Supplementary appendix to:

Bank capital, liquid reserves, and insolvency risk*

Julien Hugonnier† Erwan Morellec‡

Contents

This document gathers the proofs of all the results in the main text. To simplify the presentation the numbering of equations is continued from the main text.

*This version: August 31, 2016. The latest version of this document can be downloaded from the authors's homepages sfi.epfl.ch/hugonnier and sfi.epfl.ch/morellec.

†Ecole Polytechnique Fédérale de Lausanne, Swiss Finance Institute and CEPR. Postal: Extranef, Quartier UNIL Dorigny, 1015 Lausanne, Switzerland. Email: Julien.Hugonnier@epfl.ch

‡Ecole Polytechnique Fédérale de Lausanne, Swiss Finance Institute and CEPR. Postal: Extranef, Quartier UNIL Dorigny, 1015 Lausanne, Switzerland. Email: Erwan.Morellec@epfl.ch

Appendix A. The auxiliary problem

A.1. Notation

To facilitate the presentation we start by fixing some notation that will be of repeated use throughout the appendix. A strategy is a pair $\pi = (P^\pi, R^\pi)$ of adapted, non decreasing, *left* continuous and *right* limited processes with initial value zero such that

$$R_t^\pi = \sum_{n=1}^{\infty} 1_{\{t>\xi_n\}} r_n \tag{65}$$

for some increasing sequence of stopping times $(\xi_n)_{n=1}^\infty$ and some sequence of nonnegative random variables $(r_n)_{n=1}^\infty$ such that r_n is measurable with respect to $\mathcal{F}_{\xi_n}$. The liquid reserves process and liquidation time associated with the use of a given payout and financing strategy are defined by

$$S_t^\pi = s + C_t - P_t^\pi + R_t^\pi = s + \bar{\mu} t + \sigma B_t - \sum_{n=1}^{N_t} Y_n - P_t^\pi + R_t^\pi \tag{66}$$

and

$$\tau_\pi = \inf \left\{ t \geq 0 : S_{t+}^\pi = \lim_{u \downarrow t+} S_u^\pi \leq 0 \right\}. \tag{67}$$

with the strictly positive constant $\bar{\mu} = (1-\theta)(\mu - c)$. The set $\Pi(s)$ of strategies that are admissible starting from $s \in \mathbb{R}$ is defined as the set of strategies such that

$$\Delta^+ P_t^\pi \leq S_t^\pi + \Delta^+ R_t^\pi \tag{68}$$

and

$$\mathbb{E}_s \left[\int_0^{\tau_\pi} e^{-\rho t} (dP_t^\pi + d\Phi_t(R^\pi)) \right] < \infty. \tag{69}$$

where the nondecreasing process

$$\Phi_t(R^\pi) = R^\pi_t + \sum_{n=1}^{\infty} 1_{\{t>\xi_n\}}\phi \tag{70}$$

represents the total contribution of shareholders to the bank, and

$$\Delta^+ Z_t = Z_{t+} - Z_t = \lim_{u\downarrow t} Z_u - Z_t \tag{71}$$

denotes the jump that occurs immediately after time $t \geq 0$. The inequality constraint imposed by Eq.(68) prevents shareholders from distributing dividends that exceed the available liquid reserves and is necessary to guarantee that the optimization problems in Eqs.(10) and (24) are well-defined. Otherwise the bank would be able to generate infinite value by simply paying out amounts that it does not hold.

A.2. Immediate liquidation

Lemma A.1. *Denote by $v_0^* > 0$ the unique solution to*

$$z(v) = -(\rho+\lambda)v + \bar{\mu} + \lambda\mathbb{E}\left[(v - Y_1)^+\right] = 0 \tag{72}$$

If the liquidation value is such that $\alpha \geq v_0^*$ then it is optimal for shareholders to shut down the bank immediately.

Proof. The function $z(v)$ is continuous, non increasing, starts out from $\bar{\mu} > 0$ at the origin and satisfies

$$\lim_{v\to\infty} z(v) = -\infty. \tag{73}$$

Therefore it crosses the horizontal axis at a unique point $v_0^* > 0$. Given the result of Lemma A.8 below it now suffices to show that the equity value function $u_0(s) = (\alpha + s)^+$ associated with the strategy of immediate liquidation satisfies

$$\mathcal{H}u(s) = \max\{1 - u'(s), \mathcal{D}u(s)\} \leq 0, \qquad s > 0, \tag{74}$$

with the integro-differential operator defined by

$$\mathcal{D}u(s) = -\rho u(s) + \bar{\mu}u'(s) + \frac{1}{2}\sigma^2 u''(s) + \lambda\mathbb{E}\left[u(s - Y_1) - u(s)\right]. \tag{75}$$

Substituting the function $u_0(s)$ into the right hand side of Eq.(74) gives

$$\mathcal{H}u_0(s) = \max\left(0, z(\alpha + s)\right), \qquad s > 0. \tag{76}$$

Since the term inside the bracket is decreasing we have that $\mathcal{H}u_0(s) \leq 0$ for all $s > 0$ if and only if $\mathcal{H}u_0(0) = 0$. This is equivalent to $z(\alpha) \leq 0$ and the required result follows from the definition of the constant v_0^*. ■

A.3. Value of a barrier strategy

In order to compute the equity value of the auxiliary bank under a given barrier strategy let us first start by fixing some notation. Let

$$B_1 < -\beta < B_2 < 0 < B_3 \tag{77}$$

denote the three real roots of the cubic equation

$$\rho = B_i\left(\bar{\mu} + B_i\frac{\sigma^2}{2} - \frac{\lambda}{\beta + B_i}\right), \tag{78}$$

set

$$A(\alpha) = \lambda\mathbb{E}\left[(\alpha - Y_1)^+\right] = \lambda\left(\alpha - \frac{F(\alpha)}{\beta}\right) \in [0, \alpha\lambda] \tag{79}$$

and define

$$W(x) = \sum_{i=1}^{3} 1_{\{x \geq 0\}} \frac{2(\beta + B_i)}{\sigma^2 \prod_{k \neq i}(B_i - B_k)} e^{B_i x}. \tag{80}$$

The function $W(x)$ is referred to as the $\rho-$scale function of the uncontrolled liquid reserves process (see Kuznetsov, Kyprianou and Rivero (2013) for a comprehensive survey of the theory of scale functions) and the following result shows that, as stated in Eqs.(27) and (28) of the main text, this function can be used as a building block to derive explicit expressions for the functions appearing on the right hand side of equation Eq.(27).

Lemma A.2. *We have*

$$w(s;0;b) = \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b\right] = (s-b)^+ + \frac{W(s\wedge b)}{W'(b)} \tag{81}$$

and

$$\psi(s;\alpha) = \sum_{i=1}^{2} a_i(\alpha)e^{B_i s}, \qquad s \geq 0, \tag{82}$$

with the coefficients defined by

$$a_i(\alpha) = (-1)^i\left[\frac{2A(\alpha)+\alpha(\beta+B_i)(\beta+B_3)\sigma^2}{(B_2-B_1)(\beta+B_3)\sigma^2}\right] > 0 \qquad i=1,2. \tag{83}$$

The function $\psi(s;\alpha)$ is completely monotone with respect to $s \geq 0$ and therefore decreasing and convex.

Proof. The first part follows directly from Avram, Palmowski and Pistorius (2007, Proposition 1). To establish the second part we start by decomposing the function into two components according to whether the uncontrolled process enters the negative real line continuously (first term), or through a jump (second term):

$$\psi(x;\alpha) = \alpha\mathbb{E}_x\left[e^{-\rho\zeta_0}1_{\{\Delta X_{\zeta_0}=0\}}\right] + \mathbb{E}_x\left[e^{-\rho\zeta_0}(\alpha+X_{\zeta_0})^+1_{\{\Delta X_{\zeta_0}\neq 0\}}\right]. \tag{84}$$

Combining Corollary 2 and Equation (6) of Bertoin (1997) shows that the first term can be computed in terms of the scale function as

$$\alpha\mathbb{E}_x\left[e^{-\rho\zeta_0}1_{\{\Delta X_{\zeta_0}=0\}}\right] = \frac{\alpha\sigma^2}{2}\left(W'(x)-B_3W(x)\right). \tag{85}$$

On the other hand, Bertoin (1997, Corollary 2) shows that the potential measure of the uncontrolled liquid reserves process killed at ζ_0 is given by

$$U(x,dy) = \mathbb{E}_x\left[\int_0^\infty e^{-\rho t}1_{\{\zeta_0>t\}\cap\{X_t\in dy\}}dt\right] = \left(e^{-B_3 y}W(x)-W(x-y)\right)dy \tag{86}$$

and it thus follows from the compensation formula for point processes that second term can be calculated as

$$\mathbb{E}_x\left[e^{-\rho\zeta_0}(\alpha+X_{\zeta_0})^+1_{\{\Delta X_{\zeta_0}\neq 0\}}\right]=\int_0^\infty \lambda U(x,dy)\int_y^\infty(\alpha+y-u)^+dF(u). \tag{87}$$

Substituting Eqs.(85) and (87) into Eq.(84), calculating the integral, and simplifying the result gives the formula reported in the statement. The nonnegativity of $A(\alpha)$ and Eq.(77) imply that $a_2(\alpha)>0$. On the other hand, using Eq.(77) and

$$\prod_{i=1}^{3}(\beta+B_i)+\frac{2\lambda\beta}{\sigma^2}=0<A(\alpha)-\lambda\alpha \tag{88}$$

we deduce that

$$a_1(\alpha)=\frac{2}{(B_1-B_2)(\beta+B_3)}\left(A(\alpha)-\frac{\lambda\beta\alpha}{\beta+B_2}\right)>0 \tag{89}$$

and the required complete monotonicity of the function $\psi(s;\alpha)$ now follows from the fact that the constants B_1 and B_2 are negative. ■

A.4. The optimal barrier

In the earnings retention region the value of a barrier strategy depends on the barrier level only through the function

$$H(b;\alpha)=\frac{1-\psi'(b;\alpha)}{W'(b)}. \tag{90}$$

Therefore, since the scale function is strictly positive by Lemma A.6 below, we have that the existence and uniqueness of an optimal barrier amount to the existence and uniqueness of a global maximizer for this function.

