Internet Appendix

Early Option Exercise: Never Say Never

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A. Appendix: Derivations and proofs

Proof of Proposition 1

If the agent wants cash and exercises to immediately sell the stock in the market, the net proceeds would be $S_0(1-\lambda^{i,S})-X$. If the investor instead places $\frac{X}{1+r^f}$ in the risk-free asset and sells short the stock, the cash flow would now be $S_0(1-\lambda^{i,S})-\frac{X}{1+r^f}$ (as there are no funding or short-sale costs). In the next period the option can be exercised paid by the money from the risk-free asset and the received stock can be used to close the short position. This has a net payment of zero. Thereby the cash flow of the strategy dominates the one created by exercising and selling the stock (since $r^f>0$).

If the investor wants the exposure to the stock and exercises to get the stock, the cash flow is $-X$. The alternative dominating strategy is to place $\frac{X}{1+r^f}$ in the risk-free asset and wait and exercise later. This costs less today (as $r^f>0$), and the stock purchase can be made cash flow neutral at time 1 by the money from the risk-free asset. No value is forgone by not owning the stock in the meantime as there is no lending fee, $l^i=0$, and no funding costs are incurred.

Consider *(ii)* where the sale revenue of the option is above the intrinsic value for agent i, $C_0(1-\lambda^{i,C})>S_0-X$. If agent i wants cash, exercising the option and selling the stock obviously gives less than selling the option. If agent i wants stock, selling the option and buying the stock has a net cost of $S_0 - C_0(1 - \lambda^{i,C})$ which is less than what it costs to gain the stock through exercise, X. Hence, no matter if agent i wants stock or cash, early exercise is dominated.

A sufficient condition for $C_0(1-\lambda^{i,C}) > S_0 - X$ is that agent i faces no option transaction costs, $\lambda^{i,C} = 0$, and the existence of another type of agents j with zero short-sale costs, funding costs, and stock transaction costs. We have just shown above that for an investor without short-sale costs, funding costs or stock transaction costs the value of the option will be at least $S_0 - \frac{X}{1+r^f}$. If such unconstrained agents j exist, the option price must be at least $S_0 - \frac{X}{1+r^f}$ to avoid arbitrage. If $\lambda^{i,C} = 0$ we get: $C_0(1 - \lambda^{i,C}) = C_0 \geq S_0 - \frac{X}{1+r^f} > S_0 - X$.

Proof of Proposition 2

This proof shows that there exists a strategy involving early exercise that is not dominated by any strategy not involving early exercise if frictions are large enough. We introduce the following notation for the initial portfolio held by an agent: The number of stocks, α_0, the amount invested in the risk-free asset, β_0, and the number of options, $\gamma_0 > 0$. Similarly, we use the subscript 1 for the portfolio that the agent trades to (e.g., α_1 is the number of stocks held by agent i after trading).

Consider an agent who wants as much cash as possible at time 0 without any negative cash flow in any possible state at time 1. One way to do this is to exercise the option position early and close the positions in stock and risk-free asset. If stocks needs to be sold to close the position, stock transaction costs apply. However, in the event that more can be earned through lending out the stock and borrow the present value of the future earned lending fee than through selling the stock, this is done instead. If stock must be acquired this can be done either by buying stock or buying options and exercise immediately. The cheapest way is chosen. The cash-flow at time 0 of this "liquidating early exercise strategy" is:

$$
\text{CF} = \begin{cases} (\alpha_0+\gamma_0)S_0(1-\lambda^S)+\beta_0-\gamma_0 X & \text{if } \alpha_0+\gamma_0 \geq 0 \text{ and } 1-\lambda^S \geq \frac{l^i}{1+r^f} \\ (\alpha_0+\gamma_0)\frac{S_0 l^i}{1+r^f}+\beta_0-\gamma_0 X & \text{if } \alpha_0+\gamma_0 \geq 0 \text{ and } 1-\lambda^S < \frac{l^i}{1+r^f} \\ (\alpha_0+\gamma_0)S_0+\beta_0-\gamma_0 X & \text{if } \alpha_0+\gamma_0 < 0 \text{ and } S_0 \leq C_0+X \\ (\alpha_0+\gamma_0)(C_0+X)+\beta_0-\gamma_0 X & \text{if } \alpha_0+\gamma_0 < 0 \text{ and } S_0 > C_0+X \end{cases} \tag{A.1}
$$

We want to show that this early exercise strategy is not dominated for agent i.

