Online Apendixes for

Portfolio Rebalancing in General Equilibrium

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Appendix A: Stationary Markov Equilibrium

A.1 Definition of Equilibrium

This appendix defines the equilibrium and discusses its solution. For convenience, Section 4.5.3 is repeated here. As discussed in Section 4.5.3, the time-invariant equilibrium functions associated with the model's variables are denoted in **bold**. Consistent with the notation tradition of the literature on Markov equilibria, the prime ($'$) denotes the next period variables or functions.

The idiosyncratic dividend growth rate shock G has a finite support $\{G_n\}_{n=1}^{N}$ with probability π_n for each G_n. The middle-aged post-shock wealth $w = (w_M^L, w_M^H)$ is the state variable of the economy. We will use lower-case letters of variables to indicate the values of the variables divided by same-dated dividends: $x = X/D$. ρ_Y and ρ_M denote the marginal propensities to consume for the young and the middle aged, respectively. ψ_j denotes the constant fraction of agents of type j in each cohort. γ_j is the risk aversion parameter of type j. α denotes the capital share, so the labor income of the young is $w_Y^j = w_Y = \frac{1-\alpha}{\alpha}$ for both $j = H, L$.

We first define the recursive equilibrium by collecting the equations that fully characterize the equilibrium, and then discuss these equations in more detail. Equation numbers in the square brackets denote the corresponding equations in the main text. (Note that the cross-referenced equations are not normalized by dividends [capital letters instead of lower case letters].)

The recursive equilibrium consists of price **functions** $\{\boldsymbol{p}(w), \boldsymbol{R}(w), \boldsymbol{Z}(w, G')\}$, allocations $\{\boldsymbol{c_Y^j}(w),\ \boldsymbol{c_M^j}(w),\ \boldsymbol{c_O}(w),\ \boldsymbol{\theta_Y^j}(w),\ \boldsymbol{\theta_M^j}(w)\}_{j=H,L}$, associated return functions $\{\boldsymbol{\phi_M^j}(w)\}_{j=H,L}$, and the law of motion of wealth $\boldsymbol{w'}(w, G') = (\boldsymbol{w_M^{L\prime}}(w, G'), \boldsymbol{w_M^{H\prime}}(w, G'))$, such that

1. Given prices and laws of motion of wealth, the allocations solve the household problems:

$$\boldsymbol{c_Y^j}(w) = \rho_Y w_Y, \quad [26] \tag{SM.1}$$

$$\boldsymbol{c_M^j}(w) = \rho_M w_M^j, \quad [19] \tag{SM.2}$$

$$\boldsymbol{c_O}(w) = 1 + \boldsymbol{p}(w) - (\psi_H w_M^H + \psi_L w_M^L), \quad [34] \tag{SM.3}$$

$$\boldsymbol{Z}(w, G') = \frac{(\boldsymbol{p}(\boldsymbol{w'}(w, G')) + 1)G'}{\boldsymbol{R}(w)\boldsymbol{p}(w)}, \quad [8] \tag{SM.4}$$

$$\boldsymbol{\theta_M^j}(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\{1 - \theta + \theta \boldsymbol{Z}(w, G')\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [15] \tag{SM.5}$$

$$\boldsymbol{\phi_M^j}(w) = \boldsymbol{R}(w) \left(E_{G'} \left[\left\{ 1 - \boldsymbol{\theta_M^j}(w) + \boldsymbol{\theta_M^j}(w) \boldsymbol{Z}(w, G') \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}, \quad [16] \tag{SM.6}$$

$$\boldsymbol{\theta_Y^j}(w) = \arg\max_{\theta \in \Theta} \left(E_{G'} \left[\left\{ [1 - \theta + \theta \boldsymbol{Z}(w, G')] \boldsymbol{\phi_M^j}(\boldsymbol{w'}(w, G'))^{1-\rho_M} \right\}^{1-\gamma_j} \right] \right)^{\frac{1}{1-\gamma_j}}. \quad [27] \tag{SM.7}$$

2. Prices clear goods markets:

$$\boldsymbol{p}(w) = (1 - \rho_Y) w_Y + (1 - \rho_M)(\psi_H w_M^H + \psi_L w_M^L), \quad [36] \tag{SM.8}$$

and bond markets:

$$0 = \sum_{j=H,L} \psi_j \left\{ (1 - \rho_M) w_M^j (1 - \boldsymbol{\theta_M^j}(w)) + (1 - \rho_Y) w_Y (1 - \boldsymbol{\theta_Y^j}(w)) \right\}. \quad [32] \tag{SM.9}$$

Note that R_t is determined implicitly in the bond market equation through the dependency of the portfolio shares on it.

3. Wealth evolves consistent with portfolio allocations and prices:

$$\boldsymbol{w_M^{H\prime}}(w, G') = \left[(1 - \boldsymbol{\theta_Y^H}(w)) \frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta_Y^H}(w) \frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y) w_Y. \quad [38] \tag{SM.10}$$

$$\boldsymbol{w_M^{L\prime}}(w, G') = \left[(1 - \boldsymbol{\theta_Y^L}(w)) \frac{\boldsymbol{R}(w)}{G'} + \boldsymbol{\theta_Y^L}(w) \frac{1 + \boldsymbol{p}(\boldsymbol{w'}(w, G'))}{\boldsymbol{p}(w)} \right] (1 - \rho_Y) w_Y, \quad [39] \tag{SM.11}$$

The first three consumption equations are relatively straightforward. Equations (SM.1) and (SM.2) specify optimal consumption of young and middle-aged households, derived from their savings-consumption tradeoffs under inter-temporal log utility, and old households consume everything as in equation (SM.3).

The next four equations are related to the portfolio problems. To streamline the portfolio problems of the young and the middle-aged, we define two return functions in equations (SM.4) and (SM.6). The excess return function $\boldsymbol{Z}(w, G')$ in equation (SM.4) specifies the excess return of risky tree investment conditional on the dividend growth shock G' next period. The total return function $\boldsymbol{\phi^j_M}(w)$ in equation (SM.6) gives the total return on savings under the optimal portfolio choice for the middle-aged, which are in the utility form given the Epstein-Zin-Weil preferences. Equation (SM.5) gives the optimal risky asset share for the middle-aged households. Equation (SM.7) gives the optimal risky asset shares of the young.