Lemma A.3. *For any fixed $\alpha\in[0,v_0^*)$ the function $H(b;\alpha)$ achieves its supremum over the positive half line at a unique point $b^*(\alpha)$.*

Proof. Since the case $\alpha=0$ follows directly from the result of Lemma A.6 we will assume that $\alpha\in(0,v_0^*)$. Using the definition of the functions $z(\alpha)$ and $A(\alpha)$ in conjunction with the fact that the roots of Eq.(78) satisfy

$$\beta + \sum_{i=1}^{3} B_i + \frac{2\bar{\mu}}{\sigma^2} = \prod_{i=1}^{3} B_i - \frac{2\beta\rho}{\sigma^2} = 0, \tag{91}$$

we deduce that

$$H'(0;\alpha) = \bar{\mu} + A(\alpha) - \alpha(\rho+\lambda) = z(\alpha) > 0, \tag{92}$$

where the inequality follows from the fact that $\alpha < v_0^*$ if and only if $z(\alpha) > 0$ as established in the proof of Lemma A.1. To study the behavior of the derivative away from the origin consider the function

$$G(b;\alpha) = -\frac{\psi''(b;\alpha)}{W''(b)}. \tag{93}$$

A direct calculation using the definition of the function $H(b;\alpha)$ shows that

$$H'(b;\alpha) = \left(G(b;\alpha) - H(b;\alpha)\right)\frac{W''(b)}{W'(b)} \tag{94}$$

and it thus follows from Lemma A.6 that we have

$$H'(b;\alpha) \geq 0 \iff (x^* - b)(H(b,\alpha) - G(b,\alpha)) \geq 0 \tag{95}$$

for any $b \neq x^*$ where the constant $x^* > 0$ is the unique solution to $W''(x^*) = 0$ provided by Lemma A.6 below. Now consider the threshold defined by

$$b^* = b^*(\alpha) \equiv \inf\{b \geq 0 : H(b;\alpha) \leq G(b;\alpha)\}. \tag{96}$$

Since $G(x^*, \alpha) = \infty$ by construction and $H(0;\alpha) > G(0;\alpha)$ due to Eqs.(92) and (95), we have that this threshold lies in $(0, x^*)$ and satisfies

$$H'(b^*;\alpha) = H(b^*;\alpha) - G(b^*;\alpha) = 0 \leq H'(b;\alpha), \qquad 0 \leq b \leq b^*. \tag{97}$$

Therefore, the proof will be complete once we show that function $H(b;\alpha)$ is decreasing on the interval $[b^*, \infty)$. Combining Lemmas A.2 and A.6 we deduce that

$$G(b;\alpha) \leq 0 \leq H(b;\alpha), \qquad b > x^*, \tag{98}$$

and it thus follows from Eq.(95) that $H(b;\alpha)$ is decreasing over (x^*, ∞). On the other hand, the fact that $b^* > 0$ implies that we have $G'(b^*;\alpha) > 0$ and therefore

$$G'(b;\alpha) > 0, \qquad b \geq b^*, \tag{99}$$

by Lemma A.7. Combining this property with Eq.(97) shows that there exists an $\epsilon_0 > 0$ such that the function

$$\Phi(b;\alpha) = H(b;\alpha) - G(b;\alpha) \tag{100}$$

is strictly negative on the interval $N_0 = (b^*, b^* + \epsilon_0]$ and it now follows from Eq.(95) that we have $H'(b;\alpha) < 0$ on N_0. Repeating the same argument at $b^* + \epsilon_0$ then allows to propagate this property to the whole interval $(b^*, x^*]$ and completes the proof. ■

Intuitively we expect that the incentives of the bank to retain earnings decrease as the liquidation value of assets increases. The next result confirms this intuition by showing that the optimal barrier is decreasing in the liquidation value of assets.

Lemma A.4. *The function $b^*(\alpha)$ is strictly decreasing on the interval $[0, v_0^*]$ with $b^*(0) = x^*$ and $b^*(v_0^*) = 0$.*

Proof. As shown in the proof of Lemma A.3 we have that for any $\alpha \in [0, v^*)$ the optimal barrier satisfies

$$H(b^*(\alpha);\alpha) - G(b^*(\alpha);\alpha) = 0. \tag{101}$$

Therefore it follows from the implicit function theorem that the function $b^*(\alpha)$ is once continuously differentiable with

$$\frac{db^*(\alpha)}{d\alpha} = -\frac{G_\alpha(b^*(\alpha);\alpha) - H_\alpha(b^*(\alpha);\alpha)}{G_b(b^*(\alpha);\alpha) - H_b(b^*(\alpha);\alpha)} = \frac{H_\alpha(b^*(\alpha);\alpha) - G_\alpha(b^*(\alpha);\alpha)}{G_b(b^*(\alpha);\alpha)} \tag{102}$$

where the second equality follows from Eq.(97). By Eq.(99) we have that the denominator is strictly positive and so it only remains to show that the numerator is negative on $[0, v_0^*)$. To this end it suffices to show that the function

$$K(b;\alpha) = \frac{1}{4}\sigma^4 W'(b)W''(b)(H_\alpha(b;\alpha) - G_\alpha(b;\alpha)) \tag{103}$$

is strictly positive. A direct calculation shows that this function can be decomposed into a sum of exponentials as

$$K(b;\alpha) = c_{12}(\alpha)e^{(B_1+B_2)b} + c_{13}(\alpha)e^{(B_1+B_3)b} + c_{23}(\alpha)e^{(B_2+B_3)b} \tag{104}$$

with the coefficients

$$c_{ij}(\alpha) = -\frac{B_iB_j((\beta+B_i)(\beta+B_j)\sigma^2 + 2A'(\alpha))}{2(B_i - B_{-ij})(B_i - B_{-ij})}, \qquad i \neq j \in \{1,2,3\}. \tag{105}$$

The increase of the function $A(\alpha)$ and Eq.(77) imply that $c_{23}(\alpha)$ is strictly positive. On the other hand, using Eq.(77) in conjunction with

$$\prod_{i=1}^{3}(\beta + B_i) + \frac{2\lambda\beta}{\sigma^2} = 0 < A'(\alpha) - \lambda \tag{106}$$

we deduce that

$$c_{13}(\alpha) = \frac{-2B_1B_3}{(B_1-B_2)(B_3-B_2)}\left(A'(\alpha) - \frac{\lambda\beta}{(\beta+B_2)}\right) > 0. \tag{107}$$

If the remaining coefficient is nonnegative then $K(b;\alpha) > 0$ for all $b \geq 0$ and the proof is complete. If instead the remaining coefficient is strictly negative then it follows from Eq.(77) and Lemma A.5 that there exist $\gamma > 0$ such that $K(b;\alpha) > 0$ if and only if $b > \gamma$. Since

$$K(0;\alpha) = c_{12}(0) + c_{13}(0) + c_{23}(0) = \rho + \lambda - A'(\alpha) > 0. \tag{108}$$

by Eq.(106) we have that $\gamma < 0$ and required result follows. The limit value at zero follows from the definition of the constant x^*. On the other hand, Eqs.(92), (94) and the definition of the constant v_0^* imply that

$$\lim_{\alpha \to v_0^*} H'(0;\alpha) = \lim_{\alpha \to v_0^*} (H(0;\alpha) - G(0;\alpha)) = 0 \tag{109}$$

and the result now follows from the fact that, as shown in the proof of Lemma A.3, the function $H(b;\alpha)$ is decreasing in b after the point where it crosses $G(b;\alpha)$. ■

Lemma A.5. *Let $a_1 < 0 < a_3$ and $b_1 < b_2 < b_3$ be constants. Then there exists a unique constant γ^* such that the function*

$$f(x) = \sum_{i=1}^{3} a_i e^{b_i x} \tag{110}$$

is positive if and only if $x \geq \gamma^*$.

Proof. Under the conditions of the statement we have that

$$k(x) = e^{-b_1 x} f(x) = a_1 + a_2 e^{(b_2 - b_1)x} + a_3 e^{(b_3 - b_1)x} \tag{111}$$

tends to $a_1 < 0$ as $x \to -\infty$ and to ∞ as $x \to \infty$. If a_2 is nonnegative, then this function is nondecreasing and therefore crosses the origin only once. On the other hand, if a_2 is negative then $k(x)$ is ∪-shaped and therefore attains a minimum at the unique point where its derivative equals zero. In either case, the equation $k(x) = 0$ admits a unique solution at which $k'(x) > 0$ and the desired result follows. ■

Lemma A.6. *The scale function is strictly increasing on $[0,\infty)$ and there exists a constant $x^* > 0$ such that $W''(x) \leq 0$ if and only if $x \leq x^*$.*

Proof. Differentiating Eq.(81) and using Eq.(77) shows that for any $x \geq 0$ the derivative of the scale function is given by

$$W'(x) = \sum_{i=1}^{3} c_i e^{B_i x} \tag{112}$$

with the coefficients

$$c_i = \frac{2(\beta + B_i)B_i}{\sigma^2 \prod_{k \neq i}(B_i - B_k)} > 0, \qquad i = 1, 2, 3. \tag{113}$$

This shows that the scale function is strictly increasing over the positive real line and hence positive since $W(0) = 0$. The second part follows from an application of Lemma A.5 and the fact that

$$W''(0) = \frac{2}{\sigma^2}(B_1 + B_2 + B_3 + \beta) = -\frac{4\bar{\mu}}{\sigma^4} \tag{114}$$

is strictly negative. ■

Lemma A.7. *There exists an* $x_0 \in [0, x^*)$ *such that* $G(x; \alpha)$ *is decreasing on* $[0, x_0]$ *and increasing on* (x_0, x^*).

Proof. Consider the function defined by

$$K(x; \alpha) = \frac{1}{2}(\sigma W''(x))^2 G'(x; \alpha) \tag{115}$$

A direct calculation using Eq.(82) and the definition of the scale function shows that this function is explicitly given by

$$K(x; \alpha) = a_{13}(\alpha)e^{(B_1+B_3)x} + a_{23}(\alpha)e^{(B_2+B_3)x} + a_{12}(\alpha)e^{(B_1+B_2)x} \tag{116}$$

with the coefficients

$$a_{13}(\alpha) = \frac{B_1^2 B_3^2(\beta + B_3)}{B_3 - B_2} a_1(\alpha) \tag{117}$$

$$a_{23}(\alpha) = \frac{B_2^2 B_3^2(\beta + B_3)}{B_3 - B_1} a_2(\alpha) \tag{118}$$

and

$$a_{12}(\alpha) = \frac{B_1^2 B_2^2(2A(\alpha) + \alpha\sigma^2(\beta + B_1)(\beta + B_2))}{(B_3 - B_1)(B_3 - B_2)\sigma^2}. \tag{119}$$

As shown in the proof of Lemma A.2, we have that $a_1(\alpha), a_2(\alpha) > 0$ and combining this with Eq.(77) shows that $a_{13}(\alpha), a_{23}(\alpha) > 0$.

To complete the proof we distinguish two cases depending on the sign of the last coefficient. If we have that $a_{12}(\alpha) \geq 0$ then $K(x; \alpha) \geq 0$ and the required result holds with the constant $x_0 = 0$. On the contrary, if we have that $a_{12}(\alpha) < 0$ then it follows from Eqs.(77),

(115) and Lemma A.5 that there exists an k such that

$$x \le k \Longleftrightarrow G'(x;\alpha) \ge 0. \tag{120}$$

In this case we let $x_0 = k^+$ and it now only remains to show that $x_0 < x^*$. Since

$$\lim_{x\uparrow x^*} G(x;\alpha) = \lim_{x\uparrow x^*} -\frac{\psi''(x;\alpha)}{W''(x)} = \infty \tag{121}$$

by definition of x^*, we have that $G'(x;\alpha) > 0$ in a left neighborhood of x^* and the desired result now follows from the definition of the constant k. ■

A.5. Verification

By construction we have that the barrier strategy at $b^*(\alpha)$ is optimal in the restricted class of barrier strategies. To show that this strategy is in fact optimal among all admissible strategies we will rely on the following verification result.

Lemma A.8. *Assume that $u : \mathbb{R} \to \mathbb{R}$ is a continuous function that is twice continuously differentiable on $(0,\infty)$ and such that*

$$u(s) - (\alpha + s)^+ = 0, \quad s \le 0, \tag{122}$$

$$\max\{1 - u'(s), \mathcal{D}u(s)\} \le 0, \quad s > 0. \tag{123}$$

where the operator $\mathcal{D}$ is defined in Eq.(75). Then $u(s) \ge w(s;\alpha)$ for all $s \in \mathbb{R}$.