Any strategy without early exercise is identified by the allocation of wealth between the three assets.[1] This allocation can be denoted by $(\alpha_1, \beta_1, \gamma_1)$ for the number of stocks, amount invested in risk-free asset, and number of options respectively. The cash-flow at time 0 for an arbitrary strategy not involving early exercise can be described by the function J_1 where:

$$
\begin{aligned} J_1(\alpha_1,\beta_1,\gamma_1) :=& (\alpha_0-\alpha_1)(S_0-\mathbf{1}_{(\alpha_0-\alpha_1>0)}S_0\lambda^S)+(\beta_0-\beta_1) \\ &+(\gamma_0-\gamma_1)(C_0-\mathbf{1}_{(\gamma_0-\gamma_1>0)}\lambda^{i,C}C_0)-F^i(\alpha_1 S_0,\gamma_1 C_0) \end{aligned} \tag{A.2}
$$

In order for a strategy to dominate the liquidating early exercise strategy, the strategy must have non-negative values at time 1 in all possible states and at least as large a cash flow at time zero. To ensure a non-negative value at time 1, for every stock agent i is short, agent i must also be long at least 1 option and have no less than $\frac{X+S_0 L^i}{1+r^f}$ in the risk-free asset (the stock has unlimited upside so the option is needed to secure a non-negative payoff, and the cash is needed to pay the possible future exercise of the option and the short-sale fee of the stock). Similarly, for every option that agent i is short, i must be at least long 1 stock, and not use the risk-free asset to borrow more than $\frac{S_0 l^i}{1+r^f}$ (where the stock is needed to be able to honor a possible future exercise of the option in all states and, since the stock could also turn out to be worthless at time 1, there cannot be borrowed more than $\frac{S_0 l^i}{1+r^f}$, the present value of the only secure income raised from lending out the stock in the period). The maximal

[1] Admittedly, the same portfolio can be obtained through several ways of trading. E.g. the number of stocks held can be reduced by 1 either by selling one stock or selling two stocks and buying one back immediately, corresponding to money burning when stock transaction costs are positive. For our purpose it is sufficient to consider strategies with the cheapest way to obtain the portfolio. If such strategies cannot dominate early exercise neither can strategies in which money is given up for nothing.

cash-flow that can be obtained at time $t = 0$ without early exercise and with non-negative value at time $t = 1$ is therefore the solution to:

$$\max_{\alpha_1,\beta_1,\gamma_1} J_1(\alpha_1,\beta_1,\gamma_1) \tag{A.3}$$

$$\text{s.t. } \alpha_1+\gamma_1 \geq 0 \tag{A.4}$$

$$\beta_1 + \mathbf{1}_{(\alpha_1<0)}\alpha_1\frac{X+S_0L^i}{1+r^f} + \mathbf{1}_{(\alpha_1>0)}\alpha_1\frac{S_0 l^i}{1+r^f} \geq 0 \tag{A.5}$$

The early exercise strategy cannot be dominated by selling the option due to the premise of the proposition that doing so is less profitable ($C_0(1-\lambda^{i,C}) \leq S(1-\lambda^{i,S})-X$). Not selling the option corresponds to $\gamma_1 \geq \gamma_0$. Combining this with (A.4) we get the inequality $\gamma_1 \geq \max(-\alpha_1,\gamma_0)$. This inequality must bind since J_1 is decreasing in γ_1 for $\gamma_1 \geq 0$. Likewise, J_1 is decreasing in β_1 which means that (A.5) must bind. We can now substitute both β_1 and γ_1 and reformulate the problem as:

$$\max_{\alpha_1} J_2(\alpha_1) \tag{A.6}$$

where we define the function J_2 by:

$$\begin{aligned} J_2(\alpha_1) :=& J_1(\alpha_1, -\mathbf{1}_{(\alpha_1<0)}\alpha_1\frac{X+S_0L^i}{1+r^f} - \mathbf{1}_{(\alpha_1>0)}\alpha_1\frac{S_0l^i}{1+r^f}, \max(-\alpha_1,\gamma_0)) \\ =&(\alpha_0-\alpha_1)(S_0 - \mathbf{1}_{(\alpha_0-\alpha_1>0)}S_0\lambda^S) + \beta_0 + \mathbf{1}_{(\alpha_1<0)}\alpha_1\frac{X+S_0L^i}{1+r^f} \\ &+\mathbf{1}_{(\alpha_1>0)}\alpha_1\frac{S_0l^i}{1+r^f} + (\gamma_0 - \max(-\alpha_1,\gamma_0))C_0 - F^i(\alpha_1S_0, -\max(-\alpha_1,\gamma_0)C_0) \end{aligned} \tag{A.7}$$

We are ready to show part *a.* in the proposition under the condition that $L^i > \frac{Xr^f}{S_0}$. We show that the optimal cash-flow (A.6) is smaller than the cash flow from the liquidating early exercise strategy (A.1), regardless of funding costs. Recall that the early exercise strategies incur no funding costs and funding costs for other strategies are non-negative so in the following we consider the case in which $F^i = 0$. Then, J_2 is piece-wise linear in α_1 with kinks at $\{-\gamma_0, \alpha_0, 0\}$. So a global maximum for the J_2 will either be at $-\gamma_0$, α_0, 0 or at plus or minus infinity. We check the last first:

$$\begin{aligned}\lim_{\alpha_1\to-\infty} J_2(\alpha_1) = &\lim_{\alpha_1\to-\infty}\left[(\alpha_0-\alpha_1)S_0(1-\lambda^{i,S})+\beta_0+\alpha_1\frac{X+S_0L^i}{1+r^f}+(\gamma_0+\alpha_1)C_0\right]\\ =&\alpha_0S_0(1-\lambda^{i,S})+\beta_0+\gamma_0C_0+\lim_{\alpha_1\to-\infty}\left[\alpha_1[-S_0(1-\lambda^{i,S})+\frac{X+S_0L^i}{1+r^f}+C_0]\right]\end{aligned} \tag{A.8}$$

It must hold that $C_0+X\geq S_0(1-\lambda^{i,S})$, otherwise there would be an arbitrage strategy by buying options, exercise immediately and sell the stock. Given this and that $L^i>\frac{Xr^f}{S_0}$ the entire expression (A.8) is diverging to $-\infty$. We next check $\alpha_1\to\infty$:

$$\begin{aligned}\lim_{\alpha_1\to\infty} J_2(\alpha_1) = &\lim_{\alpha_1\to\infty}\left[(\alpha_0-\alpha_1)S_0+\beta_0+\alpha_1\frac{S_0l^i}{1+r^f}\right]\\ =&\alpha_0S_0+\beta_0+\lim_{\alpha_1\to\infty}\left[\alpha_1(-S_0+\frac{S_0l^i}{1+r^f})\right]=-\infty\end{aligned} \tag{A.9}$$

since no-arbitrage implies that $S_0>\frac{S_0l^i}{1+r^f}$ (otherwise an arbitrage gain could be made through buying stocks and lending them out). We now evaluate the expression in $\alpha_1=\alpha_0\leq-\gamma_0$:

$$J_2(\alpha_0)=\beta_0+\alpha_0\frac{X+S_0L^i}{1+r^f}+(\gamma_0+\alpha_0)C_0<CF \tag{A.10}$$

using that $L^i>\bar{L}=\frac{Xr^f}{S_0}$ and $\alpha_0+\gamma_0\leq 0$. Next, evaluate where $\alpha_1=\alpha_0\in(-\gamma_0,0]$:

$$J_2(\alpha_0)=\beta_0+\alpha_0\frac{X+S_0L^i}{1+r^f}<CF \tag{A.11}$$

using that $\alpha_0+\gamma_0>0$ and $L^i>\bar{L}=\frac{Xr^f}{S_0}$. Evaluating J_2 where $\alpha_1=\alpha_0>0$:

$$J_2(\alpha_0)=\beta_0+\alpha_0\frac{S_0l^i}{1+r^f}<CF \tag{A.12}$$

Next, we evaluate where $\alpha_1=0$:

$$J_2(0)=\alpha_0(S_0-\mathbf{1}_{(\alpha_0>0)}S_0\lambda^S)+\beta_0<CF \tag{A.13}$$

