/d\lambda < 0$. Note that $D_2^A \neq D_2^E$ for $\lambda > 0$, and recall that $q_2^E = \phi_2 [M_2^{V*}/M_2^{E*}]^{0.5} = \phi_2^2 D_2^V / \sigma_2^V$. Thus, using Eq. (12) we define

$$F \equiv U_2^E (r + \delta_2) + \sigma - a_2 D_2^V [D_2^E - U_2^E] = 0 \quad (105)$$

$$G \equiv U_2^A (r + \delta_2) + \sigma - a_2 D_2^V [D_2^A - U_2^A] = 0, \quad (106)$$

where $a_2 = \phi_2^2 / \sigma_2^V$. Using Cramer's rule we get

$$\frac{dU_2^A}{d\lambda} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial U_2^E} \\ -\frac{\partial G}{\partial \lambda} & \frac{\partial G}{\partial U_2^E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} \\ \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E} \end{vmatrix}} = \frac{-\frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E} + \frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial U_2^E}}{\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} - \frac{\partial G}{\partial U_2^A} \frac{\partial F}{\partial U_2^E}}. \quad (107)$$

The denominator is negative if

$$\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} < \frac{\partial G}{\partial U_2^A} \frac{\partial F}{\partial U_2^E}, \quad (108)$$

which is equivalent to

$$\begin{aligned} & \frac{1}{6}a_2 [2 [D_2^E - U_2^E] - D_2^V] \frac{1}{6}a_2 [2 [D_2^A - U_2^A] - D_2^V] \\ < & \left[r + \delta_2 + \frac{1}{6}a_2 [2 [D_2^A - U_2^A] + 5D_2^V] \right] \left[r + \delta_2 + \frac{1}{6}a_2 [2 [D_2^E - U_2^E] + 5D_2^V] \right] \end{aligned} \quad (109)$$

If this condition holds for $r + \delta_2 = 0$, then it also holds for all $r + \delta_2 > 0$. Setting $r + \delta_2 = 0$ we get

$$-2 [D_2^E - U_2^E] D_2^V - 2 [D_2^A - U_2^A] D_2^V < 10 [D_2^A - U_2^A] D_2^V + 10 D_2^V [D_2^E - U_2^E] + 24 [D_2^V]^2. \quad (110)$$

This condition is satisfied as $D_2^E > U_2^E$ and $D_2^A > U_2^A$. Thus, the denominator of $dU_2^A/d\lambda$ is strictly negative. Likewise, the numerator is positive if

$$\frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial U_2^E} > \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E}, \quad (111)$$

which is equivalent to

$$\begin{aligned} & \frac{1}{6}\pi a_2 [[D_2^A - U_2^A] - 2D_2^V] \left[r + \delta_2 + \frac{1}{6}a_2 [2 [D_2^E - U_2^E] + 5D_2^V] \right] \\ < & \frac{1}{6}\pi a_2 [[D_2^E - U_2^E] + D_2^V] \frac{1}{6}a_2 [2 [D_2^A - U_2^A] - D_2^V]. \end{aligned} \quad (112)$$

This condition can be written as

$$\frac{2}{a_2} (r + \delta_2) [[D_2^A - U_2^A] - 2D_2^V] + [D_2^A - U_2^A] D_2^V - D_2^V [D_2^E - U_2^E] < 3 [D_2^V]^2. \quad (113)$$

From F and G we know that

$$D_2^V [D_2^E - U_2^E] = \frac{U_2^E (r + \delta_2) + \sigma}{a_2} \quad \text{and} \quad D_2^V [D_2^A - U_2^A] = \frac{U_2^A (r + \delta_2) + \sigma}{a_2}, \quad (114)$$

so that we can write condition (113) as follows:

$$\frac{2}{a_2} (r + \delta_2) [[D_2^A - U_2^A] - 2D_2^V] + \frac{U_2^A (r + \delta_2) + \sigma}{a_2} - \frac{U_2^E (r + \delta_2) + \sigma}{a_2} < 3 [D_2^V]^2 \quad (115)$$

$$\Leftrightarrow (r + \delta_2) \underbrace{[2D_2^A - U_2^A - 4D_2^V - U_2^E]}_{\equiv T} < 3 [D_2^V]^2 a_2. \quad (116)$$