Proof. Let the function $u(s)$ be as in the statement, fix a strategy $\pi \in \Pi_0(s)$ and consider the nonnegative process defined by

$$Y_t = e^{-\rho t\wedge\tau_\pi} u(S_{t\wedge\tau_\pi}) + \int_0^{t\wedge\tau_\pi} e^{-\rho s} dP^\pi_s. \tag{124}$$

Applying Itô's formula for semimartingales (see for example Dellacherie and Meyer (1980, Theorem VIII.25)) shows that there is a local martingale M_t such that

$$Y_t - M_{t\wedge\tau_\pi} = u(s) + \sum_{0\leq s<t\wedge\tau_\pi} e^{-\rho s}\left(\Delta^+ P_s^\pi + \Delta^+ u(S_s^\pi)\right) \tag{125}$$

$$+ \int_0^{t\wedge\tau_\pi} e^{-\rho s}\mathcal{D}u(S_{s-}^\pi)ds + \int_0^{t\wedge\tau_\pi} e^{-\rho s}(1-u'(S_{s-}))dP_s^{\pi,c} \tag{126}$$

where $P_t^{\pi,c}$ denotes the continuous part of the cumulative payout process. Using the definition of the liquidation time together with the fact that the function $u(s)$ satisfies Eqs.(122) and (123), we deduce that

$$1_{\{s<\tau_\pi\}}\left(\Delta^+ P_s^\pi + \Delta^+ u(S_s^\pi) + \mathcal{D}u(S_{s-}^\pi)ds + (1-u'(S_{s-}))dP_s^{\pi,c}\right) \tag{127}$$

$$\leq 1_{\{s<\tau_\pi\}}\left(\Delta^+ P_s^\pi + \Delta^+ u(S_s^\pi)\right) \tag{128}$$

$$= 1_{\{s<\tau_\pi\}}\left(u(S_s^\pi - \Delta^+ P_s^\pi) + \Delta^+ P_s^\pi - u(S_s^\pi)\right) \leq 0 \tag{129}$$

and it follows that the process on the right hand side of Eq.(125) is decreasing. This in turn implies that Y_t is local supermartingale and hence a supermartingale since it is nonnegative. In particular, combining Eq.(122) with the the optional sampling theorem for nonnegative supermartingales shows that

$$u(s) = Y_0 \geq \mathbb{E}_s\left[Y_{\tau_\pi}\right] = \mathbb{E}_s\left[e^{-\rho\tau_\pi}u(S_{\tau_\pi}^\pi) + \int_0^{\tau_\pi} e^{-\rho s}dP_s^\pi\right] \tag{130}$$

$$= \mathbb{E}_s\left[e^{-\rho\tau_\pi}(\alpha + S_{\tau_\pi}^\pi)^+ + \int_0^{\tau_\pi} e^{-\rho s}dP_s^\pi\right] \tag{131}$$

and the required result now follows from the arbitrariness of the strategy by taking the supremum over $\pi \in \Pi_0(s)$ on both sides. ■

Lemma A.9. *For $\alpha < v_0^*$ the continuous function*

$$u(s;\alpha) = w(s;\alpha, b^*(\alpha)), \qquad s \in \mathbb{R}, \tag{132}$$

is concave and twice continuously differentiable on $(0,\infty)$ with $u'(s;\alpha) \geq 1$ for all $s > 0$.

Proof. By construction we have that $u(s;\alpha)$ is twice continuously differentiable on the set $(0,\infty)\backslash b^*(\alpha)$ and so it suffices to establish the required smoothness at the barrier.

Differentiating on both sides of Eq.(81) we deduce that

$$\lim_{s\uparrow b^*(\alpha)} u'(s;\alpha) = \lim_{s\uparrow b^*(\alpha)} \left(\psi'(s;\alpha) + W'(s)H(b^*(\alpha);\alpha)\right) = 1 \tag{133}$$

and it now follows from Eq.(27) that the function $u(s;\alpha)$ is continuously differentiable across the barrier. Similarly, differentiating twice with respect to s on both sides of Eq.(81) and using the definition of the optimal barrier gives

$$u''(s;\alpha) = W''(s)\left(G(b^*(\alpha);\alpha) - G(s;\alpha)\right), \qquad s < b^*(\alpha), \tag{134}$$

and the required smoothness now follows from the continuity of $G(b;\alpha)$. By Lemmas A.3 and A.6 we have

$$W''(b) \leq 0 \leq G(b;\alpha) \leq H(b;\alpha) \leq H(b^*(\alpha);\alpha) = G(b^*(\alpha),\alpha) \tag{135}$$

for all $b \leq b^*(\alpha) \leq x^*$. Therefore, it follows from Eq.(134) that $u(s;\alpha)$ is concave on $[0, b^*(\alpha))$ and hence on the positive half line since it is linear outside of this interval. Finally, using this concavity and the fact that $u'(s;\alpha) = 1$ above the point $b^*(\alpha)$, we deduce that $u'(s;\alpha) \geq 1$ for all $s \geq 0$ and the proof is complete. ■

The next result establishes the global optimality of the barrier strategy at $b^*(\alpha)$ for the auxiliary problem Eq.(24) and concludes the proof of Proposition 2.

Lemma A.10. *Let $b^*(\alpha)$ denote the unique solution to $H'(b;\alpha) = 0$ for $\alpha < v_0^*$ and $b^*(\alpha) = 0$ otherwise. Then we have have $w(s;\alpha) = u(s;\alpha)$ and the optimal dividend strategy is a barrier strategy at $b^*(\alpha)$.*

Proof. When $\alpha \geq v_0^*$ the result follows directly from Lemma A.1 so let us assume that $\alpha < v_0^*$. Since by construction $u(s;\alpha) \leq w(s;\alpha)$, it suffices to show that $u(s;\alpha)$ satisfies the conditions of Lemma A.8. By Lemma A.9 we have that this function is continuous on $\mathbb{R}$, twice continuously differentiable on $(0,\infty)$ and satisfies

$$u'(s;\alpha) \geq 1, \qquad s > 0, \tag{136}$$

as well as Eq.(122). Therefore it only remains to show that $\mathcal{D}u(s;\alpha) \leq 0$ for all $s > 0$. A direct calculation using Eq.(78) shows that we have

$$\mathcal{D}\psi(s;\alpha) = \mathcal{D}W(s) = 0, \qquad s > 0. \tag{137}$$

Combining this identity with Eqs.(27) and (132) we conclude that

$$\mathcal{D}u(s;\alpha) = \mathcal{D}\psi(s;\alpha) + H(b^*(\alpha);\alpha)\mathcal{D}W(s) = 0, \qquad s < b^*(\alpha). \tag{138}$$

On the other hand, using Eq.(137) in conjunction with Eq.(27), the result of Lemma A.9 and the definition of $\mathcal{D}$ shows that for $s \geq b^*(\alpha)$

$$\mathcal{D}u(s;\alpha) = \mathcal{D}u(s;\alpha) - \mathcal{D}\psi(s;\alpha) - H(s;\alpha)\mathcal{D}W(s) \tag{139}$$

$$= \rho\left(w(s;\alpha,s) - u(s;\alpha)\right) - \frac{\sigma^2}{2}\left(\psi''(s;\alpha) + H(s;\alpha)W''(s)\right) \tag{140}$$

$$+ \lambda \int_0^\infty \left[\left(u(s-y;\alpha) - w(s-y;\alpha,s)\right) - \left(u(s;\alpha) - w(s;\alpha,s)\right)\right] dF(y). \tag{141}$$

Fix an arbitrary $s \geq b^*(\alpha)$ and consider the function defined by

$$\varphi(x;\alpha,s) = w(x;\alpha,s) - u(x;\alpha) \tag{142}$$

Using Eq.(27) together with Eq.(132) we deduce that

$$\varphi'(x;\alpha,s) = W'(x)\left(H(s;\alpha) - H(b^*(\alpha);\alpha)\right) \leq 0, \qquad x \in (0, b^*(\alpha)) \tag{143}$$

where the inequality follows from Lemmas A.6 and A.3. On the other hand, using Eq.(27) together with Eq.(132) we deduce that

$$\varphi'(x;\alpha,s) = \psi'(x;\alpha) + W'(x)H(s;\alpha) - 1, \qquad b^*(\alpha) \leq x \leq s \tag{144}$$

$$= W'(x)\left(H(s;\alpha) - H(x;\alpha)\right) \leq 0 \tag{145}$$

where the inequality follows from Lemma A.6 and the fact that, as established in the proof of Lemma A.3, the function $H(x;\alpha)$ is decreasing over $[b^*(\alpha), \infty)$. Combining Eqs.(143) and

(145) shows that we have

$$\varphi'(x;\alpha,s) \leq 0, \qquad 0 \leq x \leq s, \tag{146}$$

and since

$$\varphi(x;\alpha,s) = w(x;\alpha,s) - u(x;\alpha) = 0, \qquad x \leq 0, \tag{147}$$

because of Eq.(27) we conclude that the function $\varphi(x;\alpha,s)$ is non positive on $(-\infty, s]$. This implies that the first and last term in Eq.(140) are non positive. Finally, it follows from the definition of the function $H(s;\alpha)$ that the second term satisfies

$$-\frac{\sigma^2}{2}\left(\psi''(s;\alpha) + H(s;\alpha)W''(s)\right) = \frac{\sigma^2}{2}W'(s)H'(s;\alpha) \tag{148}$$

and the desired conclusion follows from Lemma A.6 and the fact that $H(x;\alpha)$ is decreasing over the interval $[b^*(\alpha), \infty)$ as shown in the proof of Lemma A.3. ■

Appendix B. Proofs

B.1. *The frictionless problem*

In order to prove Proposition 1 we consider the parametrized family of optimal stopping problems defined by

$$p(\Phi) = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-\rho s} dC_s + e^{-\rho\tau} \max\{\Phi, -\Delta C_\tau\}\right] \tag{149}$$

$$\mathbb{E}\left[\int_0^\tau e^{-\rho s} dC_s + e^{-\rho\tau} \max(\Phi, -\Delta C_\tau)\right] = \mathbb{E}\left[\int_0^\tau e^{-\rho s} dC_s + e^{-\rho\tau} \max(\Phi, -\Delta C_\tau)\right] \tag{150}$$

where $\mathcal{S}$ denotes the set of all stopping times and the cumulative cash flow process evolves according to

$$C_t = A_t - (1-\theta)ct = \bar{\mu}t + \sigma B_t - \sum_{n=1}^{N_t} Y_n. \tag{151}$$

The following result provides a closed-form solution to this family of problems and allows to recover the conclusion of Proposition 1 by setting $\Phi = \ell(0)$.

Proposition B.1. *The value function and the optimal stopping time for problem Eq.(149) are explicitly given by*

$$p(\Phi) = \max\{\Phi, v_0^*\} \tag{152}$$

and

$$\tau^*(\Phi) = 1_{\{\Phi < v_0^*\}} \inf\{t \geq 0 : v_0^* + \Delta C_t \leq 0\} \tag{153}$$

where the strictly positive constant v_0^* is the unique solution to

$$\rho v_0^* = \bar{\mu} - \lambda \mathbb{E}\left[\min\{v_0^*, Y_1\}\right]. \tag{154}$$

In particular, the value function of the frictionless problem is given by $\max\{\ell(0), v_0^*\}$ and the optimal strategy is to liquidate immediately if $\ell(0) > v_0^*$ and otherwise wait until the first time that the absolute value of a jump of the cash flow process exceeds v_0^*.

Before proving the above proposition, we start by establishing a verification result for the HJB equation associated with Eq.(149).

Lemma B.1. *Assume that $q \in \mathbb{R}$ satisfies*

$$\max\left\{\Phi - q; -\rho q + \bar{\mu} + \lambda \mathbb{E}\left[\max(0, q - Y_1, \Phi - Y_1) - q\right]\right\} = 0 \tag{155}$$

Then we have $q \geq p(\Phi)$.