The market clearing conditions specify the equilibrium price functions. The goods-market clearing condition in equation (SM.8) is actually savings supply equals savings demand, and specifies the tree price as a function of the sum of middle aged wealth, and the interest rate function $\boldsymbol{R}(w)$ is implicitly given by the bond market clearing condition in equation (SM.9), where optimal portfolio shares are functions of the interest rate R.

Let's discuss in greater detail the law of motion for the middle-aged wealth vector. The next-period middle-aged wealth vector is given by the portfolio choice of the young in the current period, prices, and returns conditional on the dividend growth shock G' next period. Specifically, consider a young household of type $j = \{H, L\}$ with savings of $(1-\rho_Y)w_Y$. By investing optimally a share $\boldsymbol{\theta^j_Y}(w)$ of her savings in the risky asset, her returns at the beginning of the next period (turning middle-aged) after the dividend growth rate shock G' are given by equations (SM.11) and (SM.10). In the square brackets of the right-hand side, the first term is the return on the part of savings in bonds and the second term is the return on the part of savings in the tree. The total returns from both bonds and tree imply their wealth when middle-aged for each possible dividend growth rate shock.

To derive the law of motion of wealth in a way that makes the Markov nature more transparent, plug in the following relation into equations (SM.11) and (SM.10)

$$\boldsymbol{p}(\boldsymbol{w'}(w, G')) = (1-\rho_M)\left(w_Y + \psi_H \boldsymbol{w^{H\prime}_M}(w, G') + \psi_L \boldsymbol{w^{L\prime}_M}(w, G')\right).$$

We derive

$$\begin{aligned}
&\left(\frac{1}{\psi_L} - \frac{(1-\rho_M)(1-\rho_Y)w_Y\boldsymbol{\theta^L_Y}(w)}{\boldsymbol{p}(w)}\right)\psi_L\boldsymbol{w^{L\prime}_M}(w, G') - \frac{(1-\rho_M)(1-\rho_Y)w_Y\boldsymbol{\theta^L_Y}(w)}{\boldsymbol{p}(w)}\psi_H\boldsymbol{w^{H\prime}_M}(w, G') \\
= \quad & (1-\rho_Y)w_Y\left[\frac{(1-\boldsymbol{\theta^L_Y}(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta^L_Y}(w)\left(1+(1-\rho_M)\,w_Y\right)}{\boldsymbol{p}(w)}\right],
\end{aligned}$$

$$\begin{aligned}
&-\frac{(1-\rho_M)(1-\rho_Y)w_Y\boldsymbol{\theta^H_Y}(w)}{\boldsymbol{p}(w)}\psi_L\boldsymbol{w^{L\prime}_M}(w, G') + \left(\frac{1}{\psi_H} - \frac{(1-\rho_M)(1-\rho_Y)w_Y\boldsymbol{\theta^H_Y}(w)}{\boldsymbol{p}(w)}\right)\psi_H\boldsymbol{w^{H\prime}_M}(w, G') \\
= \quad & (1-\rho_Y)w_Y\left[\frac{(1-\boldsymbol{\theta^H_Y}(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta^H_Y}(w)\left(1+(1-\rho_M)w_Y\right)}{\boldsymbol{p}(w)}\right].
\end{aligned}$$

Thus, we have

$$(A_1 + \frac{1}{\psi_L})\psi_L\boldsymbol{w^{L\prime}_M}(w, G') + A_1\psi_H\boldsymbol{w^{H\prime}_M}(w, G') = A_2,$$

$$B_1\psi_L \boldsymbol{w}_{\boldsymbol{M}}^{\boldsymbol{L'}}(w,G') + (B_1 + \frac{1}{\psi_H})\psi_H \boldsymbol{w}_{\boldsymbol{M}}^{\boldsymbol{H'}}(w,G') = B_2,$$

where

$$A_1 = -\frac{(1-\rho_M)(1-\rho_Y)w_Y \boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{L}}(w)}{\boldsymbol{p}(w)}$$

$$A_2 = (1-\rho_Y)w_Y\left[\frac{(1-\boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{L}}(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{L}}(w)\,(1+(1-\rho_M)\,w_Y)}{\boldsymbol{p}(w)}\right],$$

$$B_1 = -\frac{(1-\rho_M)(1-\rho_Y)w_Y \boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{H}}(w)}{\boldsymbol{p}(w)},$$

$$B_2 = (1-\rho_Y)w_Y\left[\frac{(1-\boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{H}}(w))\boldsymbol{R}(w)}{G'} + \frac{\boldsymbol{\theta}_{\boldsymbol{Y}}^{\boldsymbol{H}}(w)\,(1+(1-\rho_M)w_Y)}{\boldsymbol{p}(w)}\right].$$

And we have

$$\boldsymbol{w}_{\boldsymbol{M}}^{\boldsymbol{H'}}(w,G') = \frac{A_2B_1 - (A_1+\frac{1}{\psi_L})B_2}{A_1B_1 - (A_1+\frac{1}{\psi_L})(B_1+\frac{1}{\psi_H})}\frac{1}{\psi_H} \tag{SM.12}$$

$$\boldsymbol{w}_{\boldsymbol{M}}^{\boldsymbol{L'}}(w,G') = \frac{A_2(B_1+\frac{1}{\psi_H}) - A_1B_2}{(A_1+\frac{1}{\psi_L})(B_1+\frac{1}{\psi_H}) - A_1B_1}\frac{1}{\psi_L}. \tag{SM.13}$$

The next-period middle-aged wealth w' is given by the optimal portfolio choice of the young and the prices, which, aside from the shock G', are both functions solely of current middle-aged wealth w.

A.2 Existence of Equilibria

This section establishes the existence of the recursive equilibrium for our model. We first state the result and then illustrate the proof.

PROPOSITION. There exists a recursive equilibrium that satisfies equations (SM.1)-(SM.11).