We now show that $T < 0$. Using the definitions of D_2^A and D_2^V we can write $T < 0$ as

$$\frac{2}{3} [1 - \lambda] \pi + \frac{1}{3} [U_2^E + U_2^A] - U_2^A - \frac{2}{3} [2 + \lambda] \pi + \frac{4}{3} [U_2^E + U_2^A] - U_2^E < 0 \quad (117)$$

$$\Leftrightarrow U_2^E + U_2^A < [1 + 2\lambda] \pi. \quad (118)$$

This condition is satisfied for all $\lambda \geq 0$ because $\pi > U_2^E + U_2^A$. Thus, the numerator of $dU_2^A/d\lambda$ is strictly positive. Consequently, $dU_2^A/d\lambda < 0$. Finally note that $\partial D_2^E/\partial\lambda = \pi/6 < |\partial D_2^A/\partial\lambda| = \pi/3$. Thus, $d[U_2^E + U_2^A]/d\lambda < 0$, which implies that $dD_2^V/d\lambda > 0$.

Next we analyze the effects of λ on the early stage equilibrium variables. Consider the equilibrium degree of competition θ_1^* . We get

$$\frac{d\theta_1^*}{d\lambda} = 2 \frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{d\lambda}. \quad (119)$$

Recall that

$$\frac{d}{d\lambda} (U_2^A + U_2^E) = \underbrace{\frac{dU_2^A}{d\lambda}}_{<0} + \underbrace{\frac{dU_2^E}{d\lambda}}_{>0} < 0. \quad (120)$$

This implies

$$\frac{dD_1^A}{d\lambda} + \frac{dD_1^E}{d\lambda} < 0 \quad \Rightarrow \quad \frac{dD_1^A}{d\lambda} < 0. \quad (121)$$

Thus, $d\theta_1^*/d\lambda < 0$.

Now consider the equilibrium entry of entrepreneurs m_1^{E*} . Using Eq. (28), we get

$$\frac{dU_1^E}{d\lambda} = \frac{\frac{\phi_1^2}{\sigma_1^A} \left[\frac{dD_1^A}{d\lambda} [D_1^E - U_1^E] + D_1^A \frac{dD_1^E}{d\lambda} \right]}{r + \delta_1 - \frac{\phi_1^2}{\sigma_1^A} \left[\frac{d\Gamma}{d\alpha} \frac{d\alpha^*}{dU_1^E} + \frac{\partial \Gamma}{\partial U_1^E} \right]}, \quad (122)$$

where $\Gamma = D_1^A [D_1^E - U_1^E]$. Note that $d\Gamma/d\alpha = 0$; see Eq. (18). Thus,

$$\frac{dU_1^E}{d\lambda} = \frac{\frac{\phi_1^2}{\sigma_1^A} \left[\frac{dD_1^A}{d\lambda} [D_1^E - U_1^E] + D_1^A \frac{dD_1^E}{d\lambda} \right]}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A}, \quad (123)$$

where the denominator is positive. Consequently, $dU_1^E/d\lambda < 0$ if

$$\frac{dD_1^A}{d\lambda} [D_1^E - U_1^E] + D_1^A \frac{dD_1^E}{d\lambda} < 0. \quad (124)$$

Using Eq. (18) we can derive the following expression for $D_1^E - U_1^E$:

$$D_1^E - U_1^E = -\frac{\frac{dD_1^E}{d\alpha}}{\frac{dD_1^A}{d\alpha}} D_1^A, \quad (125)$$

so that Eq. (124) can be written as

$$\frac{dD_1^A}{d\lambda} \underbrace{\left(-\frac{\frac{dD_1^E}{d\alpha}}{\frac{dD_1^A}{d\alpha}} \right)}_{\equiv X} + \frac{dD_1^E}{d\lambda} < 0. \quad (126)$$