Proof. Assume that q satisfies the assumption of the statement, fix an arbitrary stopping time $\zeta \in \mathcal{S}$ and consider the process

$$w_t = e^{-\rho t} q 1_{\{\zeta > t\}} + e^{-\rho\zeta} \max\{\Phi; -\Delta C_\zeta\} 1_{\{\zeta \le t\}} + \int_0^{t\wedge\zeta} e^{-\rho s} dC_s. \tag{156}$$

Since q satisfies Eq.(155) we have that $q \ge \Phi$ and

$$\rho q \ge \bar{\mu} + \lambda \mathbb{E}\left[\max\{0, q - Y_1, \Phi - Y_1\} - q\right] = \bar{\mu} - \lambda \mathbb{E}\left[\min\{v_0^*, Y_1\}\right]. \tag{157}$$

Combining these properties with an application of Itô's lemma shows that

$$\begin{aligned}
dw_t &= 1_{\{\zeta > t\}} e^{-\rho t} \left(\sigma dB_t + \Delta C_t + (\bar{\mu} - \rho q) dt\right) + 1_{\{\zeta = t\}} e^{-\rho t} \left((\Phi + \Delta C_t)^+ - q\right) && (158)\\
&\le 1_{\{\zeta > t\}} e^{-\rho t} \left(\sigma dB_t + \Delta C_t + (\bar{\mu} - \rho q) dt\right) && (159)\\
&\le 1_{\{\zeta > t\}} e^{-\rho t} \left(\sigma dB_t - Y_{N_t} dN_t + \lambda \mathbb{E}\left[q - \max\{0, q - Y_1, \Phi - Y_1\}\right] dt\right) && (160)\\
&= 1_{\{\zeta > t\}} e^{-\rho t} \left(\sigma dB_t - Y_{N_t} dN_t + \lambda \mathbb{E}\left[Y_1\right] dt + \lambda \mathbb{E}\left[\min\{0, q - Y_1\}\right] dt\right) && (161)\\
&\le 1_{\{\zeta > t\}} e^{-\rho t} \left(\sigma dB_t - Y_{N_t} dN_t + \lambda \mathbb{E}\left[Y_1\right] dt\right) && (162)
\end{aligned}$$

Integrating on both sides and using the fact that $q \ge \Phi$ then gives

$$w_t - q \le \int_0^t 1_{\{\zeta > s\}} e^{-\rho s} \left(\sigma dB_s - Y_{N_s} dN_s + \lambda \mathbb{E}\left[Y_1\right] ds\right) \equiv m_t. \tag{163}$$

By construction, we have that the process m_t is a local martingale and since its quadratic variation satisfies

$$\mathbb{E}[m]_\zeta \le \mathbb{E}\left[\int_0^\infty e^{-2\rho s} \left(\sigma^2 ds + Y_{N_s}^2 dN_s\right)\right] = \frac{\sigma^2 + \lambda \mathbb{E}\left[Y_1^2\right]}{2\rho} < \infty. \tag{164}$$

it follows from the Burkholder-Davis-Gundy inequality that it is a true martingale on the interval $[\![0, \zeta]\!]$. Combining this property with Eq.(163) then shows that

$$q \ge \mathbb{E}\left[w_\zeta - m_\zeta\right] = \mathbb{E}\left[w_\zeta\right] = \mathbb{E}\left[e^{-\rho\zeta} \max\{\Phi; -\Delta C_\zeta\} + \int_0^\zeta e^{-\rho s} dC_s\right] \tag{165}$$

and the desired result now follows from the arbitrariness of $\zeta \in \mathcal{S}$ by taking the supremum on both sides of this inequality. ■

Proof of Proposition B.1. Consider the function defined by

$$z(v) = \bar{\mu} + \lambda \mathbb{E}\left[(v - Y_1)^+\right] - (\rho + \lambda)v \tag{166}$$

and observe that Eq.(154) is equivalent to $z(v_0^*) = 0$. The function $z(v)$ is continuous, non increasing, starts out from $\bar{\mu} > 0$ at the origin and satisfies

$$\lim_{v \to \infty} z(v) = -\infty. \tag{167}$$

Therefore it crosses the horizontal axis at a unique point $v_0^* > 0$ and it follows that Eq.(154) admits a unique strictly positive solution. A direct calculation then shows that $q^* = \max(\Phi, v_0^*)$ satisfies Eq.(155) and it thus follows from Lemma B.1 that $q^* \geq p(\Phi)$. To establish the reverse inequality denote by

$$\bar{q} = \mathbb{E}\left[e^{-\rho\tau^*(\Phi)} \max\{\Phi; -\Delta C_{\tau^*(\Phi)}\} + \int_0^{\tau^*(\Phi)} e^{-\rho s} dC_s\right] \tag{168}$$

the value associated with the stopping time $\tau^*(\Phi)$. By definition we have $\bar{q} \leq p(\Phi)$ and we claim that $\bar{q} = q^*$. If $\Phi > v_0^*$ then the claim immediately follows from the definition of the stopping time. On the contrary, if $\Phi \leq v_0^*$ then it follows from the law of iterated expectations and the definition of the cash flow process that

$$c(\bar{q}) \equiv -\rho\bar{q} + \bar{\mu} + \lambda \mathbb{E}\left[(\bar{q} - Y_1)1_{\{Y_1 \leq v_0^*\}} - \bar{q}\right] = 0. \tag{169}$$

As is easily seen the continuous function $c(q)$ is decreasing in q and therefore crosses the horizontal axis at most once. Since

$$c(v^*) = -\rho v^* + \bar{\mu} + \lambda \mathbb{E}\left[(v_0^* - Y_1)1_{\{Y_1 \leq v_0^*\}} - v_0^*\right] \tag{170}$$

$$= -\rho v_0^* + \bar{\mu} - \lambda \mathbb{E}\left[\min\{v_0^*, Y_1\}\right] = 0 \tag{171}$$

we have that this crossing point is uniquely given by $\bar{q} = v_0^*$. ■

B.2. Value of an unregulated bank

Consider the functions defined by

$$g(\alpha, k; b) = \max\{k, w(b, \alpha; b) - b - \phi\}, \tag{172}$$

and let

$$g(\alpha, k) = \max\{k, h(\alpha)\} = \max\{k, w(b^*(\alpha), \alpha; b^*(\alpha)) - b^*(\alpha) - \phi\}. \tag{173}$$

The following lemmas will be used in the construction of the optimal strategy for the bank subject to refinancing costs.

Lemma B.2. *The function $\alpha \mapsto g(\alpha, k; b)$ admits a unique fixed point for any given b, $k > 0$.*

PROOF. The proof is similar to that of Lemma B.3 below and therefore is omitted. ■

Lemma B.3. *The function $\alpha \mapsto g(\alpha, k)$ admits a unique fixed point $a_k \geq 0$ for any given $k > 0$ and this fixed point is such that $a \equiv a_0 < v_0^*$ and $a_k = \max\{a, k\}$.*

PROOF. Fix a barrier level $b > 0$ and two liquidation values $0 \leq \alpha_2 < \alpha_1$. By definition, we have that the corresponding values satisfy

$$w(s; \alpha_2; b) \leq w(s; \alpha_1; b) = \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b + e^{-\rho\tau_{\pi_b}}(\alpha_1 + S^{\pi_b}_{\tau_{\pi_b}})^+\right] \tag{174}$$

$$\leq (\alpha_1 - \alpha_2)\mathbb{E}_s\left[e^{-\rho\tau_{\pi_b}}\right] + \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b + e^{-\rho\tau_{\pi_b}}(\alpha_2 + S^{\pi_b}_{\tau_{\pi_b}})^+\right] \tag{175}$$

$$= (\alpha_1 - \alpha_2)\mathbb{E}_s\left[e^{-\rho\tau_{\pi_b}}\right] + w(s; \alpha_2; b) \tag{176}$$

and it follows that

$$\frac{\partial w(s; \alpha; b)}{\partial \alpha} \in [0, 1), \qquad (s, \alpha, b) \in (0, \infty) \times [0, v_0^*) \times (0, \infty). \tag{177}$$

On the other hand, using Eq.(27) together with Lemma A.3 we get

$$\left.\frac{\partial w(s; \alpha; b)}{\partial b}\right|_{s=b^*(\alpha)} = 1 - \left.\frac{\partial w(s; \alpha; b)}{\partial s}\right|_{s=b=b^*(\alpha)} = 0 \tag{178}$$

and combining this with equation Eq.(177) we conclude that the function $h(\alpha)$ is non decreasing with $h'(\alpha) < 1$ for all $\alpha \in [0, v_0^*)$. If we have $h(0) \leq 0$ then this implies that $g(\alpha, 0)$ admits a unique fixed point that is located at the origin. On the contrary, if we have $h(0) > 0$ then we know from Lemmas A.3 and A.4 that the non increasing function $g(\alpha, 0) - \alpha$ is strictly positive at the origin, strictly decreasing on $[0, v_0^*)$ with

$$g(\alpha, 0) - \alpha = w(0, \alpha; 0) - \phi - \alpha = -\phi < 0, \qquad \alpha \geq v_0^*, \tag{179}$$

and it immediately follows that in this case the function $g(\alpha, 0)$ admits a unique fixed point that lies in the interval $(0, v_0^*)$.

By the first part of the proof we have that for $k > 0$ the function $g(\alpha, k) - \alpha$ is strictly decreasing on $[0, \max(k + \phi, v_0^*))$ and satisfies

$$g(\alpha, k) - \alpha = \max\{k - \alpha, -\phi\} = -\phi < 0, \qquad \alpha \geq \max(k + \phi, v_0^*). \tag{180}$$

This implies that the fixed point of the function $g(\alpha, k)$ must be unique if it exists and it now only remains to show that $\max\{a, k\}$ is such a fixed point. If $k > a$ then the result follows by observing that since the function $g(\alpha, 0)$ is non increasing we have

$$k > a = g(a, 0) \geq g(k, 0) = h(k) \tag{181}$$

by definition of a. Similarly, if $a > 0$ and $k \leq a$ then we have $k \leq a = g(a, 0) = h(a)$ and the desired fixed point property follows. ■

Proof of Proposition 3. Consider the strategy $\hat{\pi}_b$ that consists in distributing dividends to maintain liquid reserves at or below $b > 0$ and in either liquidating or raising funds back to b depending on which option is more profitable whenever liquid reserves become negative. Denote by

$$v_b(s) = \mathbb{E}_s \left[\int_0^{\tau_{\hat{\pi}_b}} e^{-\rho t} (dP_t^{\hat{\pi}_b} - d\Phi_t(R^{\hat{\pi}_b})) \right] \tag{182}$$

the value of the bank under this strategy. By definition we have that

$$v_b(b) = \left(\max\{\ell(0), v_b(b) - b - \phi\} + s \right)^+, \qquad s \leq 0 \tag{183}$$

and

$$R_t^{\hat{\pi}_b} = P_t^{\hat{\pi}_b} - P_t^b = 0, \qquad 0 \leq t \leq \tau_{\pi_b} \tag{184}$$

where the stopping time τ_{π_b} denotes the stochastic liquidation time associated with the barrier strategy for dividends at level b. Combining these properties with Eq.(182) and the law of iterated expectations gives

$$v_b(s) = \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b + e^{-\rho \tau_{\pi_b}} v(S_{\tau_{\pi_b}}^{\pi_b}; b)\right] \tag{185}$$

$$= \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b + e^{-\rho \tau_{\pi_b}} \left(\max\{\ell(0), v_b(b) - b - \phi\} + S_{\tau_{\pi_b}}^{\pi_b}\right)^+\right] \tag{186}$$

$$= \mathbb{E}_s\left[\int_0^{\tau_{\pi_b}} e^{-\rho t} dP_t^b + e^{-\rho \tau_0} \left(\hat{\alpha}(b) + S_{\tau_{\pi_b}}^{\pi_b}\right)^+\right] = w(s; \hat{\alpha}(b); b) \tag{187}$$

where $\hat{\alpha}(b)$ denotes the unique fixed point of the function $g(\alpha, \ell(0); b)$ provided by Lemma B.2. Now let $\alpha_0^* = \max\{a, \ell(0)\}$ denotes the unique fixed point of the function $g(\alpha, \ell(0))$ provided by Lemma B.3 and consider the barrier the barrier $b_0^* = b^*(\alpha_0^*)$. By uniqueness we have that $\alpha_0^* = \hat{\alpha}(b_0^*)$ and combining this with Eq.(187) shows that the value of the associated strategy satisfies

$$v_{b_0^*}(s) = w(s; \alpha_0^*; b_0^*) = w(s; \alpha_0^*), \qquad s \in \mathbb{R}. \tag{188}$$

As shown in the proof of Lemma A.10, we have that

$$\max\{1 - w'(s; \alpha_0^*); \mathcal{D}w(s; \alpha_0^*)\} \leq 0, \qquad s > 0, \tag{189}$$

and the result will follow from Lemma B.4 once we show that $w(s; \alpha_0^*)$ satisfies Eq.(193). By Lemma A.9, we have that this function is concave and twice continuously differentiable on the strictly positive real line with $w'(s; \alpha_0^*) = 1$ for $s \geq b_0^*$. Therefore, it follows from the definition of α_0^* that we have

$$\max\{\ell(0), fw(0, \alpha_0^*)\} = \max\{\ell(0), w(b_0^*; \alpha_0^*) - b_0^* - \phi\} = \alpha_0^* \tag{190}$$

and combining this with Eq.(24) gives

$$w(s;\alpha_0^*) = (\alpha_0^* + s)^+ = \left(\max\{\ell(0), fw(0,\alpha_0^*)\} + s\right)^+, \qquad s \le 0, \tag{191}$$

which is the required boundary condition. ■

Corollary B.1. *Let*

$$\phi^* = \max\{0, w(b^*(\ell(0)), \ell(0), b^*(\ell(0))) - b^*(\ell(0)) - \ell(0)\} \tag{192}$$

Then the bank raises funds if and only if $\phi < \phi^*$.

