On the Systematic Volatility of Unpriced Earnings
– Internet Appendix –

September 2013

1 Valuation of Unlevered Equity

The following matrix representation summarizes the dynamics of the three state variables in the economy.

$$X_{t+1} = \begin{bmatrix} 0 \\ w_c \\ w_d \end{bmatrix} + \begin{bmatrix} \rho & 0 & 0 \\ 0 & \alpha_c & 0 \\ 0 & 0 & \alpha_d \end{bmatrix} X_t + \begin{bmatrix} \sigma_x \, \epsilon_{x,t+1} \\ s_c \, \eta_{c,t+1} \\ s_d \, \eta_{d,t+1} \end{bmatrix} \tag{1}$$

$$= \mu + \Phi \, X_t + u_{t+1} \tag{2}$$

where $X_t \equiv \begin{bmatrix} x_t & v_{c,t} & v_{d,t} \end{bmatrix}^\top$. Let Ω denote the covariance matrix of the shocks, $\Omega \equiv E_t \left[u_{t+1} \, u_{t+1}^\top \right]$.

The log pricing kernel of the Epstein–Zin preference is given as

$$m_{t+1} = \theta \, \log \delta - \frac{\theta}{\psi} \, \Delta c_{t+1} + (\theta - 1) \, r_{c,t+1} \tag{3}$$

where $r_{c,t+1}$ denotes the log gross return on an asset that delivers aggregate consumption as its dividends each period. The parameter $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, with $\gamma \geq 0$ being the risk-aversion parameter and $\psi \geq 0$ the intertemporal elasticity of substitution (IES) parameter.

Following **?**, $r_{c,t+1}$ is log-linearly approximated as

$$r_{c,t+1} = \kappa_0 + \kappa_1 \, z_{c,t+1} - z_{c,t} + \Delta c_{t+1} \tag{4}$$

where $z_{c,t}$ is the log price-dividend ratio of the claim to aggregate consumption, and κ_0 and κ_1 are approximating constants that both depend only on the average level of z^c.

$$\kappa_1 = \frac{e^{\bar{z}_c}}{1 + e^{\bar{z}_c}} \tag{5}$$

$$\kappa_0 = \log\left(1 + e^{\bar{z}_c}\right) - \frac{e^{\bar{z}_c}}{1 + e^{\bar{z}_c}}\, \bar{z}_c \tag{6}$$

By plugging $z_{c,t} = A_0 + A^\top X_t$ and the formulas above into the Euler equation, $E_t\left[e^{m_{t+1}}\, e^{r_{c,t+1}}\right] = 1$, we can derive

$$A = \left(1 - \frac{1}{\psi}\right) \left(I - \kappa_1\, \Phi^\top\right)^{-1} \begin{bmatrix} 1 \\ \frac{1}{2}(1-\gamma) \\ 0 \end{bmatrix} \tag{7}$$

$$A_0 = \frac{1}{1 - \kappa_1} \left\{ \log\delta + \kappa_0 + \kappa_1\, A^\top \mu + \left(1 - \frac{1}{\psi}\right) \mu_c + \tfrac{1}{2}\theta\, \kappa_1^2\, A^\top \Omega\, A \right\} \tag{8}$$

Similarly, the log gross return on a firm's underlying asset (unlevered equity) can be approximated as

$$r_{d,t+1} = \kappa_2 + \kappa_3\, z_{d,t+1} - z_{d,t} + \Delta d_{t+1} \tag{9}$$

where $z_{d,t}$ represents the firm's log price-dividend ratio, and $\kappa_2 = \log\left(1 + e^{\bar{z}_d}\right) - \frac{e^{\bar{z}_d}}{1+e^{\bar{z}_d}}\, \bar{z}_d$ and $\kappa_3 = \frac{e^{\bar{z}_d}}{1+e^{\bar{z}_d}}$. By plugging $z_{d,t} = B_0 + B^\top X_t$ again into the Euler equation, its solutions can be derived as

$$B = \left\{I - \kappa_3 \Phi^\top\right\}^{-1} \begin{bmatrix} \phi - \frac{1}{\psi} \\ \frac{1}{2}\left\{(\gamma-\phi)^2 - (\gamma - 1)\left(\gamma - \frac{1}{\psi}\right)\right\} \\ \frac{1}{2} \end{bmatrix} \tag{10}$$