Recall that $d(D_1^A + D_1^E)/d\lambda < 0$, with $dD_1^A/d\lambda < 0$; thus, this condition is satisfied when $X \geq 1$. Note that $dD_1^E/d\alpha < 0$ and $dD_1^A/d\alpha > 0$. Hence, $X \geq 1$ if

$$0 \geq \frac{dD_1^A}{d\alpha} + \frac{dD_1^E}{d\alpha} = \frac{d}{d\alpha} [D_1^A + D_1^E]. \quad (127)$$

It is easy to show that the joint surplus is maximized when $\alpha = 0$ (which maximizes effort incentives for the entrepreneur); thus

$$\left. \frac{d[D_1^A + D_1^E]}{d\alpha} \right|_{\alpha=\alpha^*>0} < 0, \quad (128)$$

so that $X \geq 1$. Consequently, $dU_1^E/d\lambda < 0$, and therefore $dm_1^{E*}/d\lambda = dF(U_1^E)/d\lambda < 0$.

Next consider the equilibrium inflow of angels, m_1^{A*} , which is defined by

$$m_1^{A*} = x_1^* = \underbrace{F(U_1^E)}_{=m_1^{E*}} \underbrace{\frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}}_{\equiv T}. \quad (129)$$

Note that $dT/d\sqrt{\theta_1^*} > 0$. Our comparative statics results for m_1^{E*} and θ_1^* then imply that $dm_1^{A*}/d\lambda = dx_1^*/d\lambda < 0$.

Now consider the angel's equilibrium equity share α^* , which is defined by Eq. (18). We get

$$\frac{d\alpha^*}{d\lambda} = \frac{d\alpha^*}{dU_2^E} \frac{dU_2^E}{d\lambda} + \frac{d\alpha^*}{dU_2^A} \frac{dU_2^A}{d\lambda}, \quad (130)$$

where $dU_2^E/d\lambda > 0$ and $dU_2^A/d\lambda < 0$. Moreover, the Nash bargaining solution implies that $d\alpha^*/dU_2^E > 0$ and $d\alpha^*/dU_2^A < 0$. Thus, $d\alpha^*/d\lambda > 0$. For the equilibrium valuation $V_1^* = k_1/\alpha^*$ this concurrently implies that $dV_1^*/d\lambda < 0$. Finally we know that $dD_1^E/d\lambda > 0$ in equilibrium. Using the Envelope Theorem we get

$$\frac{dD_1^E}{d\lambda} = \rho_1(e_1) \underbrace{\frac{d}{d\lambda} [gU_2^E + (1-g)(1-\alpha^*)y_1]}_{\equiv T} > 0, \quad (131)$$

which implies that $T > 0$. Using Eq. (3) we find

$$\frac{de_1^*}{d\lambda} = - \frac{\rho_1'(e_1) \frac{d}{d\lambda} [gU_2^E + (1-g)(1-\alpha)y_1]}{\frac{d}{de_1} [\rho_1'(e_1) [gU_2^E + (1-g)(1-\alpha)y_1] - c'(e_1)]}, \quad (132)$$

where $T > 0$, and the denominator is negative due to the second-order condition for e_1^* . Thus, $de_1^*/d\lambda > 0$. This in turn implies that $d\rho_1(e_1^*)/d\lambda > 0$.

Finally we analyze the effects of λ on the late stage equilibrium variables. Note that $d(U_2^E + U_2^A)/d\lambda < 0$ also implies that $dD_2^V/d\lambda > 0$. Using the definitions of θ_2^* , β^{V*} and V_2^* , we can then infer that $d\theta_2^*/d\lambda > 0$, $d\beta^{V*}/d\lambda > 0$ and $dV_2^*/d\lambda < 0$. Moreover,

$$\frac{dm_2^{E*}}{d\lambda} = \frac{d}{d\lambda} [g\rho_1(e_1^*)x_1^*] = g \left[\rho_1'(e_1^*) \frac{de_1^*}{d\lambda} x_1^* + \rho_1(e_1^*) \frac{dx_1^*}{d\lambda} \right]. \quad (133)$$