Proof. By Proposition 3 we have that the bank raises funds if and only if $\alpha_0^* > \ell(0)$ and a direct calculation shows that this is equivalent to $\phi > \phi^*$. ■

Lemma B.4. *Assume that* $u : \mathbb{R} \to \mathbb{R}$ *is a continuous function that is twice continuously differentiable on* $(0,\infty)$ *and such that*

$$u(s) - \left(\max\{\ell(0), fu(0)\} + s\right)^+ = 0, \qquad s \le 0, \tag{193}$$

$$\max\{1 - u'(s), \mathcal{D}u(s)\} \le 0, \qquad s > 0. \tag{194}$$

Then we have $u(s) \ge v(s)$ *for all* $s \in \mathbb{R}$.

Proof. Assume that the function $u(s)$ satisfies the conditions of the statement, fix an admissible strategy $\pi \in \Pi(s)$ and consider the process

$$Y_t = e^{-\rho t\wedge\tau_\pi} u(S^\pi_{t\wedge\tau_\pi}) + \int_0^{t\wedge\tau_\pi} e^{-\rho s}(dP^\pi_s - d\Phi_s(R^\pi)). \tag{195}$$

Applying Itô's formula for semimartingales (see for example Dellacherie and Meyer (1980, Theorem VIII.25)) shows that

$$Y_t - M_{t\wedge\tau_\pi} = u(S^\pi_0) + \sum_{0\le s<t\wedge\tau_\pi} e^{-\rho s}\left(\Delta^+ u(S^\pi_s) + \Delta^+(P^\pi_s - \Phi_s(R^\pi))\right) \tag{196}$$

$$+ \int_0^{t\wedge\tau_\pi} e^{-\rho s}\mathcal{D}u(S^\pi_{s-})ds + \int_0^{t\wedge\tau_\pi} e^{-\rho s}(1 - u'(S_{s-}))dP^{\pi,c}_s \tag{197}$$

for some local martingale M_t. Using the fact that $u(s)$ satisfies Eqs.(193) and (194), together with the definition of the liquidation time, we deduce that

$$\begin{aligned}
&1_{\{s<\tau_\pi\}}\left(\Delta^+ u(S^\pi_s) + \Delta^+(P^\pi_s - \Phi_s(R^\pi)) + \mathcal{D}u(S^\pi_{s-})ds + (1-u'(S_{s-}))dP^{\pi,c}_s\right) && (198)\\
&\le 1_{\{s<\tau_\pi\}}\left(\Delta^+ u(S^\pi_s) + \Delta^+(P^\pi_s - \Phi_s(R^\pi))\right) && (199)\\
&= 1_{\{s<\tau_\pi\}}\left(u(S^\pi_s - \Delta^+(P^\pi_s - R^\pi_s)) + \Delta^+(P^\pi_s - \Phi_s(R^\pi)) - u(S^\pi_s)\right) && (200)\\
&\le 1_{\{s<\tau_\pi\}}\left(u(S^\pi_s + \Delta^+ R^\pi_s) - \Delta^+\Phi_s(R^\pi) - u(S^\pi_s)\right) && (201)\\
&= 1_{\{s<\tau_\pi\}\cap\{\Delta^+R^\pi_s>0\}}\left(u(S^\pi_s + \Delta^+ R^\pi_s) - \Delta^+ R^\pi_s - \phi - u(S^\pi_s)\right). && (202)
\end{aligned}$$

On the other hand, Eqs.(193) and (194) jointly imply that

$$\begin{aligned}
u(s) &= 1_{\{s\le 0\}}u(s) + 1_{\{s>0\}}\left(u(0) + \int_0^s u'(x)dx\right) && (203)\\
&\ge 1_{\{s\le 0\}}u(s) + 1_{\{s>0\}}\left(u(0) + s\right) && (204)\\
&= (\max\{\ell(0), fu(0)\} + s)^+ \ge (fu(0)+s)^+ && (205)
\end{aligned}$$

and combining this inequality with Eq.(202) shows that that the process on the right hand side of equation Eq.(196) is decreasing. This in turn implies that the process Y_t is a local supermartingale and it follows that there exists a non decreasing sequence of stopping times $(\theta_n)_{n=1}^\infty$ such that $\lim_n \theta_n = \infty$ and

$$\begin{aligned}
u(s) \ge \mathbb{E}_s\left[Y_{\theta_n}\right] &= \mathbb{E}_s\left[\int_0^{\theta_n\wedge\tau_\pi} e^{-\rho t}(dP^\pi_t - d\Phi_t(R^\pi)) + e^{-\rho\theta_n\wedge\tau_\pi}u(S^\pi_{\theta_n\wedge\tau_\pi})\right] && (206)\\
&\ge \mathbb{E}_s\left[\int_0^{\theta_n\wedge\tau_\pi} e^{-\rho t}(dP^\pi_t - d\Phi_t(R^\pi))\right] && (207)
\end{aligned}$$

where the last inequality follows from the fact that the function $u(s)$ is nonnegative as a result of Eqs.(193) and (194). Letting $n\to\infty$ and using Eq.(69) in conjunction with the dominated convergence theorem then gives

$$u(s) \ge \mathbb{E}_s\left[\int_0^{\tau_\pi} e^{-\rho t}(dP^\pi_t - d\Phi_t(R^\pi))\right] \tag{208}$$

and the desired conclusion now follows from the arbitrariness of the strategy by taking the supremum over $\pi\in\Pi(s)$ on both sides. ■

Corollary B.2. *The target level of liquid reserves b_0^* increases with refinancing costs and decreases with the liquidation value of assets. The value α_0^* os equity in a bank without liquid reserves decreases with refinancing costs ϕ and increases with the liquidation value of assets.*

Proof of Corollary B.2. By Lemma A.4 we have that the target level $b^*(\alpha)$ is decreasing in α and, since

$$\alpha_0^* = \max\{a, \ell(0)\} = \max\{a, (\Lambda - L - D)^+\} \tag{209}$$

it suffices to establish that the fixed point of the function

$$g(\alpha; 0; \phi) = \max\{0; h(\alpha; \phi)\} = \max\left\{0; w(b^*(\alpha); \alpha; b^*(\alpha)) - b^*(\alpha) - \phi\right\} \tag{210}$$

is non increasing in ϕ. Assume that this is not the case and fix an $\epsilon > 0$. By definition we have that the function $h(\alpha; \phi)$ is decreasing in ϕ and we know from the proof of Lemma B.3 that it is also decreasing in α. Combining these properties with the assumed increase of the fixed point gives

$$a(\phi + \epsilon) = \max\left\{h(a(\phi + \epsilon); \phi + \epsilon\right\} \tag{211}$$

$$\leq \max\left\{h(a(\phi + \epsilon); \phi\right\} \leq \max\left\{h(a(\phi); \phi\right\} = a(\phi) \tag{212}$$

and provides the required contradiction. ■

B.3. Value of a bank subject to a liquidity requirement

Let us now turn to the case of a bank that is subject to a minimal cash holding requirement at some level $T \geq 0$. The equity value of such a bank is given by

$$v(s; T) = \sup_{\pi \in \Pi(s,T)} \mathbb{E}_s \left[\int_0^{\tau_{\pi,T}} e^{-\rho t}(dP_t^\pi - d\Phi_t(R^\pi)) + e^{-\rho \tau_{\pi,T}} \ell\left(S^\pi_{\tau_{\pi,T}}\right) \right] \tag{213}$$

where the stopping time

$$\tau_{\pi,T} = \inf\{t \geq 0 : S^\pi_{t+} \leq T\} \tag{214}$$

denotes the liquidation time associated with the use of a strategy π in the presence of a cash holding requirement at level T, and $\Pi(s,T)$ denotes the set of payout and financing strategies such that Eq.(69) holds and

$$\Delta^+ P_t^\pi \le S_t^\pi - T + \Delta^+ R_t^\pi, \qquad t \ge 0. \tag{215}$$

As a first step towards the proof of Proposition 4 the following lemma establishes a verification result for this optimization problem.

Lemma B.5. *Assume that $u : \mathbb{R} \to \mathbb{R}$ is a continuous function that is twice continuously differentiable on (T,∞) and such that*

$$u(s) - \sup_{b \ge T} (\ell(s) \vee (u(b) - b + s - \phi)) = 0, \qquad s \le T, \tag{216}$$

$$\max\{1 - u'(s), \mathcal{D}u(s)\} \le 0, \qquad s > T. \tag{217}$$

Then $u(s) \ge v(s;T)$ for all $s \in \mathbb{R}$.

Proof. The proof is similar to that of Lemma B.4 and therefore is omitted. ■

Proposition B.2. *Assume that $\ell(T) \le a$. Then the equity value of a regulated bank is*

$$v(s;T) = v(s-T;0), \qquad s \in \mathbb{R}, \tag{218}$$

and the optimal strategy consists in paying dividends to maintain reserves below $T + b_0^*$, raising funds to move to $T + b_0^*$ whenever liquid reserves fall below T with a shortfall less than a, and liquidating otherwise.

Proof. Assume that $\ell(T) \le a = \alpha_0^*$ where the equality follows from the fact that the function $\ell(s)$ is non decreasing and denote by

$$u(s) = v(s-T;0) = w(s-T;a) \tag{219}$$

the candidate value function. From the proof of Lemma A.9, we know that this function satisfies Eq.(217). On the other hand, the assumption of the statement and the fact that the unconstrained value function $v(s;0)$ satisfies Eq.(193) imply that for all $s \leq T$ we have

$$u(s) = v(s-T;0) = (a+s-T)^+ \geq (\ell(T)+s-T)^+ = \ell(s), \tag{220}$$

as well as

$$\begin{aligned} u(s) = v(s-T;0) &= (\max\{\ell(0), fv(0;0)\} + s - T)^+ = (fv(0;0)+s-T)^+ && (221)\\ &= \left(\sup_{b\geq 0}\left(v(b;0) - b + s - T - \phi\right)\right)^+ && (222)\\ &= \left(\sup_{b\geq 0}\left(u(b+T) - b + s - T - \phi\right)\right)^+ && (223)\\ &= \left(\sup_{q\geq T}\left(u(q) - q + s - \phi\right)\right)^+ && (224) \end{aligned}$$

Combining these two relations shows that the candidate value function satisfies Eq.(216) and it now follows from Lemma B.5 that we have $u(s) \geq v(s,T)$ for all $s \in \mathbb{R}$.

