Internet Appendix

A Estimation via Alternating Least Squares

This appendix describes our ALS approach to optimizing IPCA objective function (5). To initialize the algorithm, we choose a starting guess for Γ_β as the left eigenvectors corresponding to the leading K eigenvalues of the characteristic-managed portfolio second moment matrix, $\sum_t x_t x_t'$. As described in Section 2, this initial guess, which amounts to the static loading matrix estimate from applying PCA to the x_t dataset, is a close approximation to the exact solution of (5) as long as $Z_t'Z_t$ is not too volatile. Initializing the optimization with this guess ensures that the algorithm converges very quickly (typically within 10 iterations, taking roughly 0.25 seconds on a standard desktop computer).

Given the initial guess for Γ_β, we evaluate the least squares regression corresponding to first-order condition (6) for all t. Then, given the resulting solutions f_t's, we evaluate the least squares regression corresponding to first-order condition (7). We iterate between evaluations of (6) and (7) until convergence, defined as the point at which the maximum absolute change in any element of Γ_β or f_t (for all t) is smaller than 10^{-6}.

B Orthonormal Characteristics: An Exact Analytical Estimator

An alternative approach to characteristic standardization is to cross-sectionally orthonormalize characteristics each period, so that $Z_t'Z_t = \mathbb{I}_L$ for all t. A convenient feature of this normalization is that the IPCA estimator of Γ_β becomes directly calculable via singular value decomposition with no need for numerical optimization.

This result follows from the concentrated IPCA objective described in equation (8) of Section 2. When instruments are orthonormal period-by-period, (8) reduces to

$$\max_{\Gamma_\beta} \quad \mathrm{tr}\left(\sum_{t=1}^{T-1} \left(\Gamma_\beta'\Gamma_\beta\right)^{-1} \Gamma_\beta' Z_t' r_{t+1} r_{t+1}' Z_t \Gamma_\beta\right). \tag{17}$$

That is, the objective function collapses to a sum of homogeneous Rayleigh quotients. As a result, the K leading eigenvectors of $\sum_t Z_t' r_{t+1} r_{t+1}' Z_t$ satisfy the maximization problem and thus estimate Γ_β, a solution well known from the PCA literature.

More specifically, we derive the algebraic solution for Γ_β via the following eigenvalue decomposition:

$$USU' = \sum_t Z_t' r_{t+1} r_{t+1}' Z_t,$$

The IPCA estimator of Γ_β is

$$\hat{\Gamma}_\beta = U_K$$

where the columns of U are arranged in decreasing eigenvalue order and U_K denotes the first K columns of U. The factor estimates are likewise analytical, and simplify to

$$\hat{f}_{t+1} = \hat{\Gamma}_\beta'(Z_t' r_{t+1}).$$

One can directly impose characteristic orthonormality in the data. In particular, given some matrix of "raw" instruments $\tilde{Z}_t$, we recommend constructing orthonormal instruments Z_t using the Gram-Schmidt process. This uses regression to sequentially orthogonalize instruments in the cross section each period, then cross-sectionally volatility-standardizes the residuals. This orthogonalization is not invariant to the ordering of characteristics. As a result, our tests of individual characteristics can be influenced by the order of the instruments.