In general, the effect on m_2^{E*} is ambiguous as $de_1^*/d\lambda > 0$ and $dx_1^*/d\lambda < 0$. However, we can see that $dm_2^{E*}/d\lambda < 0$ when $\rho_1'(e_1^*) \rightarrow 0$. Moreover, for $\delta_1 \rightarrow 0$ we have $m_1^{A*} = m_1^{E*}$; with m_1^{E*} being sufficiently inelastic, we have $dx_1^*/d\lambda \rightarrow 0$, so that $dm_2^{E*}/d\lambda > 0$. Next, recall that m_2^{V*} is defined by

$$m_2^{V*} = x_2^* = m_2^{E*} \underbrace{\frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}}_{\equiv T}. \quad (134)$$

One can show that $dT/d\sqrt{\theta_2^*} > 0$, so that $dT/d\lambda > 0$. Recall, however, that the sign of $dm_2^{E*}/d\lambda$ is ambiguous. Thus, the effect of λ on $m_2^{V*} = x_2^*$ is also ambiguous. \square

Angel protection – angel not required for VC search.

Suppose the entrepreneur can search for a VC without the angel. The entrepreneur then incurs the search cost $\gamma\sigma$, with $\gamma > 2$. Using Nash bargaining, the deal values for the VC (\widehat{D}_2^V) and the entrepreneur (\widehat{U}_2^E) are then given by

$$\widehat{D}_2^V = \frac{1}{2} [\lambda\pi - \widehat{U}_2^E] \quad \widehat{D}_2^E = \frac{1}{2} [\lambda\pi + \widehat{U}_2^E], \quad (135)$$

where \widehat{U}_2^E denotes the entrepreneur's outside option.

Now consider the bargaining problem at the late stage between entrepreneur, angel, and VC. The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda\pi$, $CV_E = \widehat{U}_2^E$, and $CV_{AV} = CV_A = CV_V = 0$. Using the Shapley value we then get the following deal values:

$$D_2^E = \frac{1}{6} [2 + \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] + \frac{1}{3} \widehat{U}_2^E \quad (136)$$

$$D_2^A = \frac{1}{3} [1 - \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] - \frac{1}{6} \widehat{U}_2^E \quad (137)$$

$$D_2^V = \frac{1}{6} [2 + \lambda] \pi - \frac{1}{3} [U_2^E + U_2^A] - \frac{1}{6} \widehat{U}_2^E \quad (138)$$

The expected utilities from search, U_2^A , U_2^E , and \widehat{U}_2^E , are then defined by

$$F \equiv U_2^A (r + \delta_2) + \sigma - a_2 D_2^V [D_2^A - U_2^A] = 0 \quad (139)$$

$$G \equiv U_2^E (r + \delta_2) + \sigma - a_2 D_2^V [D_2^E - U_2^E] = 0 \quad (140)$$

$$H \equiv \widehat{U}_2^E (r + \delta_2) + \gamma\sigma - a_2 \widehat{D}_2^V [\widehat{D}_2^E - \widehat{U}_2^E] = 0, \quad (141)$$

where $a_2 = \phi_2^2/\sigma_2^V$. Using H we find that

$$\frac{d\widehat{U}_2^E}{d\lambda} = \frac{\frac{1}{2}a_2\pi \left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V \right]}{r + \delta_2 + \frac{1}{2}a_2 \left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V \right]} > 0 \quad (142)$$

$$\frac{d\widehat{U}_2^E}{d\gamma} = -\frac{\sigma}{r + \delta_2 + \frac{1}{2}a_2 \left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V \right]} < 0 \quad (143)$$