To establish the reverse inequality consider the strategy that pays dividends to maintain liquid reserves at or below $T+b$, raises outside funds to move to $T+b$ whenever they become smaller than T with a shortfall less than α, and liquidating otherwise. Let

$$Y_t(s,T,\alpha,b) = [S_t, P_t, R_t]\,(s,T,\alpha,b) \in \mathbb{R}^3 \tag{225}$$

with

$$S_t(s,T,\alpha,b) = s + C_t - P_t(s,T,\alpha,b) + R_t(s,T,\alpha,b) \tag{226}$$

and

$$R_0(s,T,\alpha,b) = P_0(s,T,\alpha,b) = 0 \tag{227}$$

denote the liquid reserves, cumulative dividend and cumulative financing processes associated with this strategy under the assumption that the bank starts out holding $s \in \mathbb{R}$ in cash reserves. In addition, let

$$\tau(s,T,\alpha,b) = \inf\{t \geq 0 : S_{t+}(s,T,\alpha,b) \leq T\} = \inf\{t \geq 0 : S_t(s,T,\alpha,b) \leq T - \alpha\} \tag{228}$$

give the corresponding liquidation time, and denote by

$$\vartheta_t(s,T,\alpha,b) = \sup\{0 \le u \le t : T-\alpha < S_u(s,T,\alpha,b) \le T\} \qquad (\sup\emptyset = 0) \tag{229}$$

the last time before $t \ge 0$ that the bank raises funds. With this notation, the cumulative financing and dividend processes associated with the strategy satisfy

$$R_t(s,T,\alpha,b) = \sum_{0\le u<t} 1_{\{T-a<S_u(s,T,\alpha,b)\le T\}} \left(T+b-S_u(s,T,\alpha,b)\right) \tag{230}$$

$$P_t(s,T,\alpha,b) = P_{\vartheta_t(s,T,\alpha,b)}(s,T,\alpha,b) \tag{231}$$

$$+ \max_{\vartheta_t(s,T,\alpha,b)\le u<t} \left(S_{\vartheta_t(s,T,\alpha,b)+}(s,T,\alpha,b) + C_u - C_{\vartheta_t(s,T,\alpha,b)} - (T+b)\right)^+ \tag{232}$$

for all $t > 0$, and a straightforward calculation using the dynamics of the liquid reserves process then shows that we have the almost sure identities

$$Y_t(s,T,\alpha,b) = (T,0,0) + Y_t(s-T,0,\alpha,b) \tag{233}$$

and

$$\tau(s,T,\alpha,b) = \tau(s-T,0,\alpha,b). \tag{234}$$

between the state variables, controls and liquidation times with and without a liquidity requirement. As a result, the equity value function

$$v(s,T,\alpha,b) = \mathbb{E}\Bigg[\int_0^{\tau(s,T,\alpha,b)} e^{-\rho u}\big(dP_u(s,T,\alpha,b) - d\Phi_u(R(s,T,\alpha,b))\big) \tag{235}$$

$$+ e^{-\rho\tau(s,T,\alpha,b)}\ell\left(S_{\tau(s,T,\alpha,b)}(s,T,\alpha,b)\right)\Bigg] \tag{236}$$

generated by the use of the strategy associated with the triple $(T, \alpha, b) \in \mathbb{R}^3_+$ starting from the initial cash holding $s \in \mathbb{R}$ satisfies

$$v(s, T, \alpha, b) = v(s - T, 0, \alpha, b), \qquad (s, T, \alpha, b) \in \mathbb{R} \times \mathbb{R}^3_+. \tag{237}$$

The candidate for the constrained optimal strategy corresponds to $(T, a, b^*(a))$ and, since $\ell(0) \leq \ell(T) \leq a$ by assumption, we know from Proposition 3 that the unconstrained optimal strategy corresponds to $(0, a, b^*(a))$. Therefore it follows from Eq.(237) that

$$v(s, T, a, b^*(a)) = v(s - T, 0, a, b^*(a)) = v(s - T; 0) = u(s) \tag{238}$$

and the proof is complete. ■

Proposition B.3. *Assume that $\ell(T) > a$. Then the equity value of a regulated bank is*

$$v(s; T) = w(s - T; \ell(T)) \tag{239}$$

and the optimal strategy consists in paying dividends to maintain liquid reserves at or below the level $T + b^*(\ell(T))$, and liquidating the first time that they fall below T.

Proof. Assume that $\ell(T) > a$, denote by

$$u(s) = w(s - T; \ell(T)) = w\left(s - T; \ell(T); b^*(\ell(T))\right) \tag{240}$$

the candidate value function and recall that $b^*(\alpha) = 0$ for all $\alpha \geq v_0^*$. By Lemma A.9 we have that this function satisfies Eq.(217) as well as

$$u(s) = (w(0; \ell(T)) + s - T)^+ = (\ell(T) + s - T)^+ = \ell(s), \qquad s \leq T. \tag{241}$$

On the other hand, the same lemma shows that the function $w(s; \ell(T))$ is concave on the positive real line with $w'(s, \ell(T)) = 1$ for $s \geq b^*(\ell(T))$. In particular,

$$h(\ell(T)) = \max_{b \geq 0} \left(w(b, \ell(T)) - b - \phi\right) \tag{242}$$

and using this property in conjunction with the definition of the candidate value function and the fact that, since $\ell(T) > a$, the unique fixed point of $g(\alpha, \ell(T))$ is located at $\ell(T)$ we deduce that for any cash holding $s \leq T$ we have

$$\max_{q \geq T} (u(q) - q + s - \phi) = h(\ell(T)) + s - T \leq g(\ell(T), \ell(T)) + s - T \tag{243}$$

$$= \ell(T) + s - T \leq \ell(s). \tag{244}$$

Combining this inequality with Eq.(241) shows that Eq.(216) is satisfied and it now follows from Lemma B.5 that $u(s) \geq v(s, T)$ for all $s \in \mathbb{R}$.

To establish the reverse inequality consider the candidate optimal strategy described in the statement. This strategy is associated with the triple $(T, 0, b^*(\ell(T)))$ and does not involve refinancing. Therefore, it follows from Eq.(237), and the definition of the function $\ell(s)$ that its value satisfies

$$v\left(s, T, 0, b^*(\ell(T))\right) = v\left(s - T, 0, 0, b^*(\ell(T))\right) \tag{245}$$

$$= w\left(s - T, \ell(T), b^*(\ell(T))\right) = u(s) \tag{246}$$

and the proof is complete. ■

Denote by

$$G(k, a, b, h) = \mathbb{P}\left[\tau(k, 0, a, b) \leq h\right] \tag{247}$$

the probability that a bank an unregulated bank that follows a barrier strategy $(0, a, b)$ defaults prior to some fixed horizon $h \geq 0$ given that it starts out with $k \in \mathbb{R}$ in liquid reserves, and by

$$H(k, a, b, h) = \mathbb{P}\left[S_{\tau(k,0,a,b)}(k, 0, a, b) + h > 0\right] \tag{248}$$

the probability that the liquid reserves of this bank exceeds some level $-h \leq 0$ at the time of default. Our next result establishes some monotonicity properties of these two probabilities and will serve as a basis for the proof of Proposition 5.

Lemma B.6. *The functions $G(k, a, b, h)$ and $F(k, a, b, h)$ defined by Eqs.(247) and (248) are non decreasing in h and satisfy*

$$F(k, a, b, h) \leq F(k, 0, b, h) \leq F(k, 0, b', h), \qquad F \in \{G, H\} \tag{249}$$

for all $(k,h,a) \in \mathbb{R}\times\mathbb{R}^2_+$ *and* $0 \le b' \le b$.

Proof. The first part of the statement is immediate from the definition so let us focus on the second part and start with the case of the function $G(k,a,b,h)$. By definition we have that $\tau(k,0,a,b) = \tau(k,0,0,b)$ on the set

$$\{S_{\tau(k,0,0,b)}(k,0,0,b) + a \le 0\} \tag{250}$$

and $\tau(k,0,a,b) > \tau(k,0,0,b)$ otherwise. Therefore, $\tau(k,0,a,b) \ge \tau(k,0,0,b)$ and the first inequality in Eq.(249) follows. To establish the second inequality denote by X^0_t the uncontrolled liquid reserves of the bank starting from zero and observe that

$$S_t(k,0,0,b) = k + X^0_t - \sup_{0<u\le t}\left(k + X^0_u - b\right)^+. \tag{251}$$

Since the right hand side of this equality is non decreasing in b for any given $k \in \mathbb{R}$, we have that the map

$$b \longmapsto \tau(k,0,0,b) = \inf\{t \ge 0 : S_{t+}(k,0,0,b) \le 0\} \tag{252}$$

is non decreasing for any given $k \in \mathbb{R}$ and the desired result follows. Let us now turn to the function $F(k,a,b,h)$, consider the first inequality and assume that $h > a$ for otherwise the result is trivial since we have $H(k,a,b,h) = 1$ for $h \le a$. Using the fact that

$$\{\tau(k,0,a,b) = \tau(k,0,0,b)\} = \{S_{\tau(k,0,a,b)}(k,0,0,b) + a \le 0\} \tag{253}$$

in conjunction with the assumption that $h > a$ gives

$$1 - H(k,a,b,h) = \mathbb{P}\left[\{S_{\tau(k,0,a,b)}(k,0,a,b) + h \le 0\}\right] \tag{254}$$

$$\ge \mathbb{P}\left[\{S_{\tau(k,0,a,b)}(k,0,a,b) + h \le 0\} \cap \{\tau(k,0,a,b) = \tau(k,0,0,b)\}\right] \tag{255}$$

$$= \mathbb{P}\left[\{S_{\tau(k,0,0,b)}(k,0,a,b) + h \le 0\} \cap \{S_{\tau(k,0,a,b)}(k,0,0,b) + a \le 0\}\right] \tag{256}$$

$$= \mathbb{P}\left[S_{\tau(k,0,0,b)}(k,0,a,b) + h \le 0\right] = 1 - H(k,0,b,h) \tag{257}$$

which is the first inequality in Eq.(249). Assume from now on that $h \geq 0$. Using the result of Lemma B.8 below we have that

$$1 - H(k,0,b,h) = \mathbb{P}[S_{\tau(k,0,0,b)}(k,0,0,b) + h \leq 0] = 1_{\{h=0\}} \tag{258}$$

$$+ 1_{\{h>0\}}\, \lambda \int_0^b e^{-\beta(z+h)} \left[W_0'(b-z) \frac{W_0(x)}{W_0'(b)} - W_0(x-z) \right] dz \bigg|_{x=k\wedge b} \tag{259}$$

where the undiscounted scale function is defined by

$$W_0(x) = 1_{\{x\geq 0\}} \left[\frac{2\beta}{\sigma^2 A_1 A_2} + \frac{2(\beta + A_1)e^{A_1 x}}{\sigma^2 A_1 (A_1 - A_2)} + \frac{2(\beta + A_2)e^{A_2 x}}{\sigma^2 A_2 (A_2 - A_1)} \right] \tag{260}$$

for some constants $\beta + A_1 < 0 < \beta + A_2$. Denote by $I(x,b,h)$ the integral on the right hand side of Eq.(258) and observe that the desired result will follow once we show that this function is non decreasing in both b and $x \leq b$. Differentiating with respect to b and using the fact that the scale function is nonnegative and increasing we obtain that

$$\text{sign}\left\{ \frac{\partial I(x,b,h)}{\partial b} \right\} = \text{sign}\left\{ W_0'(0) + \int_0^b e^{-\beta(z-b)} \left[W_0''(b-z) - \frac{W_0''(b)}{W_0'(b)} W_0'(b-z) \right] dz \right\} \tag{261}$$

$$= \text{sign}\left\{ \frac{2(A_1 - A_2)e^{\beta h + b(A_1 + A_2)}}{\sigma^2((\beta + A_1)e^{bA_1} - e^{bA_2}(\beta + A_2))} \right\} \tag{262}$$

where the second equality follows from Eq.(260) and the computation of the integral on the first line. Since $\beta + A_1 < 0 < \beta + A_2$ the fraction is positive and the required increase in b follows. On the other hand, differentiating with respect to x and using Eq.(260) together with the fact that the scale function is strictly shows that

$$\text{sign}\left\{ \frac{\partial I(x,b,h)}{\partial x} \right\} = \text{sign}\left\{ \int_0^b e^{-\beta(z+h)} \frac{W_0'(b-z)}{W_0'(b)} dz - \int_0^x e^{-\beta(z+h)} \frac{W_0'(x-z)}{W_0'(x)} dz \right\}. \tag{263}$$