Duffie, Geanokoplos, Mas-Colell, and McLennan (1994) have a result (Proposition 1.2) under quite general conditions that if an equilibrium always exists for any finite-horizon, then an ergodic Markov equilibrium also exists in the infinite horizon. In this section, we will show that their general proof applies to our specific setting. Their proposition is stated as follows.

> **PROPOSITION 1.2**: Let $G : S \to \mathcal{P}(S)$ be an expectations correspondence and K be a compact subset of S. If, for every $T \in N$, there exists a T-horizon equilibrium $\{S_1, ...S_T\}$ for G such that $S_t \in K$ almost surely for all t, then G has an ergodic Markov equilibrium.

Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) establish the existence of ergodic Markov equilibria for a stochastic overlapping generations model in Theorem 2.1 as an example of their general theory. Their overlapping generations model is an endowment economy with very general assumptions on fundamentals. It allows for l goods, n assets and m types of agents in each cohort with an exogenous Markov shock process. They first establish that an equilibrium exists for any finite horizon of their overlapping generations model, and then invoke Proposition 1.2 to establish that a Markov equilibrium exists in the infinite horizon.

Their example nests our overlapping generations model as a special case with $l = 1$, $n = 2$, $m = 2$, and an i.i.d. shock process, except for two minor differences. One minor difference is that agents live for two periods in their setup instead of three periods in ours. The other difference is that they assume that the utility is von Neumann-Morgenstern, while ours is Epstein-Zin-Weil. These two minor modifications in our model do not affect the arguments of their proof. The difference between two and three periods does not change the basic argument of the proof. Furthermore, their proof uses only the continuity, monotonicity, and concavity of the utility function, and the Epstein-Zin-Weil preferences have all these three properties. The Epstein-Zin-Weil preferences do not satisfy the *Independence of Irrelevant Alternatives* (IIA) of the von Neumann-Morgenstern utility, but this property is not required for the proof. Thus, the generalization of the preferences does not change the proof either.

This appendix therefore proceeds by (a) giving a formal proof closely following an OLG theorem of Duffie, Geanokoplos, Mas-Colell, and McLennan that an equilibrium exists for any finite horizon, and then (b) formally applying Proposition 1.2 to establish that a Markov equilibrium exists in the infinite horizon.

Below we show that their proof that equilibria exist for any finite horizon $T \in \mathbb{N}$ goes through with our two minor modifications. The original proof was omitted in their paper. We thank Andrew McLennan for providing us a copy of the omitted proof, which is attached with this document for your reference. We go through their proof line by line, taking care of any minor modifications that we have to make for our setup. We adopt their notation in this section of the appendix for an easy comparison.

We now outline a proof, similar to Radner's (1972), that T-horizon equilibria exist for any T. Fix $T \geq 1$. The space of T-period deterministic price paths is P, the set of pairs (p, q) where $p = (p_1, ..., p_T)$ and $q = (q_1, ..., q_T)$ are T-tuples of functions $p_t : G^t \to [0, 1]^2$ and $q_t : G^t \to [0, 1]$ with $(p_t(G_1, G_2, ..., G_t), q_t(G_1, G_2, ..., G_t)) \in \Delta^3$ for all t and $(G_1, G_2, ..., G_t)$. p is asset price and q is goods price. Δ^3 denotes the price simplex. In our model, there are two types of assets, risk-free bonds and risky trees, and one good, implying that p_t has two dimensions and q_t has one dimension.

The truncated consumption set is $\aleph = [0, L]$ where L is a number large enough to include the total endowment in any period. In our setup, $L > \frac{1}{\alpha} > 1$, since $\frac{1}{\alpha}$ is total endowment of goods in every period. The truncated portfolio set is $\Gamma = [-L, L]^2$. For the risky tree, the demand is bounded in $[0, 1]$, and for the risk-free bond, it is bounded by $[-L, L]$. These bounds are introduced to ensure that excess demand is bounded and compact-valued. In equilibrium, these constraints do not bind.

<u>The first difference</u> between our setting and that of Duffie et al. is that we have a three-

period rather than two-period generations model. That implies that we need to take care of an extra cohort in each period and one extra period for each generation. As is shown below, this generalization is straightforward.

An excess demand is a tuple $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$, where $\zeta_Y = (\zeta_{Y1}^H, \zeta_{Y1}^L, ..., \zeta_{YT}^H, \zeta_{YT}^L)$, $\zeta_M = (\zeta_{M1}^H, \zeta_{M1}^L, ..., \zeta_{MT}^H, \zeta_{MT}^L)$, $\zeta_O = (\zeta_{O1}^H, \zeta_{O1}^L, ..., , \zeta_{OT}^H, \zeta_{OT}^L)$, $\xi_Y = (\xi_{Y1}^H, \xi_{Y1}^L, ..., \xi_{YT}^H, \xi_{YT}^L)$, and $\xi_M = (\xi_{M1}^H, \xi_{M1}^L, ..., \xi_{MT}^H, \xi_{MT}^L)$ are T-tuples of functions $\zeta_{Yt}^i : G^t \to \mathbb{R}$, $\zeta_{Mt}^i : G^t \to \mathbb{R}$, $\zeta_{Ot}^i : G^t \to \mathbb{R}$, $\xi_{Yt}^i : G^t \to \Gamma$, and $\xi_{Mt}^i : G^t \to \Gamma$. Here ζ_{jt}^i denotes the excess demand for goods for cohort $j = \{Y, M, O\}$ and type $i = \{H, L\}$ in period t, and ξ_{jt}^i denotes the excess demand for assets for cohort $j = \{Y, M\}$ and type $i = \{H, L\}$ in period t.

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *(truncated) individually feasible* if, for all i, t, and G^t,

$$e_{Yt}^i(G_t) + \zeta_{Yt}^i(G^t) \in \aleph,$$

$$e_{Mt}^i(G_t) + \sum_a \xi_{Yat-1}^i(G_t) d_a(G_t) + \zeta_{Mt}^i(G^t) \in \aleph,$$

$$e_{Ot}^i(G_t) + \sum_a \xi_{Mat-1}^i(G_t) d_a(G_t) + \zeta_{Ot}^i(G^t) \in \aleph,$$

where e_{jt}^i denotes the endowment of agent of type i and age j in period t; ξ_{jat}^i is the excess demand of asset $a = 1, 2$ of type i and age j in period t, and d_a denotes the dividend paid out by asset a in period t. In our setup, only the young have the labor endowment $e_{Yt}^i(G_t)$, which is proportional to the dividend, so $e_{Mt}^i(G_t) = e_{Ot}^i(G_t) = 0$.