$$B_0 = \frac{1}{1-\kappa_3} \Big[\theta\, \log\delta + (\phi - \gamma)\, \mu_c + (\theta - 1)\left\{\kappa_0 + \kappa_1(A_0 + A^\top \mu) - A_0\right\}$$

$$+ \kappa_2 + \kappa_3 B^\top \mu + \tfrac{1}{2}\left\{(\theta-1)\kappa_1 A + \kappa_3 B\right\}^\top \Omega \left\{(\theta-1)\kappa_1 A + \kappa_3 B\right\}\Big] \tag{11}$$

Note that B_3 is not zero because of Jensen's inequality.

2 Analytic Solution of Debt and Levered Equity

Let's begin with a lemma that is critical to derive the analytic solution of debt and levered equity.

Lemma 1. *If i is a (2×1) constant vector and $X = (X_1, X_2)^\top$ is a bivariate normal with mean vector μ and covariance matrix Σ, then*

$$\int^{a<X_2<b} e^{X^\top i}\, dX = e^{\mu^\top i + \frac{1}{2} i^\top \Sigma\, i}\, \mathrm{P}\left(a < X_2 < b\,;\, \mu^*, \Sigma\,\right) \tag{12}$$

where $\mathrm{P}\left(\,\cdot\,;\, \mu^*, \Sigma\,\right)$ is the bivariate density function with mean vector $\mu^* = \mu + \Sigma\, i$ and covariance matrix Σ.

In this internet appendix, we assume the traditional **?**'s payoff functions.

$$F_T = \begin{cases} R\,(V_T + D_T) \\ K \end{cases}, \quad S_T = \begin{cases} 0 & \text{if } V_T + D_T < K \\ V_T + D_T - K & \text{if } V_T + D_T \geq K \end{cases} \tag{13}$$

where F_T denotes the payoff to corporate debts at maturity, S_T the payoff to levered equity, V_T the value of underlying assets, D_T dividends, K the principal amount of debts, and R the recovery rate in case of bankruptcy.

Suppose the maturity comes in one time period, $T = t + 1$. Today's price of these assets can be derived as

$$\begin{aligned} F_t &= E_t\left[e^{m_{t+1}} F_{t+1}\right] \\ &= K e^{-r_{f,t}}\, G\left(\omega - \sigma_r\right) + R\, V_t\, G\left(-\omega\right) \end{aligned} \tag{14}$$

$$\begin{aligned} S_t &= E_t\left[e^{m_{t+1}} S_{t+1}\right] \\ &= V_t\, G(\omega) - K e^{-r_{f,t}} G(\omega - \sigma_r) \end{aligned} \tag{15}$$

where $\omega \equiv \left\{\ln\left(V_t/K\right) + r_{f,t} + \frac{1}{2}\sigma_r^2\right\}/\sigma_r$, $\sigma_r^2 \equiv \mathrm{var}_t\left(r_{d,t+1}\right) = \phi^2\, v_{c,t} + v_{d,t} + \kappa_3^2\, B^\top \Omega\, B$ is the variance of unlevered asset returns, and $g(\,\cdot\,)$ and $G(\,\cdot\,)$ denote the pdf and cdf of standard normal random variable. Note that ω implies the firm's leverage relative to its volatility. Its comparative statistics are

$$\frac{\partial\,\omega}{\partial V_t} = \frac{1}{\sigma_r V_t}, \quad \frac{\partial\,\omega}{\partial \sigma_r} = 1 - \frac{\omega}{\sigma_r}, \quad \frac{\partial\,\omega}{\partial r_{f,t}} = \frac{1}{\sigma_r} \tag{16}$$

In the rest of this section, it is assumed that the firm is not underwater, i.e., $V_t \geq K\, e^{-r_{f,t}}$, thus $\omega > 0$.