Therefore it suffices to show that the integral

$$I_2(b,h) = \int_0^b e^{-\beta(z+h)} \frac{W_0'(b-z)}{W_0'(b)} dz \tag{264}$$

is increasing in b but, since

$$W_0'(b)\frac{\partial I_2(b,h)}{\partial b} = W_0'(0) + \int_0^b e^{-\beta(z-b)}\left[W_0''(b-z) - \frac{W_0''(b)}{W_0'(b)}W_0'(b-z)\right]dz \tag{265}$$

this property follows directly from the increase of the scale function, Eq.(262) and the definition of the constants A_1 and A_2. ■

PROOF OF PROPOSITION 5. Define a non increasing function by setting $a^*(T) = 1_{\{\ell(T)<a\}}a$ and let us start by observing that as a result of Eq.(234) we have

$$\mathbb{P}_{T+k}[\tau_T^* \leq h] = G\left(k, a^*(T), b^*(a \wedge \ell(T)), h\right) \tag{266}$$

$$\mathbb{P}_{T+k}\left[\mathcal{L}_T^* \geq h\right] = 1 - H\left(k, a^*(T), b^*(a \wedge \ell(T)), \ell(h+T)\right) \tag{267}$$

where

$$\mathcal{L}_T^* = \left(L + D - \Lambda - S_{\tau_T^*}\right)^+ \tag{268}$$

is the loss that the bank imposes on its creditors by defaulting. Now fix a liquidity requirement level $T \geq 0$ and let $T' \geq T$. In order to establish the required monotonicity we distinguish two cases.

Case 1: $a^*(T) = a^*(T') = a$**.** In this case we have

$$b^*(a \wedge \ell(T)) = b^*(a \wedge \ell(T')) = b^*(a) \tag{269}$$

and it thus follows from Eqs.(266), (267), the increase of the function $\ell(\cdot)$ and the first part of Lemma B.6 that

$$\mathbb{P}_{T+k}[\tau_T^* \leq h] = G\left(k, a, b^*(a \wedge \ell(T)), h\right) \tag{270}$$

$$= G\left(k, a, b^*(a \wedge \ell(T')), h\right) = \mathbb{P}_{T'+k}[\tau_{T'}^* \leq h] \tag{271}$$

$$\mathbb{P}_{T+k}\left[\mathcal{L}_T^* \geq h\right] = 1 - H\left(k, a, b^*(a), \ell(h+T)\right) \tag{272}$$

$$\geq 1 - H\left(k, a, b^*(a), \ell(h+T')\right) = \mathbb{P}_{T+k}\left[l_{T'}^* \geq h\right] \tag{273}$$

for all $(k,h) \in \mathbb{R}_+^2$ as required.

Case 2: $a^*(T) \geq a^*(T') = 0$**.** In this case the increase of the function $\ell(\cdot)$ and Lemma A.4 jointly imply that we have

$$b^*(a \wedge \ell(T)) \geq b^*(\ell(T)) \geq b^*(\ell(T')) = b^*(a \wedge \ell(T')) \tag{274}$$

and it now follows from Eqs.(266), (267) and Lemma B.6 that

$$\begin{aligned}
\mathbb{P}_{T+k}[\tau_T^* \leq h] &= G\left(k, a^*(T), b^*(a \wedge \ell(T)), h\right) &(275)\\
&\leq G\left(k, 0, b^*(a \wedge \ell(T)), h\right) \leq G\left(k, 0, b^*(\ell(T')), h\right) = \mathbb{P}_{T'+k}[\tau_{T'}^* \leq h] &(276)\\
\mathbb{P}_{T+k}\left[\mathcal{L}_T^* \geq h\right] &= 1 - H\left(k, a^*(T), b^*(a \wedge \ell(T)), \ell(h+T)\right) &(277)\\
&\geq 1 - H\left(k, 0, b^*(a \wedge \ell(T)), \ell(h+T)\right) &(278)\\
&\geq 1 - H\left(k, 0, b^*(\ell(T')), \ell(h+T)\right) &(279)\\
&\geq 1 - H\left(k, 0, b^*(\ell(T')), \ell(h+T')\right) = \mathbb{P}_{T+k}\left[l_{T'}^* \geq h\right] &(280)
\end{aligned}$$

for all $(k, h) \in \mathbb{R}_+^2$ as required. ■

B.4. Debt values

Fix some coupon rates (c_L, c_D) and some face values (L, D). Let $W(x) = W(x|c)$ with $c = c_L + c_D$ stand for the $\rho-$scale function of the corresponding uncontrolled liquid reserves process, and denote by

$$\Delta_q(x, \Theta|L, D) = \mathbb{E}\left[\delta_q\left(x - Y_1, \Theta|L, D\right)\right], \qquad q \in \{\text{junior}, \text{senior}\} \tag{281}$$

with

$$\delta_{\text{junior}}(x, \Theta|L, D) = 1_{\{x+a \leq 0\}} \min\left\{(x + T + \Lambda - D)^+, L\right\} \tag{282}$$

$$\delta_{\text{senior}}(x, \Theta|L, D) = \delta_{\text{junior}}(x, \Theta|L, 0) = 1_{\{x+a \leq 0\}} \min\left\{(x + T + \Lambda)^+, L\right\} \tag{283}$$

the expected payment that creditors receive under the strategy $\Theta = (T, a, b)$ if their debt is of type $q \in \{\text{senior}, \text{junior}\}$ and liquidation occurs following the arrival of a jump at a point where the liquid reserves of the bank exceed the minimal required level by $x \in \mathbb{R}$. With this notation we have that the value of the creditor's claim is

$$d_q(s,\Theta|c_L,L,D) = \mathbb{E}\left[\int_0^{\tau(s,\Theta)} e^{-\rho t} c_L dt + e^{-\rho\tau(s,\Theta)}\delta_q\left(S_{\tau(s,\Theta)}(s,\Theta) - T|L,D\right)\right] \tag{284}$$

and the following lemma provides a closed-form expression for this value in terms of the scale function of the uncontrolled liquid reserves process.

Lemma B.7. *For any $s \geq T$ the value of the creditor's claim under a strategy Θ can be computed as*

$$d_q\left(s,\Theta|c_L,L,D\right) = \sum_{i=1}^{2}\left[\varphi_{q,i}(s-T,\Theta|c_L,L,D) + \varphi_{q,i}(b,\Theta|c_L,L,D)\frac{\gamma(s-T,\Theta)}{1-\gamma(b,\Theta)}\right] \tag{285}$$

with the functions defined by

$$\varphi_{q,1}(x,\Theta|c_L,L,D) = c_L\left[\frac{W(b)W(x\wedge b)}{W'(b)} - \int_0^{x\wedge b} W(z)dz\right] \tag{286}$$

$$\varphi_{q,2}(x,\Theta|c_L,L,D) = (\sigma^2/2)\left[W'(x\wedge b) - \frac{W(x\wedge b)W''(b)}{W'(b)}\right]\delta_q(0,\Theta|L,D) \tag{287}$$

$$+\int_0^b \lambda\Delta_q\left(z,\Theta|L,D\right)\left[\frac{W(x\wedge b)W'(b-z)}{W'(b)} - W(x\wedge b-z)\right]dz \tag{288}$$

and

$$\gamma(x,\Theta) = (\sigma^2/2)\left[W'(x\wedge b) - \frac{W(x\wedge b)W''(b)}{W'(b)}\right]1_{\{a>0\}} \tag{289}$$

$$+\int_0^b \lambda F(a)(1-F(z))\left[\frac{W(x\wedge b)W'(b-z)}{W'(b)} - W(x\wedge b-z)\right]dz. \tag{290}$$

Proof. Fix a strategy $\Theta = (T,a,b)$ and let $\Theta_b = (0,0,b)$. The definition of Θ implies that the debt value satisfies the value matching condition

$$d_q(s,\Theta|c_L,L,D) = d_q(T+b,\Theta|c_L,L,D), \qquad s\in(T-a,T]\cup[T+b,\infty). \tag{291}$$

On the other hand, Eq.(234) and the fact that the strategy Θ_b does not involve refinancing implies that we have

$$\tau(s,\Theta) = \tau(s-T,0,a,b) \geq \tau(s-T,\Theta_b), \qquad s \in \mathbb{R}. \tag{292}$$

Combining theses identities with the definition of the debt value function and the law of iterated expectations then shows that for $s \geq T$ we have

$$d_q(s,\Theta|c_L,L,D) = \sum_{i=1}^{2} \varphi_{q,i}(s-T,\Theta|c_L,L,D) + \gamma(s-T,\Theta) d_q(T+b,\Theta|c_L,L,D), \tag{293}$$

with the functions defined by

$$\varphi_{q,2}(x,\Theta|c_L,L,D) = \mathbb{E}\left[e^{-\rho\tau(x,\Theta_b)}\delta_q(S_{\tau(x,\Theta_b)}(x,\Theta_b),\Theta|L,D)\right], \tag{294}$$

$$\varphi_{q,1}(x,\Theta|c_L,L,D) = \mathbb{E}\left[\int_0^{\tau(x,\Theta_b)} e^{-\rho t} c_L dt\right] = (c_L/\rho)\left(1-\mathbb{E}\left[e^{-\rho\tau(x,\Theta_b)}\right]\right), \tag{295}$$

and

$$\gamma(x,\Theta) = \mathbb{E}\left[e^{-\rho\tau(x,\Theta_b)} 1_{\{S_{\tau(x,\Theta_b)}(x,\Theta_b)+a>0\}}\right]. \tag{296}$$

Evaluating Eq.(293) at the point $s = T+b$, solving the resulting equation for the value at the dividend barrier and substituting the solution back into Eq.(293) shows that Eq.(285) holds and it now remains to show that Eqs.(286), (287), and (289) are also satisfied.