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *budget feasible* for price path (p, q) if, for all i, t, and G^t,

$$p_t(G^t) \cdot \xi_{Yt}^i(G^t) + q_t(G^t) \cdot \zeta_{Yt}^i(G^t) \le 0,$$

$$p_t(G^t) \cdot \xi_{Mt}^i(G^t) + q_t(G^t) \cdot \zeta_{Mt}^i(G^t) \le p_t(G^t) \cdot \xi_{Yt-1}^i(G^{t-1}),$$

$$q_t(G^t) \cdot \zeta_{Ot}^i(G^t) \le p_t(G^t) \cdot \xi_{Mt-1}^i(G^{t-1}).$$

The excess demand $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is *optimal* for price path (p, q) if it is individually feasible, budget feasible for (p, q), and no individual has another individually feasible and budget feasible plan that yields higher utility. Note that since preferences are increasing, continuous, and concave, the set of optimal excess demands for (p, q) is compact and convex.

<u>The second difference</u> between our setting and theirs is that we use the Epstein-Zin-Weil preferences, while they use the von Neumann-Morgenstern utility. Note that the utility function properties used in the proof apply equally to both their and our preferences.

An *aggregate excess demand* is a pair $\Psi = (\psi_\theta, \psi_x)$ where $\psi_\theta = (\psi_{\theta 1}, ..., \psi_{\theta T})$ and $\psi_x = (\psi_{x1}, ..., \psi_{xT})$ are T-tuples of functions $\psi_{\theta t} : G^t \to \mathbb{R}^2$ and $\psi_{xt} : G^t \to \mathbb{R}$. The aggregate excess demand (ψ_θ, ψ_x) is said to be *derived* from $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ if for all t and G^t

$$\psi_{\theta t}(G^t) = \sum_i \left(\xi_{Yt}^i(G^t) + \xi_{Mt}^i(G^t) \right),$$

and

$$\psi_{xt}(G^t) = \sum_i \left(\zeta^i_{Yt}(G^t) + \zeta^i_{Mt}(G^t) + \zeta^i_{Ot}(G^t)\right).$$

Note that if $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$ is budget feasible for (p, q) and (ψ_θ, ψ_x) is derived from $(\zeta_Y, \zeta_M, \zeta_O, \xi_Y, \xi_M)$, then

$$p_t(G^t) \cdot \psi_{\theta t}(G^t) + q_t(G^t)\psi_{xt}(G^t) \leq 0,$$

for all t and G^t. This is Walras' Law.

For each price path (p, q) let $\Psi(p, q)$ be the set of aggregate excess demands derived from excess demands that are optimal for (p, q). The usual arguments show that Ψ is a compact convex valued upper semicontinuous correspondence. In the proof McLennan provided, Duffie, Geanakoplos, Mas-Colell, and McLennan established a generalization of the Debreu-Gale-Kuhn-Nikaido lemma as follows.

> **Proposition**: Let A and B be positive integers, and let $\Delta^A = \{\rho \in \mathbb{R}^A_+ | \sum_a \rho_a = 1\}$. Suppose that $F : (\Delta^A)^B \to (\mathbb{R}^A)^B$ is an upper semicontinuous compact convex valued correspondence whose image is contained in a bounded subset of $(\mathbb{R}^A)^B$, and suppose that $r_b \cdot z_b \leq 0$, $b = 1, ..., B$, whenever $z = (z_1, ..., z_B) \in F(r)$. Then there exists $r^\star \in (\Delta^A)^B$ and $z^\star \in F(r^\star)$ with $z^\star \leq 0$.

This Proposition guarantees the existence of a price path $(p^\star, q^\star)$ and an aggregate excess demand $(\psi^\star_\theta, \psi^\star_x) \in \Psi(p^\star, q^\star)$ with $\psi^\star_\theta(G^t) \leq 0$ and $\psi^\star_x(G^t) \leq 0$ for all t and G^t. Let $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ be an optimal excess demand vector for $(p^\star, q^\star)$ whose derived aggregate excess demand is $(\psi^\star_\theta, \psi^\star_x)$. To show that $(p^\star, q^\star)$ and $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ constitute a T-period equilibrium it now suffices to show that the plans for individuals specified by $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ are optimal, i.e., the constraints $\zeta^{i\star}_{jt} \in \aleph$ and $\xi^{i\star}_{jt} \in \Gamma$ do not rule out utility-improving plans that are budget feasible and individually feasible in the absence of these artificial constraints. But $(\zeta^\star_Y, \zeta^\star_M, \zeta^\star_O, \xi^\star_Y, \xi^\star_M)$ specifies consumption bundles and portfolios that are feasible in the aggregate, so the artificial constraints are satisfied with strict inequality, and the desired result follows from the concavity of all utility functions. Thus, we establish the existence of equilibrium in any finite horizon case for our setting.

Since the amount of each good available in each state and date is bounded uniformly, both above and away from zero, there is a compact set K almost surely containing all states reached in any such T-horizon equilibrium. Therefore, Proposition 1.2 implies the existence of a stationary Markov equilibrium in our setting for the infinite horizon.

To recap the strategy of the proof, we cite Duffie et. al. page 763:

> We induce a T-period equilibria for all finite horizons $T \in \mathbb{N}$ as follows. ... For each T these preferences induce a T-period finite-state Radner (1972) style event-tree economy. Then, by an argument similar to Radner's, we establish the

> existence of an S^T-valued random variable that is a T-horizon equilibrium for G, in the sense of our central results. ... Since the amount of each good available in each state and date is bounded uniformly, both above and away from zero, there is a compact set K almost surely containing all states reached in any such T-horizon equilibrium. Proposition 1.2 now implies the existence of a compact self-justified set J for g, and the desired conclusion follows from Proposition 1.3.