Next we show that $dU_2^A/d\lambda < 0$. Using Cramer's rule we get $dU_2^A/d\lambda = A/B$, where

$$A = \begin{vmatrix} -\frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial \widehat{U}_2^E} \\ -\frac{\partial G}{\partial \lambda} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial \widehat{U}_2^E} \\ -\frac{\partial H}{\partial \lambda} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial \widehat{U}_2^E} \end{vmatrix} \quad B = \begin{vmatrix} \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial \widehat{U}_2^E} \\ \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial \widehat{U}_2^E} \\ \frac{\partial H}{\partial U_2^A} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial \widehat{U}_2^E} \end{vmatrix} \quad (144)$$

Consider first the denominator B . Since $\partial H/\partial U_2^A = 0$ and $\partial H/\partial U_2^E = 0$, we can write B as

$$B = \frac{\partial H}{\partial \widehat{U}_2^E} \left[\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} - \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial U_2^A} \right], \quad (145)$$

where

$$\frac{\partial H}{\partial \widehat{U}_2^E} = r + \delta_2 + \frac{1}{2}a_2 \left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V \right] > 0. \quad (146)$$

Thus, $B > 0$ if

$$\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} > \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial U_2^A}, \quad (147)$$

which can be written as

$$\begin{aligned} & \left[r + \delta_2 + \frac{1}{6}a_2 \left[2 \left[D_2^A - U_2^A \right] + 5D_2^V \right] \right] \left[r + \delta_2 + \frac{1}{6}a_2 \left[2 \left[D_2^E - U_2^E \right] + 5D_2^V \right] \right] \\ & > \frac{1}{6}a_2 \left[2 \left[D_2^A - U_2^A \right] - D_2^V \right] \frac{1}{6}a_2 \left[2 \left[D_2^E - U_2^E \right] - D_2^V \right]. \end{aligned} \quad (148)$$

Note that this condition holds for all $r + \delta_2 > 0$ if it holds for $r + \delta_2 = 0$. Setting $r + \delta_2 = 0$ we get

$$12 \left[D_2^A - U_2^A \right] D_2^V + 12D_2^V \left[D_2^E - U_2^E \right] + 24 \left[D_2^V \right]^2 > 0. \quad (149)$$

Note that $D_2^E > U_2^E$ and $D_2^A > U_2^A$. Thus, this condition is satisfied, so that $B > 0$. Next consider the numerator A . With $\partial H/\partial U_2^E = 0$ we can write A as

$$A = \underbrace{\left[-\frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E} + \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \lambda} \right]}_{\equiv X_1} \frac{\partial H}{\partial \widehat{U}_2^E} - \frac{\partial H}{\partial \lambda} \underbrace{\left[\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \widehat{U}_2^E} - \frac{\partial F}{\partial \widehat{U}_2^E} \frac{\partial G}{\partial U_2^E} \right]}_{\equiv X_2}. \quad (150)$$

Recall that $\partial H/\partial \widehat{U}_2^E > 0$. Moreover,

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{2}\pi a_2 \left[\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V \right] < 0. \quad (151)$$

Thus, $A < 0$ when $X_1 < 0$ and $X_2 < 0$. Note that $X_1 < 0$ if

$$\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \lambda} < \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E}, \quad (152)$$

which can be written as

$$\begin{aligned} & \frac{1}{6}a_2 \left[2 \left[D_2^A - U_2^A \right] - D_2^V \right] \frac{1}{6}\pi a_2 \left[D_2^E - U_2^E + D_2^V \right] \\ & > \frac{1}{6}\pi a_2 \left[D_2^A - U_2^A - 2D_2^V \right] \left[r + \delta_2 + \frac{1}{6}a_2 \left[2 \left[D_2^E - U_2^E \right] + 5D_2^V \right] \right]. \end{aligned} \quad (153)$$

Simplifying yields

$$\frac{2}{a_2} (r + \delta_2) \left[\left[D_2^A - U_2^A \right] - 2D_2^V \right] + \left[D_2^A - U_2^A \right] D_2^V - D_2^V \left[D_2^E - U_2^E \right] < 3 \left[D_2^V \right]^2. \quad (154)$$