The final evaluation of the expression is where $\alpha_1 = -\gamma_0$:

$$J_2(-\gamma_0) = (\alpha_0 + \gamma_0)(S_0 - \mathbf{1}_{(\alpha_0+\gamma_0>0)} S_0 \lambda^{i,S}) + \beta_0 - \gamma_0 \frac{X + S_0 L^i}{1 + r^f} < CF \tag{A.14}$$

since $L^i > \bar{L} = \frac{X r^f}{S_0}$. Hence it has been proved that for sufficiently high short-sale costs, early exercise is not dominated regardless of funding costs.

Next we show that a similar result holds for sufficiently high funding costs. Specifically, we consider funding costs with $F^i(x, y) \geq \bar{F}(|x| + |y|)$ where $\bar{F} > \frac{r^f X}{(1+r^f)(S_0+C_0)}$. Under this condition, J_3 is greater than J_2 where:

$$\begin{aligned} J_3(\alpha_1) :=& (\alpha_0 - \alpha_1)(S_0 - \mathbf{1}_{(\alpha_0-\alpha_1>0)} S_0 \lambda^S) + \beta_0 + \mathbf{1}_{(\alpha_1<0)} \alpha_1 \frac{X + S_0 L^i}{1 + r^f} + \mathbf{1}_{(\alpha_1>0)} \alpha_1 \frac{S_0 l^i}{1 + r^f} \\ &+ (\gamma_0 - \max(-\alpha_1, \gamma_0))C_0 - \bar{F}(|\alpha_1 S_0| + \max(-\alpha_1, \gamma_0)C_0) \end{aligned} \tag{A.15}$$

and we seek to show that J_3 is smaller than (A.1). Clearly, J_3 is piecewise linear with kinks in $\{-\gamma_0, \alpha_0, 0\}$. We first consider the extremes:

$$\begin{aligned} &\lim_{\alpha_1 \to -\infty} J_3(\alpha_1) \\ =& \lim_{\alpha_1 \to -\infty} (\alpha_0 - \alpha_1) S_0 (1 - \lambda^S) + \beta_0 + \alpha_1 \frac{X + S_0 L^i}{1 + r^f} + (\gamma_0 + \alpha_1) C_0 + \alpha_1 \bar{F}(S_0 + C_0) \\ =& \alpha_0 + \beta_0 + \gamma_0 C_0 + \lim_{\alpha_1 \to -\infty} \left(\alpha_1 [-S_0(1 - \lambda^S) + \frac{X + S_0 L^i}{1 + r^f} + C_0 + \bar{F}(S_0 + C_0)] \right) = -\infty \end{aligned} \tag{A.16}$$

since the expression in the squared brackets is positive since no-arbitrage implies that $S(1 - \lambda^S) \leq C + X$ (otherwise, an arbitrage strategy would be to buy options, exercise them immediately and sell the obtained stocks). Similarly:

$$\begin{aligned}
&\lim_{\alpha_1\to\infty} J_3(\alpha_1) \\
=&\lim_{\alpha_1\to\infty}(\alpha_0-\alpha_1)S_0+\beta_0+\alpha_1\frac{S_0 l^i}{1+r^f}-\bar{F}(\alpha_1 S_0+\gamma_0 C_0) \\
=&\alpha_0+\beta_0-\bar{F}\gamma_0 C_0+\lim_{\alpha_1\to\infty}\left[\alpha_1[-S_0+\frac{S_0 l^i}{1+r^f}-\bar{F}S_0]\right]=-\infty
\end{aligned} \tag{A.17}$$

since, as before, $S_0 > \frac{S_0 l^i}{1+r^f}$. Next, we evaluate J_3 where $\alpha_1 = -\gamma_0$:

$$J_3(-\gamma_0) = (\alpha_0+\gamma_0)(S_0-\mathbf{1}_{(\alpha_0+\gamma_0>0)}S_0\lambda^S)+\beta_0-\gamma_0\frac{X+S_0 L^i}{1+r^f}-\bar{F}(\gamma_0 S_0+\gamma_0 C_0) < CF \tag{A.18}$$