This section shows that their proof applies to our model. Therefore, there exists a stationary Markov equilibrium for our model.

A.3 Equilibrium Solution

Having shown the existence of Markov equilibria, we now describe how to find such an equilibrium. This section details the computation algorithm in our paper. We use backward induction to solve for time-invariant equilibrium functions. We start from a terminal period to serve as the transversality condition and to rule out sunspots. We solve the model backward from the last period. In each period t, we obtain the optimal policy functions $\{\boldsymbol{c^j_{Yt}}(w),\ \boldsymbol{c^j_{Mt}}(w),\ \boldsymbol{c^j_{Ot}}(w),\ \boldsymbol{\theta^j_{Yt}}(w),\ \boldsymbol{\theta^j_{Mt}}(w),\ \boldsymbol{R_t}(w),\ \boldsymbol{p_t}(w),\ \boldsymbol{Z_t}(w, G_{t+1}),\ \boldsymbol{w'_t}(w, G_{t+1})\}$. We move backward until the optimal policy functions in period t and period $t-1$ converge to obtain the time-invariant equilibrium functions $\{\boldsymbol{c^j_Y}(w),\ \boldsymbol{c^j_M}(w),\ \boldsymbol{c^j_O}(w),\ \boldsymbol{\theta^j_Y}(w),\ \boldsymbol{\theta^j_M}(w),$ $\boldsymbol{R}(w),\ \boldsymbol{p}(w),\ \boldsymbol{Z}(w, G'),\ \boldsymbol{w'}(w, G')\}$. Note that we need to index functions by time since the solution iterates backwards until the policy functions converge to the stationary equilibrium. Now we describe this backward induction.

A.3.1 Last Period T

Optimal choices at time T are simple: all agents consume their wealth and the stock price is zero. Since there is no saving, the portfolio share is irrelevant. Thus, for $j = H$ and L, we have

$$\boldsymbol{p_T}(w^H_M, w^L_M) = 0,$$
$$\boldsymbol{c^j_{YT}}(w^H_M, w^L_M) = (1-\alpha)/\alpha,$$
$$\boldsymbol{c^j_{MT}}(w^H_M, w^L_M) = w^j_M,$$
$$\boldsymbol{c_{OT}}(w^H_M, w^L_M) = 1 - \psi_H w^H_M - \psi_L w^L_M.$$

Nonnegative consumption for all agents implies the set of state variables in period T is $\Omega^w_T = \{(w^H_M, w^L_M) \in \mathbb{R}^2_+ | \psi_H w^H_M + \psi_L w^L_M \in [0,1]\}$.

A.3.2 Period $T-1$

In Period $T-1$, the asset market equilibrium conditions and goods-market equilibrium condition are modified: the marginal propensity to consume and the optimal risky asset share for the young become the same as for the middle aged. The stock price is given by

$$\boldsymbol{p_{T-1}}(w^H_M, w^L_M) = (1-\rho_M) w_Y + (1-\rho_M)\left(\psi_H w^H_M + \psi_L w^L_M\right). \tag{A.1}$$

It is easy to derive the optimal consumption:

$$\boldsymbol{c^j_{YT-1}}(w^H_M, w^L_M) = \rho_M w_Y,$$

$$\boldsymbol{c^j_{MT-1}}(w^H_M, w^L_M) = \rho_M w^j_M,$$

$$\begin{aligned}\boldsymbol{c_{OT-1}}(w_M^H, w_M^L) &= 1 + \boldsymbol{p_{T-1}}(w_M^H, w_M^L) - (\psi_H w_M^H + \psi_L w_M^L) \\ &= 1 + (1-\rho_M)w_Y - \rho_M\left(\psi_H w_M^H + \psi_L w_M^L\right). \end{aligned} \tag{A.2}$$

We now derive bounds for variables of interest. First, nonnegative consumption implies the set of state variables in period $T-1$ is

$$\Omega^w_{T-1} = \left\{ (w_M^H, w_M^L) \in \mathbb{R}^2_+ | \psi_H w_M^H + \psi_L w_M^L \in \left[0, \frac{\alpha + (1-\alpha)(1-\rho_M)}{\alpha\rho_M}\right]\right\}.$$

This implies $(1-\rho_M)\frac{1-\alpha}{\alpha} \leq \boldsymbol{p}_{T-1}(w) \leq (1-\rho_M)\frac{1}{\alpha\rho_M}$.

Second, $\frac{G_1}{\boldsymbol{p_{T-1}}(w)} \leq \boldsymbol{R_{T-1}}(w) \leq \frac{G_N}{\boldsymbol{p_{T-1}}(w)}$. If the tree return is larger than R_{T-1} even for the lowest shock realization, i.e., $\frac{G_1}{\boldsymbol{p_{T-1}(w)}} > \boldsymbol{R_{T-1}}(w)$, the households will all borrow in bonds to invest in the tree, and the excess demand of bonds is negative. If the excess return is lower than R_{T-1} even for the highest realization, i.e., $\frac{G_N}{\boldsymbol{p_{T-1}}(w)} < \boldsymbol{R_{T-1}}(w)$, the households will all buy bonds, and the excess demand of bonds is positive. To clear the market, it must be the case that $\frac{G_1}{\boldsymbol{p_{T-1}}(w)} \leq \boldsymbol{R_{T-1}}(w) \leq \frac{G_N}{\boldsymbol{p_{T-1}}(w)}$.

In period $T-1$, $\boldsymbol{\theta^j_{MT-1}}(w) = \boldsymbol{\theta^j_{YT-1}}(w)$ since both the young and the middle-aged have only one more period to live. Consider the middle-aged portfolio problem for each w:

$$\boldsymbol{\theta^j_{MT-1}}(w) = \arg\max_{\theta\in\Theta} \left(\sum_{n=1}^N \pi_n \left[1 - \theta + \theta\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{1-\gamma_j}\right)^{\frac{1}{1-\gamma_i}}.$$

We directly plug in the excess return $\boldsymbol{Z_{T-1}}(w, G_n) = \frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}$ since $\boldsymbol{p_T}(w) = 0$. Nonnegative consumption when old implies that $1-\theta+\theta\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} \geq 0$ for all G_n. This implies $-\frac{1}{\frac{G_N}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}-1} \leq \theta \leq \frac{1}{1-\frac{G_1}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}}$.