The required result for $\varphi_{q,1}(x,\Theta|c_L,L,D)$ follows from Kuznetsov et al. (2013, Theorem 2.8.ii). To compute the other two functions we proceed as in the proof of Lemma A.2: Applying the dividend/penalty identity (see Gerber, Lin and Yang (2006)) and using the same notation as in Section A.3. we find that

$$\varphi_{q,2}(x,\Theta|c_L,L,D) = \sum_{i=1}^{2}\left[\varphi_{q,2,i}(x\wedge b,\Theta|c_L,L,D) - \varphi'_{q,2,i}(b,\Theta|c_L,L,D)\frac{W(x\wedge b)}{W'(b)}\right] \tag{297}$$

with the auxiliary functions defined by

$$\varphi_{q,2,1}(x,\Theta|c_L,L,D) = \mathbb{E}_x\left[e^{-\rho\zeta_0}1_{\{\Delta X_{\zeta_0}=0\}}\delta_q(0,\Theta|L,D)\right] \tag{298}$$

$$\varphi_{q,2,2}(x,\Theta|c_L,L,D) = \mathbb{E}_x\left[e^{-\rho\zeta_0}1_{\{\Delta X_{\zeta_0}\neq 0\}}\delta_q(X_{\zeta_0},\Theta|L,D)\right]. \tag{299}$$

By Eq.(85) we know that the first of these auxiliary functions can be computed explicitly in terms of the generalized scale function as

$$\varphi_{q,2,1}(x,\Theta|c_L,L,D) = \frac{\sigma^2}{2}\left(W'(x) - B_3 W(x)\right)\delta_q(0,\Theta|L,D) \tag{300}$$

where the constant B_3 is the strictly positive root of the cubic equation Eq.(78). On the other hand, using the compensation formula for point processes in conjunction with the potential density given in Eq.(86) we obtain that

$$\varphi_{q,2,2}(x,\Theta|c_L,L,D) = \int_0^\infty \lambda\left(e^{-B_3 z}W(x) - W(x-z)\right)\Delta_q(z,\Theta|L,D)dz. \tag{301}$$

Differentiating Eqs.(300) and (301), substituting into Eq.(298), and simplifying shows that Eq.(287) is satisfied. A similar argument shows that the function $\gamma(x,\Theta)$ can be computed as indicated in Eq.(289) and completes the proof. ■

Remark B.1. The integrals in the definition of the functions $\gamma(x,\Theta)$ and $\varphi_{q,i}(x,\Theta|c_L,L,D)$ are left unevaluated to simplify the presentation but can easily be expressed as combinations of exponentials by using the explicit expression of the scale function in Eq.(80) and the fact the jumps of the cash flow process are exponentially distributed.

Remark B.2. The formula of Lemma B.7 can also be used to compute the market value of deposits. Indeed, if depositors are senior then the liquidation payment that they receive if the bank defaults at a point where liquid reserves are equal to $x \leq T - a$ is

$$\delta_{\text{senior}}(x-T,\Theta|D,0) = \min\left\{(x+\Lambda)^+, D\right\} \tag{302}$$

and it thus follows from the proof of Lemma B.7 that the market value of their claim can be computed as $d_{\text{senior}}(s,\Theta|c_D,D,0)$. Similarly, if depositors are junior to creditors in default then the market value of their claim can be computed as $d_{\text{junior}}(s,\Theta|c_D,D,L)$. In either case the market value of the combined claim of the bank's depositors and creditors can be computed as $d_{\text{senior}}(s, \Theta | c, D + L, 0)$.

B.5. Default probability and the distribution of default losses

To quantify the effect of liquidity requirements on the default risk of the bank we need to calculate the probability

$$f(s, y, t, \Theta) = \mathbb{P}[\{\tau(s, \Theta) \leq t\} \cap \{S_{\tau(s,\Theta)}(s, \Theta) \leq T - y\}] \tag{303}$$

that the bank is liquidated prior to a fixed horizon in a state where the shortfall of its liquid reserves relative to the required level exceeds a given amount $y \geq 0$. Unfortunately, this probability cannot be computed in closed form due to the time dependence induced by the presence of a fixed horizon. To circumvent this difficulty we consider instead the Laplace transform

$$\widehat{f}(s, y, k, \Theta) = \int_0^\infty e^{-kt} f(s, y, t, \Theta) dt = (1/k)\, \mathbb{E}\left[e^{-k\tau(s,\Theta)} 1_{\{S_{\tau(s,\Theta)}(s,\Theta) \leq T - y\}}\right]. \tag{304}$$

Relying on arguments similar to those we used in the computation of the debt value function allows to obtain this Laplace transform in closed form. To state the result let

$$W_k(x) = \sum_{i=1}^{3} 1_{\{x \geq 0\}} \frac{2(\beta + B_{i,k}) e^{B_{i,k} x}}{\sigma^2 \prod_{j \neq i} (B_{i,k} - B_{j,k})} \tag{305}$$

where the constants $(B_{i,k})_{i=1}^3$ denote the real roots of the cubic equation

$$k = B_{i,k} \left(\bar{\mu} + B_{i,k} \frac{\sigma^2}{2} - \frac{\lambda}{\beta + B_{i,k}} \right), \tag{306}$$

denote the scale function of the uncontrolled liquid reserves process associated with the discount rate $k \geq 0$.

Lemma B.8. *For any $s \geq T$ the Laplace transform of the bank's default probability can be computed as*

$$\widehat{f}(s, y, k, \Theta) = (1/k) \left[\kappa(s - T, y, k, \Theta) + \kappa(b, y, k, \Theta) \frac{\gamma_k(s - T, \Theta)}{1 - \gamma_k(b, \Theta)} \right] \tag{307}$$

where we have set

$$\kappa(x,y,k,\Theta) = (\sigma^2/2)\left[W_k'(x\wedge b) - \frac{W_k(x\wedge b)W_k''(b)}{W_k'(b)}\right]1_{\{y=a=0\}} \tag{308}$$

$$+\int_0^b \lambda(1-F(z+a\vee y))\left[\frac{W_k(x\wedge b)W_k'(b-z)}{W_k'(b)} - W_k(x\wedge b-z)\right]dz \tag{309}$$

and the function $\gamma_k(x,\Theta)$ *is defined as in Eq.*(289) *but with* $W_k(x)$ *instead of* $W(x)$*.*

Proof. The computation follows the same steps as that of the debt value function in the proof Lemma B.7 and therefore is omitted. ■

To obtain the default probability we will numerically invert the Laplace transform using the Gaver-Stehfest formula (Gaver (1966), Stehfest (1970)):

$$f(s,y,t,\Theta) \approx \sum_{n=1}^{N} \omega_n(t,N)\widehat{f}\left(s,y,\frac{1}{t}\log 2^n,\Theta\right) \tag{310}$$

where $N \in 2\mathbb{N}$ is an even constant chosen to insure the convergence of the approximation, and the weights are defined by

$$\omega_n(t,N) = \sum_{m=[\![\frac{n+1}{2}]\!]}^{n\wedge\frac{N}{2}} \frac{(-1)^{n+\frac{N}{2}} m^{N/2}(2m)!(\log 2)(1/t)}{m!(m-1)!(n-m)!(2m-n)!(N/2-m)!}. \tag{311}$$

where $[\![x]\!]$ denotes the integer part of a real number x. The main advantage of this method is that it does not require the evaluation of the Laplace transform in the complex plane and, therefore, allows to avoid solving Eq.(306) at complex values of the transform parameter. Its main disadvantage is that requires a high accuracy to deal with the fact that the weights and the approximations include factorials and alternating signs. This however is not a problem in modern computer softwares such as Mathematica®. For example, in our numerical implementation we achieve a precision of 6 digits by using $N = 10$ and an accuracy of 100 digits in the computations.

Remark B.3. While the joint distribution of the default time and loss in default in Eq.(303) cannot be computed in closed form, the marginal distribution of the loss in default can be explicitly computed as

$$\mathbb{P}[\{S_{\tau(s,\Theta)}(s,\Theta) \leq T - y\}] = \lim_{k\to 0} k\widehat{f}(s,y,k,\Theta) \tag{312}$$

$$= \kappa_0(s-T,y,\Theta) + \kappa_0(b,y,\Theta)\frac{\gamma_0(s-T,\Theta)}{1-\gamma_0(b,\Theta)} \tag{313}$$

where the functions $\kappa_0(s,y,\Theta)$ and $\gamma_0(s,\Theta)$ are defined as in the statement of Lemma B.8 but with the undiscounted scale function $W_0(x) = \lim_{k\to 0} W_k(x)$ instead of the discounted scale function $W_k(x)$.

B.6. *Optimal capital structure*

The following proposition provides condition under which the statically optimal capital structure of section 4.2. is dynamically optimal.

Proposition B.4. *If deposits and market debt can be repurchased at face value and if liability adjustments are subject to the same fixed cost as equity issuance then it is never optimal for the bank to modify its liabilities once it has issued the amount of deposits and market debt prescribed by the solution to Eq.(53)–Eq.(57).*

Proof of Proposition B.4. Assume that the pair (c_L^*, D^*) solves the static optimization problem Eq.(53)–Eq.(57) and denote by

$$F(c_L,D) = D + L(c_L,D) \tag{314}$$

with

$$L(c_L,D) \equiv L = \mathbb{E}_{b^*_{T(c_L,D)}(c_L,D)}\left[\int_0^{\tau^*_{T(c_L,D)}(c_L,D)} e^{-\rho t} c_L dt\right] \tag{315}$$

$$+ \mathbb{E}_{b^*_{T(c_L,D)}(c_L,D)}\left[e^{-\rho\tau^*_{T(c_L,D)}(c_L,D)} \min\left\{L, \left(S_{\tau^*_{T(c_L,D)}(c_L,D)} + \Lambda - qD\right)^+\right\}\right] \tag{316}$$

the combined face value of the bank's deposits and market debt associated with an arbitrary pair (c_L, D). Verification arguments similar to those of Appendix B.2. imply that in order to establish the desired result it is sufficient to show that

$$v(s;T(c_L^*,D^*)|c_L^*,D^*) \geq \sup_{(e,c_L,D)\in\mathbb{R}_+\times\mathcal{A}(\Omega)} \Big\{-e-1_{\{e>0\}\cup\{D^*\neq D\}\cup\{c_L\neq c_L^*\}}\phi$$
$$+ v\left(s+e+F(c_L,D)-F(c_L^*,D^*);T(c_L,D)\right| c_L,D)\Big\} = G(s) \quad (317)$$

for all $s \in \mathbb{R}$ where $\mathcal{A}(\Omega)$ is the set of pairs $(c_L, D) \in \mathbb{R}_+^2$ that satisfy Eq.(57). The right hand side of this inequality can be written as

$$G(s) = \max\left\{v\left(s;T(c_L^*,D^*)\right| c_L^*,D^*\right), H(s)\} \quad (318)$$

with the function

$$H(s) = \sup_{(e,c_L,D)\in\mathbb{R}_+\times\mathcal{A}(\Omega)} \Big\{v\left(s+e+F(c_L,D)-F(c_L^*,D^*);T(c_L,D)\right| c_L,D) - e - \phi\Big\}. \quad (319)$$

From the properties of the equity value function derived in Appendices B.2. and B.3., we know that the target level of liquid reserves satisfies

$$\sup_{e\geq 0}\left\{v\left(s+e;T(c_L,D)\right| c_L,D) - e\right\} = s - b^*_{T(c_L,D)}(c_L,D) \quad (320)$$
$$+ v\left(b^*_{T(c_L,D)}(c_L,D);T(c_L,D)\middle| c_L,D\right) \quad (321)$$

for all $s \in \mathbb{R}$. Therefore, we have

$$H(s) = \sup_{(c_L,D)\in\mathcal{A}(\Omega)} \Big\{v\left(b^*_{T(c_L,D)}(c_L,D);T(c_L,D)\middle| c_L,D\right)$$
$$+ s - b^*_{T(c_L,D)}(c_L,D) + F(c_L,D) - F(c_L^*,D^*) - \phi\Big\}, \quad (322)$$

and combining this with the assumption that the pair (c_L^*, D^*) solves the static optimization problem Eq.(53)–Eq.(57), we deduce that

$$H(s) = v\left(b^*_{T(c_L^*,D^*)}(c_L^*,D^*);T(c_L^*,D^*)\middle| c_L^*,D^*\right) + s - b^*_{T(c_L^*,D^*)}(c_L^*,D^*) - \phi. \quad (323)$$

This in turn implies that we have

$$G(s) = \sup_{e \geq 0} \left\{ v\left(s + e; T(c_L^*, D^*) \middle| \, c_L^*, D^* \right) - e - 1_{\{e>0\}}\phi \right\} \tag{324}$$

and the desired result now follows by observing that, since it is always possible for the bank to raise outside equity, the right hand of the inequality is dominated by the equity value function evaluated at the point s. ■

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