since $\bar{F} > \frac{r^f X}{(1+r^f)(S_0+C_0)}$. For $\alpha_1 = 0$ we get:

$$J_3(0) = \alpha_0(S_0-\mathbf{1}_{(\alpha_0>0)}S_0\lambda^S)+\beta_0-\bar{F}\gamma_0 C_0 < CF \tag{A.19}$$

since $S(1-\lambda^{i,S})-X>0$. Next, evaluate where $\alpha_1=\alpha_0\le-\gamma_0$:

$$J_3(\alpha_0)=\beta_0+\alpha_0\frac{X+S_0 L^i}{1+r^f}+(\gamma_0+\alpha_0)C_0+\alpha_0\bar{F}(S_0+C_0) < CF \tag{A.20}$$

Evaluating where $\alpha_1=\alpha_0\in(-\gamma_0,0]$:

$$\begin{aligned}
J_3(\alpha_0) &= \beta_0+\alpha_0\frac{X+S_0 L^i}{1+r^f}-\bar{F}(-\alpha_0 S_0+\gamma_0 C_0) \\
&= \beta_0+\alpha_0\frac{X+S_0 L^i}{1+r^f}+\alpha_0\bar{F}(S_0+C_0)-\bar{F}C_0(\alpha_0+\gamma_0) < CF
\end{aligned} \tag{A.21}$$

Finally, we evaluate where $\alpha_1=\alpha_0>0$:

$$J_3(\alpha_0)=\beta_0+\alpha_0\frac{S_0 l^i}{1+r^f}-\bar{F}(\alpha_0 S_0+\gamma_0 C_0) < CF \tag{A.22}$$

Proof of Proposition 3

Both the problem for the lower boundary (6)–(7) and the problem for the upper boundary (8) are mathematical equivalent to problem of pricing American call options in the BSM model with continuous dividend yield, allowing us to utilize results established when proving *(ii)* and *(iii)*.

Kim (1990) shows that the exercise boundary is increasing in time to expiration, and, in accordance with Merton (1973), identifies the closed-form limit value of the exercise boundary as time to expiration goes to infinity. For a positive dividend yield this limit value is finite, and hence the enitre boundary is finite as remarked by Dewynne et al. (1993). So if what corresponds to the dividend yield in our model (i.e., $L^{\underline{i}} + \psi^{\underline{i}}(m^{\underline{i},C} + m^{\underline{i},S})$ in part *(ii)* or $l^{\bar{i}} + \psi(m^{\bar{i},C} - m^{\bar{i},S})$ in *(iii)*) is positive, the exercise boundary is finite. If what corresponds to the dividend yield is zero or negative, then Merton's lower bound still holds and early exercise is always dominated, implying infinite exercise boundaries for $t < T$.

Next, we prove *(i)*. If $S(t) < \underline{B}(T-t)$ then $\underline{C} > S(t) - X$ so it is dominated for agent $\underline{i}$ to exercise early and (6) must hold. We next consider any other agent i who owns one option, showing that his valuation must be above $\underline{C}$ and, therefore, above the intrinsic value, leading us to conclude that exercise is dominated. Suppose that i assigned the same value to the option as $\underline{i}$ and also hedged the option by selling off $\underline{C}_S$ stocks and financed the equity amount at $r^f + \psi^i$. The risk of the strategy would be zero, leaving only a deterministic dt-term times this value:

$$\begin{aligned}&\underline{C}_t + \frac{1}{2}\sigma^2 S^2 \underline{C}_{SS} - [K^i(x - \underline{C}_S, 1) - K^i(x, 0) + \underline{C}](r^f + \psi^i)\\&+ \underline{C}_S S(r^f - \tilde{l}^i) + [K^i(x - \underline{C}_S, 1) - K^i(x, 0)]r^f\end{aligned} \tag{A.23}$$