The partial derivatives of the levered claims can be derived like those of **?**,

$$\frac{\partial S_t}{\partial V_t} = G(\omega) \qquad \frac{\partial F_t}{\partial V_t} = R\, G(-\omega) + \frac{1-R}{\sigma_r} g(\omega) \tag{17}$$

$$\frac{\partial S_t}{\partial \sigma_r} = V_t\, g(\omega) \qquad \frac{\partial F_t}{\partial \sigma_r} = -V_t\, g(\omega) \left\{ R + (1-R)\frac{\omega}{\sigma_r} \right\} \tag{18}$$

$$\frac{\partial S_t}{\partial r_{f,t}} = K\, e^{-r_{f,t}}\, G(\omega - \sigma_r) \qquad \frac{\partial F_t}{\partial r_{f,t}} = -K\, e^{-r_{f,t}}\, G(\omega - \sigma_r) + \frac{1}{\sigma_r} V_t\, g(\omega)(1-R) \tag{19}$$

And the comparative statistics of the Black-Scholes inputs with regard to our state variables are

$$\frac{\partial V_t}{\partial x_t} = V_t\, B_{(1)} \qquad \frac{\partial V_t}{\partial v_{c,t}} = V_t\, B_{(2)} \qquad \frac{\partial V_t}{\partial v_{d,t}} = V_t\, B_{(3)} \tag{20}$$

$$\frac{\partial \sigma_r}{\partial x_t} = 0 \qquad \frac{\partial \sigma_r}{\partial v_{c,t}} = \frac{\phi^2}{2\,\sigma_r} \qquad \frac{\partial \sigma_r}{\partial v_{d,t}} = \frac{1}{2\,\sigma_r} \tag{21}$$

$$\frac{\partial r_{f,t}}{\partial x_t} = \frac{1}{\psi} \qquad \frac{\partial r_{f,t}}{\partial v_{c,t}} = \frac{1-\gamma\,(1+\psi)}{2\psi} \qquad \frac{\partial r_{f,t}}{\partial v_{d,t}} = 0 \tag{22}$$

where $B_{(i)}$ denotes the i-th element of the column vector B in equation (**??**).

One interesting implications is that $\frac{\partial V_t}{\partial v_{c,t}} = V_t\, B_{(2)} < 0$ if $1 \leq \phi \leq 2\gamma - 1$ and $\psi > 1$, that is, (i) ($\phi \leq 2\gamma - 1$) if Jensen's inequality is not large enough to overcome the increase in risk premium and (ii) ($\phi \geq 1$) the claim to dividends is not a hedge to consumption risk. In other words, the unlevered asset value V_t can actually increase with $v_{c,t}$ if ϕ is either negative or excessively large. We will assume that ϕ is within the given range so that $B_{(2)} < 0$.

Now, let's study S_t's comparative statistics with each of the state variables.

(1) S_t and x_t

$$\frac{\partial S_t}{\partial x_t} = G(\omega)\, V_t\, B_{(1)} + K\, e^{-r_{f,t}}\, G(\omega - \sigma_r)\, \frac{1}{\psi} > 0 \quad \text{always} \tag{23}$$

(2) S_t and $v_{c,t}$

$$\frac{\partial S_t}{\partial v_{c,t}} = \underbrace{G(\omega)}_{(+)}\,\underbrace{V_t\,B_{(2)}}_{(-)} + \underbrace{V_t\,g(\omega)}_{(+)}\,\underbrace{\frac{\phi^2}{2\sigma_r}}_{(+)} + \underbrace{K\,e^{-r_{f,t}}\,G(\omega-\sigma_r)}_{(+)}\,\underbrace{\frac{1-\gamma(1+\psi)}{2\psi}}_{(-)\text{ if }\gamma>1} \tag{24}$$

$$\leq V_t\left\{G(\omega)B_{(2)} + g(\omega)\frac{\phi^2}{2\sigma_r}\right\} \tag{25}$$

$$\leq V_t\left\{G(\omega)B_{(2)} + g(\omega)\frac{\phi^2}{2\kappa_3\sqrt{B^\top\Omega\,B}}\right\} \tag{26}$$

$$\leq V_t\left\{G(\omega)B_{(2)} + g(\omega)\frac{\phi^2}{2\kappa_3\,\left(-s_c\,B_{(2)}\right)}\right\} \tag{27}$$

$$\therefore\quad \frac{\partial S_t}{\partial v_{c,t}} < 0 \quad \text{if } B_{(2)} < -\frac{\phi}{\sqrt{2\kappa_3 s_c}} \tag{28}$$