Label the objective function as $\boldsymbol{F}^j_M$:

$$\boldsymbol{F^j_M}(\theta) \equiv \left(\sum_{n=1}^N \pi_n \left[1 - \theta + \theta\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{1-\gamma_j}\right)^{\frac{1}{1-\gamma_i}}.$$

We solve the optimal θ as the solution to the first order condition

$$(\boldsymbol{F^j_M})'(\theta) = \boldsymbol{F^j_M}(\theta)^{\gamma_j} \sum_{n=1}^N \pi_n \left[1-\theta+\theta\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{-\gamma_j}\left[\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} - 1\right] = 0. \tag{A.3}$$

The solution to the first order condition $\hat{\theta}$ satisfies

$$(\boldsymbol{F^j_M})''(\hat{\theta}) = -\boldsymbol{F^j_M}(\hat{\theta})^{\gamma_j} \sum_{n=1}^N \pi_n\gamma_j \left[1-\hat{\theta}+\hat{\theta}\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)}\right]^{-\gamma_j-1}\left[\frac{G_n}{\boldsymbol{R_{T-1}}(w)\boldsymbol{p_{T-1}}(w)} - 1\right]^2 < 0.$$

The two power terms on the right hand side of the above equation are positive, implying the second derivative of $\boldsymbol{F_M^j}$ at $\hat{\theta}$ is negative. Thus, the solution $\hat{\theta}$ to the first-order condition is also globally optimal and thus unique. This result can be established straightforwardly by contradiction.

Then we use the market clearing condition for bonds to obtain $\boldsymbol{R_{T-1}}(w)$:

$$\sum_{j=H,L} \psi_j \left[1 - \boldsymbol{\theta^j_{MT-1}}(w; R_{T-1})\right] \left(w_M^j + w_Y\right) = 0. \tag{A.4}$$

To establish that $\boldsymbol{\theta^j_{MT-1}}(w; R_{T-1})$ is continuous in R_{T-1}, we invoke the implicit function theorem. Since $\boldsymbol{(F_M^j)'}$ is continuous and differentiable in both θ and R, and $\boldsymbol{(F_M^j)}''$ is nonzero, the policy function $\boldsymbol{\theta^j_{MT-1}}(w; R_{T-1})$ given by equation (A.3) is continuous in R_{T-1}. Since $\boldsymbol{\theta^j_{MT-1}}(w; R_{T-1})$ is continuous in R_{T-1}, the excess demand of bonds in the left hand side of equation (A.4) is continuous in R_{T-1}. As R_{T-1} approaches $\frac{G_1}{p_{T-1}}$ from above, the excess returns go up, and θ^j_{MT-1} approaches positive infinity, implying a negative excess demand of bonds. On the other side, as R_{T-1} approaches $\frac{G_N}{p_{T-1}}$ from below, the excess returns go down, and θ^j_{MT-1} approaches positive infinity, implying a positive excess demand of bonds. The continuity of the excess demand function over R_{T-1} implies that there exists at least one R_{T-1} which implies zero excess demand in the bond markets. Therefore, in period $T-1$, we have proven the existence of the interest rate function $\boldsymbol{R_{T-1}}(w)$.

The certainty equivalent total next-period return function for the middle-aged is

$$\boldsymbol{\phi^j_{MT-1}}(w) = \boldsymbol{R_{T-1}}(w) \left(\sum_{n=1}^{N} \pi_n \left[1 - \boldsymbol{\theta^j_{MT-1}}(w) + \boldsymbol{\theta^j_{MT-1}}(w) \frac{G_n}{\boldsymbol{p_{T-1}}(w)\boldsymbol{R_{T-1}}(w)}\right]^{1-\gamma_j} \right)^{\frac{1}{1-\gamma_j}}.$$

The young's decision on portfolio choice $\boldsymbol{\theta^j_{YT-1}} = \boldsymbol{\theta^j_{MT-1}}$ implies that his next-period wealth $\{\boldsymbol{w'_{T-1}}(w, G_n)\}$ is given by equations (SM.12) and (SM.13).

A.3.3 Period $T-k$ with $k \geq 2$ (A Generic Period)

Now consider a generic period: $T-k$ with $k \geq 2$. The tree price is given by

$$\boldsymbol{p_{T-k}}(w) = (1-\rho_Y)w_Y + (1-\rho_M)(\psi_H w_M^H + \psi_L w_M^L).$$

It is easy to derive the optimal consumption:

$$\boldsymbol{c^j_{YT-k}}(w) = \rho_Y w_Y,$$

$$\boldsymbol{c^j_{MT-k}}(w) = \rho_M w_M^j,$$

$$\boldsymbol{c_{OT-k}}(w) = 1 + \boldsymbol{p_{T-k}}(w) - (\psi_H w_M^H + \psi_L w_M^L).$$

Again, nonnegative consumption implies that the set of state variables in period $T-k$ is

$$\Omega^w_{T-k} = \left\{ (w^H_M, w^L_M) \in \mathbb{R}^2_+ | \psi_H w^H_M + \psi_L w^L_M \in \left[0, \frac{\alpha + (1-\alpha)(1-\rho_Y)}{\alpha\rho_M} \right] \right\}.$$

This implies $\frac{(1-\rho_Y)(1-\alpha)}{\alpha} \le \boldsymbol{p_{T-k}}(w) \le \frac{\alpha(1-\rho_M)+(1-\alpha)(1-\rho_Y)}{\alpha\rho_M}$.