where $x \in \mathbb{R}$ is an arbitrary number of stocks held by i, and $\tilde{l}^i$ is equal to L^i if $x \leq 0$, equal to l^i if $x \geq \underline{C}_S$, and otherwise equal to $\frac{x}{\underline{C}_S} l^i + \frac{\underline{C}_S - x}{\underline{C}_S} L^i$ if $0 < x < \underline{C}_S$. The final term $[K^i(x - \underline{C}_S, 1) - K^i(x, 0)]$ represents the increase in required amount on the margin account (relative to not holding and hedging one option), and we add $\underline{C}$ in the third term to capture the change in required equity (as the hedge must have the same cash flow as if the option was sold for $\underline{C}$). Because of (4), the drift (A.23) is greater or equal to:

$$\begin{aligned}
&\underline{C}_t + \frac{1}{2}\sigma^2 S^2 \underline{C}_{SS} - [m^{\underline{i},S} S\underline{C}_S + (m^{\underline{i},C} - 1)\underline{C}]\psi^i - \underline{C}(r^f + \psi^i) + \underline{C}_S S(r^f - \tilde{l}^i) \\
=&\underline{C}_t + \frac{1}{2}\sigma^2 S^2 \underline{C}_{SS} - [m^{\underline{i},S} S\underline{C}_S + m^{\underline{i},C}\underline{C}]\psi^i - \underline{C}r^f + \underline{C}_S S(r^f - \tilde{l}^i) \\
\geq&\underline{C}_t + \frac{1}{2}\sigma^2 S^2 \underline{C}_{SS} - [m^{\underline{i},S} S\underline{C}_S + m^{\underline{i},C}\underline{C}]\psi^{\underline{i}} - \underline{C}r^f + \underline{C}_S S(r^f - L^{\underline{i}}) \\
=&0
\end{aligned} \tag{A.24}$$

Here, the inequality follows from $\psi^i \leq \psi^{\underline{i}}$, $L^i \leq L^{\underline{i}}$, $L^i \geq l^i \geq 0$, $\psi^i \geq 0$, and $r^f > 0$. The last equality follows from (6). This non-negative drift means that i must assign a value to the option no smaller than agent $\underline{i}$ does, because the option could be hedged and generate a risk-free payment of at least $\underline{C}$ while for sure not give any negative cash flow in the future; i.e. $C^i \geq \underline{C}$. Hence, exercise is dominated for any i.

Turning to the upper bound part of *(i)*, we first note that failing to exercise is dominated for $\bar{i} := \bar{i}^{\psi^i}$ when the stock price is above the upper bound. We show that failing to exercise is not dominated for i then the same is true for $\bar{i}$, implying that it is dominated for i not to exercise whenever $S(t) > \bar{B}^{\psi^i}(T-t)$ (proof by contrapositive).

At any time for which it is not dominated for i not to exercise early, i can hedge the option position by selling C^i_S stocks, leaving a deterministic dt term that must be equal zero:

$$\begin{aligned}
&C^i_t + \frac{1}{2}\sigma^2 S^2 C^i_{SS} - [K^i(x - C^i_S, 1) - K^i(x, 0) + C^i](r^f + \psi^i) \\
&+ C^i_S S(r^f - \tilde{l}^i) + [K^i(x - C^i_S, 1) - K^i(x, 0)]r^f = 0
\end{aligned} \tag{A.25}$$

If $\bar{i}$ assigned the value C^i to the option and also hedged an option by selling off C^i_S stocks, then the corresponding drift for $\bar{i}$ can be seen to be non-negative using (A.27) below. This non-negative drift implies that $\bar{i}$ must assign a value to the option that is not immediately exercised no smaller than the value assigned by i, $\bar{C}^{\psi^i} \geq C^i$. Hence, it is not dominated for $\bar{i}$ not to exercise the option early when it is not dominated for i not to exercise the option early.

Derivation of PDE for agent $\bar{i}^\psi$

For any $\psi \in \mathbb{R}_+$ we consider agent $\bar{i}^\psi$ with $\psi^{\bar{i}^\psi} = \psi$, who we show have the highest exercise boundary and option valuation among all agents with $\psi^i = \psi$. Agent $\bar{i}^\psi$ is always long stock. Let the function for the required amount on the margin account have the form

$$K^{\bar{i}^\psi}(x, y) = m^{\bar{i}^\psi,S} S|x| + (m^{\bar{i}^\psi,C} - 1)\bar{C}^\psi y \tag{A.26}$$

where $x \in \mathbb{R}$ is the number of stocks held, $y \in \mathbb{R}_+$ is the number of options held, and $\bar{C}^\psi$ is $\bar{i}^\psi$'s valuation of the option. Also, $m^{\bar{i}^\psi,S} \in [0, 1]$ and $m^{\bar{i}^\psi,C} \in [0, 1]$.