(3) S_t and $v_{d,t}$

$$\frac{\partial S_t}{\partial v_{d,t}} = G(\omega)\,V_t\,B_{(3)} + V_t\,g(\omega)\,\frac{1}{2\sigma_r} > 0 \quad \text{always} \tag{29}$$

Comparative statistics of corporate debts, F_t, are more complicated as they depend on the firm's leverage. For example, the long-run growth rate, x_t, can move F_t in either direction since it raises not only the underlying asset value (V_t) but also the riskfree interest rate ($r_{f,t}$). Let's assume $R = 1$ for the sake of simplicity. F_t's comparative statistics are derived as

(1) F_t and x_t

$$\frac{\partial F_t}{\partial x_t} = G(-\omega)\,V_t\,B_{(1)} - Ke^{-r_{f,t}}G(\omega-\sigma_r)\,\frac{1}{\psi} \tag{30}$$

$$= \underbrace{\frac{1}{\psi}K\,e^{-r_{f,t}}G(-\omega)}_{(+)}\left\{\frac{\phi\,\psi-1}{1-\kappa_3\rho}\,e^{\sigma_r\omega-\frac{1}{2}\sigma_r^2} - \frac{G(\omega-\sigma_r)}{G(-\omega)}\right\} \tag{31}$$

Since $\frac{G(\omega-\sigma_r)}{G(-\omega)} \in (0,\infty)$ is monotonically increasing in ω and rises faster than $e^{\sigma_r\omega}$, there exists ω^* such that $\frac{\partial F_t}{\partial x_t}(\omega^*) = 0$. Thus,

$$\begin{cases} \frac{\partial F_t}{\partial x_t} > 0 & \text{if leverage is high, i.e.,} \quad \ln(V_t/K) < \sigma_r\,\omega^* - r_{f,t} - \frac{1}{2}\sigma_r^2 \\ \frac{\partial F_t}{\partial x_t} < 0 & \text{if leverage is low, i.e.,} \quad \ln(V_t/K) > \sigma_r\,\omega^* - r_{f,t} - \frac{1}{2}\sigma_r^2 \end{cases} \tag{32}$$

(2) F_t and $v_{c,t}$

$$\frac{\partial F_t}{\partial v_{c,t}} = G(-\omega)\, V_t\, B_{(2)} - V_t\, g(\omega)\, \frac{\phi^2}{2\sigma_r} - K e^{-r_{f,t}} G(\omega - \sigma_r)\, \frac{1-\gamma\,(1+\psi)}{2\,\psi} \tag{33}$$

$$\lim_{\omega\to-\infty} \frac{\partial F_t}{\partial v_{c,t}} = V_t\, B_{(2)} < 0 \tag{34}$$

$$\lim_{\omega\to\infty} \frac{\partial F_t}{\partial v_{c,t}} = K e^{-r_{f,t}}\, \frac{\gamma\,(1+\psi)-1}{2\,\psi} > 0 \tag{35}$$

Since $\lim_{\omega\to-\infty} \frac{\partial F_t}{\partial v_{c,t}} < 0$, $\lim_{\omega\to\infty} \frac{\partial F_t}{\partial v_{c,t}} > 0$ and $\frac{\partial F_t}{\partial v_{c,t}}$ is continuous in ω, there exists $\hat{\omega}^*$ such that $\frac{\partial F_t}{\partial v_{c,t}}(\hat{\omega}^*) = 0$. Therefore,

$$\begin{cases} \frac{\partial F_t}{\partial v_{c,t}} < 0 & \text{if leverage is high, i.e.,} \quad \ln(V_t/K) < \sigma_r\, \hat{\omega}^* - r_{f,t} - \frac{1}{2}\sigma_r^2 \\ \frac{\partial F_t}{\partial v_{c,t}} > 0 & \text{if leverage is low, i.e.,} \quad \ln(V_t/K) > \sigma_r\, \hat{\omega}^* - r_{f,t} - \frac{1}{2}\sigma_r^2 \end{cases} \tag{36}$$

(3) F_t and $v_{d,t}$

Two opposite channels are effective: (i) $v_{d,t} \uparrow \Rightarrow V_t \uparrow$ (Jensen's inequality) $\Rightarrow F_t \uparrow$ and (ii) $v_{d,t} \uparrow \Rightarrow \sigma_r \uparrow \Rightarrow F_t \downarrow$.