Here is some intuition for the bounds on the tree price. The lower bound is given by the minimum possible demand for the tree. Suppose the middle-aged have zero financial wealth, so they have a zero demand for savings. The old always have zero demand for savings. In that case, demand for net savings, which can only be provided by the tree, is equal to young's demand for savings, and the young's demand for savings will determine the tree price. Since the young have to live through the next two periods without labor income, they need savings of $\frac{(1-\rho_Y)(1-\alpha)}{\alpha}$, which gives the lower bound for the tree price. Now consider the upper bound for the tree price. The young still demand $\frac{(1-\rho_Y)(1-\alpha)}{\alpha}$ in savings, but now suppose the middle-aged own all the financial wealth of the economy $1+p_t$ and so have savings demand of $(1-\rho_M)(1+p_t)$. Then the maximum tree price is given by $p_t = \frac{(1-\rho_Y)(1-\alpha)}{\alpha} + (1-\rho_M)(1+p_t)$, which can be solved to give the upper bound on the tree price.

Now we need to show the existence of the price function $\boldsymbol{R_{T-k}}(w)$, the law of motion of wealth $\boldsymbol{w'_{T-k}}(w, G_n)$ and the portfolio function $\boldsymbol{\Theta_{T-k}}(w) = \{\boldsymbol{\theta^H_{YT-k}}(w), \boldsymbol{\theta^L_{YT-k}}(w), \boldsymbol{\theta^H_{MT-k}}(w), \boldsymbol{\theta^L_{MT-k}}(w)\}$ such that (i) given prices and the law of motion for wealth, the portfolio choices $\boldsymbol{\Theta_{T-k}}(w)$ are optimal; (ii) the law of the motion is consistent with the agents' choices; and (iii) the prices clear the markets.

Given the interest rate $\boldsymbol{R_{T-k}}(w)$, and the next-period state variable $\boldsymbol{w'_{T-k}}(w, G_n)$, the agents know the next period tree price $\boldsymbol{p_{T-k+1}}(\boldsymbol{w'_{T-k}}(w, G_n))$, and the total return for the middle-aged next period, $\boldsymbol{\phi^j_{MT-k+1}}(\boldsymbol{w'_{T-k}}(w, G_n))$ by solving the problem of period $T-k+1$. The implied excess returns are

$$\boldsymbol{Z_{t-k}}(w, G_n) = \frac{G_n \left(\boldsymbol{p_{T-k+1}}(\boldsymbol{w'_{T-k}}(w, G_n)) + 1 \right)}{\boldsymbol{p_{T-k}}(w)\boldsymbol{R_{T-k}}(w)}.$$

We have $\boldsymbol{Z_{t-k}}(w, G_1) \le 1 \le \boldsymbol{Z_{t-k}}(w, G_N)$ following similar arguments as in period $T-1$. This implies $\frac{G_1\left(\boldsymbol{p_{T-k+1}}(\boldsymbol{w'_{T-k}}(w,G_1))+1\right)}{\boldsymbol{p_{T-k}}(w)} \le \boldsymbol{R_{T-k}}(w) \le \frac{G_N\left(\boldsymbol{p_{T-k+1}}(\boldsymbol{w'_{T-k}}(w,G_N))+1\right)}{\boldsymbol{p_{T-k}}(w)}$. Thus, using the bounds for the tree prices, we have

$$\Omega^R_{T-k} = \left[\frac{G_1\rho_M \left(1-\rho_Y(1-\alpha)\right)}{1-\alpha\rho_M-(1-\alpha)\rho_Y}, \frac{G_N \left(1-\rho_Y(1-\alpha)\right)}{\rho_M(1-\rho_Y)(1-\alpha)} \right].$$

With the information they have, the middle-aged solve the following problem:

$$\boldsymbol{\theta^j_{MT-k}}(w) = \arg\max_{\theta\in\Theta} \left\{ \sum_{n=1}^N \pi_n \left[1-\theta+\theta \boldsymbol{Z_{T-k}}(w, G_n)\right]^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}. \tag{A.5}$$

Again as in period $T-1$, we have a unique global optimizing solution. Nonnegative consumption when old implies that $1-\theta+\theta \boldsymbol{Z_{T-k}}(w,G_n) \geq 0$ for all G_n. This implies $-\frac{1}{\boldsymbol{Z_{T-k}}(w,G_N)-1} \leq \theta \leq \frac{1}{1-\boldsymbol{Z_{T-k}}(w,G_1)}$. The optimal portfolio choices give $\boldsymbol{\phi^j_{MT-k}}(w)$, which is needed to solve the problem of period $T-k-1$.

The young's portfolio problem is given by equation (SM.7). Let's rewrite the objective function as follows:

$$\boldsymbol{F^j_Y}(\theta) = \left\{ \sum_{n=1}^{N} \pi_n \left\{ \left[1-\theta+\theta \boldsymbol{Z_{T-k}}(w,G_n)\right] \boldsymbol{\phi^j_{MT-k+1}} \left(\boldsymbol{w'_{T-k}}(w,G_n) \right)^{1-\rho_M} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}.$$

Solve for the optimal θ as the solution to the first order condition

$$\boldsymbol{(F^j_Y)'}(\theta) = \boldsymbol{F^j_Y}(\theta)^{\gamma_j} \sum_{n=1}^{N} \pi_n \boldsymbol{\phi^j_{MT-k+1}} \left(\boldsymbol{w'_{T-k}}(w,G_n) \right)^{(1-\rho_M)(1-\gamma_j)} \frac{\boldsymbol{Z_{T-k}}(w,G_n)-1}{\left(1-\theta+\theta \boldsymbol{Z_{T-k}}(w,G_n)\right)^{\gamma_j}} = 0. \quad \text{(A.6)}$$

The solution to the first order condition $\hat{\theta}$ also satisfies

$$\boldsymbol{(F^j_Y)''}(\hat{\theta}) = -\boldsymbol{F^j_Y}(\hat{\theta})^{\gamma_j} \sum_{n=1}^{N} \pi_n \gamma_j \boldsymbol{\phi^j_{MT-k+1}} \left(\boldsymbol{w'_{T-k}}(w,G_n) \right)^{(1-\rho_M)(1-\gamma_j)} \frac{\left(\boldsymbol{Z_{T-k}}(w,G_n)-1\right)^2}{\left(1-\hat{\theta}+\hat{\theta} \boldsymbol{Z_{T-k}}(w,G_n)\right)^{\gamma_j+1}} < 0.$$