Let $l^{\bar{i}^\psi} = \min_{i\in\{j:\psi^j=\psi\}} l^i$ and for $i \in \{j : \psi^j = \psi\}$ let

$$\begin{aligned} &K^i(x_2, y) - K^i(x_1, y) \leq m^{\bar{i}^\psi,S}(x_2 - x_1)S \quad \text{for } x_1 \leq x_2 \text{ and } \forall y \in \mathbb{R} \\ &K^i(x, y_2) - K^i(x, y_1) \geq (m^{\bar{i}^\psi,C} - 1)(y_2 - y_1)\bar{C}^\psi \quad \text{for } y_2 \geq y_1 \geq 0 \text{ and } \forall x \in \mathbb{R} \end{aligned} \tag{A.27}$$

The first line expresses that an increase in the number of stocks does not increase the required amount on the margin account for agent i by more than it increases the required amount on the margin account for agent $\bar{i}^\psi$. The second line expresses that an increase in the number of options increases the amount $\bar{i}^\psi$ can borrow through the margin account by at least as much as it increases the amount agent i can borrow.

Next, we want to derive the PDE for agent $\bar{i}^\psi$ exercise boundary, $\bar{B}^\psi$, and option valuation, $\bar{C}^\psi$. Consider the portfolio dynamics for agent $\bar{i}^\psi$ of buying one (additional) option at price $\bar{C}^\psi$, hedging by selling (additional) $\bar{C}^\psi_S$ shares of the stock, and fully financing the strategy based on margin loans and the use of equity capital. The value of this fully-financed strategy evolves as according to:

$$\begin{aligned} &\left(\bar{C}^\psi_t + \frac{1}{2}\sigma^2 S^2 \bar{C}^\psi_{SS}\right) \mathrm{d}t + \bar{C}^\psi_S \mathrm{d}S(t) - (1 - m^{\bar{i}^\psi,C})\bar{C}^\psi r^f \mathrm{d}t - m^{\bar{i}^\psi,C}\bar{C}^\psi(r^f + \psi)\mathrm{d}t \\ &- \bar{C}^\psi_S \mathrm{d}S(t) + \bar{C}^\psi_S S\left(-m^{\bar{i}^\psi,S} r^f + m^{\bar{i}^\psi,S}(r^f + \psi) + (r^f - l^{\bar{i}^\psi})\right) \mathrm{d}t \end{aligned} \tag{A.28}$$

The first two terms simply represent the dynamics of the option (as seen in Eqn. (2)).

The next two terms represent the funding of the option. Specifically, $(1 - m^{\bar{i}^\psi,C})\bar{C}^\psi$ can be borrowed against the option at the money-market funding cost r^f. The remaining option value, the margin requirement $m^{\bar{i}^\psi,C}\bar{C}^\psi$, must be financed as equity at a rate of $r^f + \psi^{\underline{i}}$.

The second line of (A.28) represents the terms stemming from the stock position and its financing. The first term is the stock dynamics, given the $\bar{C}^\psi_S$ number of shares sold. The last three terms capture the various financing costs. The stock sold to hedge the option decreases the long stock position thereby decreasing the required cash on the margin account by $\bar{C}^\psi_S S m^{\bar{i}^\psi,S}$ which earns the interest r^f. This cash can instead be invested at the rate $r^f + \psi$. Agent $\bar{i}^\psi$ reduces the cash that it borrows on the stock lending account by $\bar{C}^\psi_S S$ thereby saving the interest paid on this amount. The interest rate on the stock lending account is $r^f - l^{\bar{i}^\psi}$. The stochastic terms cancel out. The fully financed strategy with deterministic drift must have drift equal zero for the option valuation to be correct. This leads to the PDE stated in (8). The conditions follow from the fact that the option is an American call option and ensure that the solution will entail the exercise boundary that optimizes the option value, c.f. Merton (1973) and Kim (1990).