$$\frac{\partial F_t}{\partial v_{d,t}} = G(-\omega)\, V_t\, B_{(3)} - V_t\, g(\omega)\, \frac{1}{2\sigma_r} \tag{37}$$

$$= V_t\, G(-\omega) \left\{ B_{(3)} - \frac{g(\omega)}{1-G(\omega)} \frac{1}{2\sigma_r} \right\} \tag{38}$$

Since $\frac{g(\omega)}{1-G(\omega)} \in (0,\infty)$ is monotonically increasing in ω, there exists $\tilde{\omega}^*$ such that $\frac{\partial F_t}{\partial v_{d,t}}(\tilde{\omega}^*) = 0$. However, since $\frac{g(\omega)}{1-G(\omega)} > 0.79$ for $\omega > 0$ as assumed previously,

$$\therefore \quad \frac{\partial F_t}{\partial v_{d,t}} < 0 \quad \text{if the Jensen's inequality is not dominant, i.e., } B_{(3)} \le \frac{0.79}{2\sigma_r} \tag{39}$$

In contrast to the previous two cases, the comparative statics of credit spreads, $cr \equiv -\ln(F_t/K) - r_{f,t}$, are much simpler.

(1) cr and x_t

$$\frac{\partial cr}{\partial x_t} = -\frac{V_t\, G(-\omega)}{F_t} \left\{ B_{(1)} + \frac{1}{\psi} \right\} < 0 \tag{40}$$

(2) cr and $v_{c,t}$

$$\frac{\partial cr}{\partial v_{c,t}} = -\frac{V_t}{F_t} \left\{ \left(\underbrace{B_{(2)}}_{(-)} + \underbrace{\frac{1-\gamma(1+\phi)}{2\,\psi}}_{(-)} \right) G(-\omega) \underbrace{- \frac{\phi^2}{2\sigma_r}}_{(-)} g(\omega) \right\} > 0 \tag{41}$$

(3) cr and $v_{d,t}$

$$\frac{\partial cr}{\partial v_{d,t}} = -\frac{1}{F_t} \frac{\partial F_t}{\partial v_{d,t}} > 0 \tag{42}$$

Now let's move on to expected excess returns. The excess returns of unlevered assets are determined by the covariance of its return with the pricing kernel,

$$-\,\sigma_{mr} \equiv -\,\mathrm{cov}_t\left(m_{t+1}, r_{d,t+1}\right) = \phi\,\gamma\, v_{c,t} + constant \tag{43}$$

Those of levered equity can be derived as

$$eer_S \equiv \ln\left(\frac{E_t[S_{t+1}]}{S_t}\right) - r_{f,t} \tag{44}$$

$$\exp(eer_S) = \frac{V_t\, e^{-\sigma_{mr}} G\left(\omega - \frac{\sigma_{mr}}{\sigma_r}\right) - K e^{-r_{f,t}} G\left(\omega - \sigma_r - \frac{\sigma_{mr}}{\sigma_r}\right)}{V_t\, G\left(\omega\right) - K e^{-r_{f,t}} G\left(\omega - \sigma_r\right)} \tag{45}$$

$$\therefore \quad eer_S \approx -\sigma_{mr} \underbrace{\frac{V_t}{S_t} G(\omega)}_{\text{leverage effect}} \qquad \text{by the Taylor approximation} \tag{46}$$

And below are the excess returns of corporate bonds.

$$eer_F \equiv \ln\left(\frac{E_t[F_{t+1}]}{F_t}\right) - r_{f,t} \tag{47}$$

$$\exp(eer_F) = \frac{V_t\, e^{-\sigma_{mr}} G\left(-\omega + \frac{\sigma_{mr}}{\sigma_r}\right) + K e^{-r_{f,t}} G\left(\omega - \sigma_r - \frac{\sigma_{mr}}{\sigma_r}\right)}{V_t\, G\left(-\omega\right) + K e^{-r_{f,t}} G\left(\omega - \sigma_r\right)} \tag{48}$$

$$\therefore \quad eer_F \approx -\sigma_{mr}\, \frac{V_t}{F_t}\, G(-\omega) \tag{49}$$