From F and G we know that

$$D_2^V \left[D_2^A - U_2^A \right] = \frac{U_2^A (r + \delta_2) + \sigma}{a_2} \quad \text{and} \quad D_2^V \left[D_2^E - U_2^E \right] = \frac{U_2^E (r + \delta_2) + \sigma}{a_2}, \quad (155)$$

so that condition (154) can be written as

$$(r + \delta_2) \underbrace{\left[2D_2^A - U_2^A - 4D_2^V - U_2^E \right]}_{\equiv T} < 3 \left[D_2^V \right]^2 a_2. \quad (156)$$

It remains to prove that $T < 0$. Using the definitions of D_2^A and D_2^V we can write $T < 0$ as

$$U_2^E + U_2^A < [1 + 2\lambda] \pi. \quad (157)$$

This condition is satisfied for all $\lambda \geq 0$ as $\pi > U_2^E + U_2^A$. Thus, $X_1 < 0$. Moreover, $X_2 < 0$ if

$$\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \widehat{U}_2^E} < \frac{\partial F}{\partial \widehat{U}_2^E} \frac{\partial G}{\partial U_2^E}, \quad (158)$$

which is equivalent to

$$\begin{aligned} & \frac{1}{6}a_2 [2 [D_2^A - U_2^A] - D_2^V] \frac{1}{6}a_2 [D_2^E - U_2^E - 2D_2^V] \\ < \frac{1}{6}a_2 [D_2^A - U_2^A + D_2^V] \left[r + \delta_2 + \frac{1}{6}a_2 [2 [D_2^E - U_2^E] + 5D_2^V] \right]. \end{aligned} \quad (159)$$

Again, $D_2^A > U_2^A$ and $D_2^E > U_2^E$. Thus, if this condition holds for $r + \delta_2 = 0$, then it also holds for all $r + \delta_2 > 0$. Setting $r + \delta_2 = 0$ we get

$$0 < 3D_2^V \underbrace{[D_2^E - U_2^E]}_{>0} + 9D_2^V \underbrace{[D_2^A - U_2^A]}_{>0} + 3D_2^V D_2^V. \quad (160)$$

Hence, $X_2 < 0$, so that $A < 0$. Consequently, $dU_2^A/d\lambda < 0$. Moreover, note that $\partial D_2^E/\partial\lambda = \pi/6 < |\partial D_2^A/\partial\lambda| = \pi/3$. Thus, $d[U_2^E + U_2^A]/d\lambda < 0$. Finally, using H we get

$$\frac{d\widehat{U}_2^E}{d\lambda} = \pi \frac{\frac{1}{2}a_2 [\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V]}{\underbrace{r + \delta_2 + \frac{1}{2}a_2 [\widehat{D}_2^E - \widehat{U}_2^E + \widehat{D}_2^V]}_{\equiv Z}}, \quad (161)$$

where $Z \in (0, 1)$. Thus,

$$\frac{dD_2^V}{d\lambda} = \frac{1}{6}\pi \underbrace{[1 - Z]}_{>0} - \frac{1}{3} \underbrace{\frac{d}{d\lambda} [U_2^E + U_2^A]}_{<0}. \quad (162)$$

Consequently, $dD_2^V/d\lambda > 0$.

All this implies that the results from Proposition 5 continue to hold when the entrepreneur can search for a VC without the angel.

Proof of Proposition 6.

Recall that $U_2^E = U_2^A$ in equilibrium. Moreover, as shown in Proof of Proposition 3, $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\delta_2$, $dU_2^E/d\sigma_2^V$, $dU_2^E/dk_2 < 0$. Consequently, $d\gamma^*/d\phi_2 <$

0, and $d\gamma^*/d\sigma_2, d\gamma^*/d\delta_2, d\gamma^*/d\sigma_2^V, d\gamma^*/dk_2 > 0$. □

Proof of Proposition 7.

Recall from Proof of Proposition 5 that $d[U_2^E + U_2^A]/d\lambda < 0$. Thus, $d\gamma^*/d\lambda > 0$. □