Note that the value-weighted average of stock and corporate bond excess returns is equal to the risk premium of the unlevered asset.

$$\frac{S_t}{V_t}\, eer_S + \frac{F_t}{V_t}\, eer_F = -\sigma_{mr} \tag{50}$$

Let's define the leverage factor of stocks as follows.

$$L_S(\omega) \equiv \frac{V_t}{S_t}\, G(\omega) = \frac{1}{1 - l_S(\omega)} \tag{51}$$

$$l_S(\omega) \equiv e^{-\sigma_r \omega + \frac{1}{2}\sigma_r^2}\, \frac{G(\omega - \sigma_r)}{G(\omega)} \tag{52}$$

Both $L_S(\omega)$ and $l_S(\omega)$ are monotonically decreasing in ω. They will turn useful in the derivation of the comparative statistics of eer_S that follow.

(1) eer_S and x_t

$$\frac{\partial\, l_S(\omega)}{\partial x_t} = \underbrace{\frac{\partial\, l_S(\omega)}{\partial\, \omega}}_{(-)} \cdot \underbrace{\frac{1}{\sigma_r}\left\{B_{(1)} + \frac{1}{\psi}\right\}}_{(+)} < 0 \tag{53}$$

Thus,

$$\frac{\partial\, eer_S}{\partial x_t} < 0 \tag{54}$$

(2) eer_S and $v_{c,t}$

it is difficult to tell the sign of $\frac{\partial\, eer_S}{\partial v_{c,t}}$ as it not only raises risk premium ($-\sigma_{mr}$) but also affects the leverage.

(3) eer_S and $v_{d,t}$

$$\frac{\partial\, eer_S}{\partial v_{d,t}} < 0 \quad \text{if } \omega > \sigma_r \tag{55}$$

Similarly, let's define the leverage factor of corporate debts as follows.

$$L_F(\omega) \equiv \frac{V_t}{F_t} G(-\omega) = \frac{1}{1 + l_F(\omega)} \tag{56}$$

$$l_F(\omega) \equiv e^{-\sigma_r \omega + \frac{1}{2}\sigma_r^2} \frac{G(\omega - \sigma_r)}{G(-\omega)} \tag{57}$$

$l_F(\omega)$ is monotonically increasing in ω and $L_F(\omega)$ monotonically decreasing.

(1) eer_F and x_t

$$\frac{\partial\, l_F(\omega)}{\partial x_t} = \underbrace{\frac{\partial\, l_F(\omega)}{\partial \omega}}_{(+)} \cdot \underbrace{\frac{1}{\sigma_r}\left\{B_{(1)} + \frac{1}{\psi}\right\}}_{(+)} > 0 \tag{58}$$

Thus,

$$\frac{\partial\, eer_F}{\partial x_t} < 0 \tag{59}$$

At first glance, this result may look counter-intuitive since $\frac{S_t}{V_t} eer_S + \frac{F_t}{V_t} eer_F = -\sigma_{mr}$, $\frac{\partial\, eer_S}{\partial x_t} < 0$, $\frac{\partial\, eer_F}{\partial x_t} < 0$ but $\frac{\partial \sigma_{mr}}{\partial x_t} = 0$. Note that, however, the value-weights themselves, $\frac{S_t}{V_t}$ and $\frac{F_t}{V_t}$, are also determined by x_t. In other words, x_t is related to the spread between eer_S and eer_F.

(2) eer_F and $v_{c,t}$

$$\frac{\partial\, l_F}{\partial v_{c,t}} < 0 \quad \text{if } \omega > \sigma_r \tag{60}$$

Thus,

$$\frac{\partial\, eer_F}{\partial v_{c,t}} = \phi\, \gamma\, L_F(\omega) - \sigma_{mr} \frac{\partial\, L_F}{\partial v_{c,t}} > 0 \tag{61}$$

(3) eer_F and $v_{d,t}$

$$\frac{\partial\, l_F}{\partial v_{d,t}} < 0 \quad \text{if } \omega > \sigma_r \text{ and } B_{(3)}(\text{Jensen's inequality}) \text{ is not strong} \tag{62}$$

Thus,

$$\frac{\partial\, eer_F}{\partial v_{d,t}} > 0 \tag{63}$$