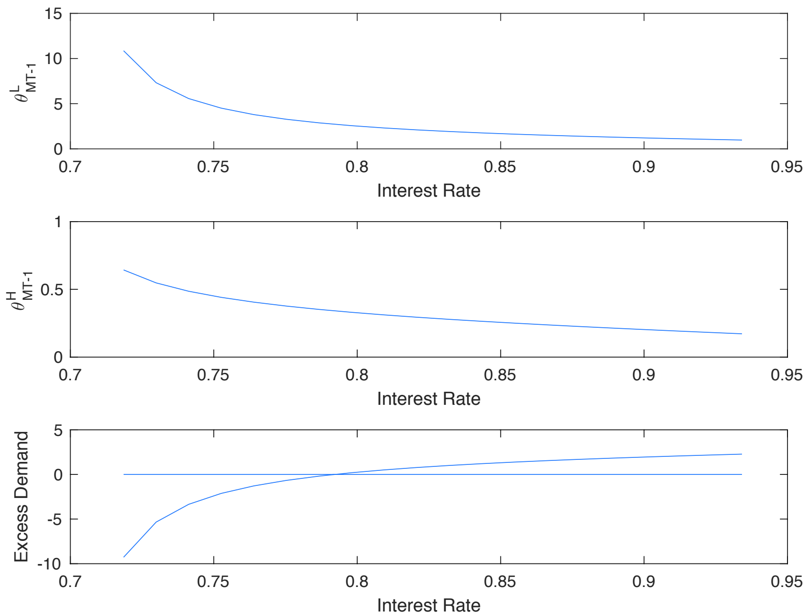
Thus, there is a unique optimal portfolio share $\boldsymbol{\theta^j_{YT-k}}(w)$. Nonnegative consumption when middle-aged implies that $-\frac{1}{\boldsymbol{Z_{T-k}}(w,G_N)-1} \leq \theta \leq \frac{1}{1-\boldsymbol{Z_{T-k}}(w,G_1)}$.

Thus, given functions $\{\boldsymbol{R_{T-k}}(w), \boldsymbol{w'_{T-k}}(w,G_n)\}$, we can solve for the optimal portfolio $\boldsymbol{\Theta_{T-k}}(w)$ from equations (A.5) and (A.6). For the optimal portfolio choices to be an equilibrium allocation, they must be consistent with the law of motion of wealth given by (SM.12) and (SM.13). Also, they need to satisfy the bond market clearing condition given $\boldsymbol{R_{T-k}}(w)$:

$$\sum_{j=H,L} \psi_j \left[(1-\rho_Y) \left[1-\boldsymbol{\theta^j_{YT-k}}(w)\right] w_Y + (1-\rho_M) \left[1-\boldsymbol{\theta^j_{MT-k}}(w)\right] w^j_M \right] = 0.$$

We cannot establish the uniqueness of the interest rate that clears the bond markets. However, numerically for our parameters, θ^j_{MT-1} decreases with R_{T-1} and the excess demand decreases with R_{T-1}, as illustrated in Figure 9. For a reasonably typical value of w, the top two panels plot the optimal portfolio shares over different values of the risk-free interest rate for the low-risk-tolerant and the high-risk-tolerant households, and the bottom panel plots the excess demand. As the risk-free rate increases and the excess tree return decreases, households decrease their risky asset shares. The excess demand increases with the interest rate. Thus, there exists a unique R_{T-1} that clears the bond market. One stable method for solving $\boldsymbol{R}(w)$ is thus the bisection technique, starting with the lower bound set at a value slightly higher than $\frac{G_1}{\boldsymbol{p_{T-1}}(w)}$ and the upper bound set at a value slightly lower than $\frac{G_N}{\boldsymbol{p_{T-1}}(w)}$.

Figure 9: Numerical Solution Illustration 1



A.3.4 Stationary Equilibrium

We use the above procedure for each period going backward until the policy functions converge. We then have the stationary Markov equilibrium for our model.

Appendix B: Price and Return Dynamics

In this appendix we describe the price and return dynamics implied by the model in more detail.

The dependence of the risky asset share of the young cohort as a whole on the shock right before they begin investing makes it possible to explain the behavior of the price-dividend ratio. Recall that the price-dividend ratio is determined exactly by the total middle-aged wealth in each period. See Equation (36). Because of the Cobb-Douglas production function and the constant average propensity to consume, the invested wealth of the young cohort is a fixed fraction of output, or equivalently, in a fixed ratio to the dividend. Their later wealth in middle age in comparison with the dividend at that time depends on their overall portfolio return compared to the growth rate of the dividend. Let us explain the general level of the price-dividend ratio after each of the four possible histories of the last two shocks.

A good shock leading into the youth of the current middle-aged cohort depressed their risky asset share when young, and they were significantly underlevered relative to the economy as a whole. Then a second good shock will actually cause the growth rate of their wealth to fall behind the growth rate of the dividend (which equals the growth rate of economic production as a whole). Middle-aged wealth that is smaller relative to the expanded size of the economy will then lead to a low price-dividend ratio (in part because dividends are large). In contrast, a bad shock as they go into middle age causes the economy to grow more slowly than their relatively safer portfolio, leading to a high middle-aged wealth to dividend ratio, which in turn implies a high price-dividend ratio.

The bad shock leading into the youth of the current middle-aged cohort led that cohort to have slightly elevated risky asset holdings and thus to be slightly overlevered relative to the economy as a whole. A good shock at middle age then leads them to have a high level of wealth relative to the economy as a whole, leading in turn to a high price-dividend ratio. In contrast, a bad shock at middle age causes them to have a low level of wealth relative to the economy as a whole, leading in turn to a low price-dividend ratio.

Note that the last shock makes a bigger difference for the price-dividend ratio between the (Good, Good) and (Good, Bad) histories than it does between the (Bad, Good) and (Bad, Bad) histories. The reason is that the daring middle-aged agents in the previous period pulled more dramatically ahead of the economy after a good shock leading into their middle age than they fell behind the economy after a bad shock leading into their middle age. This led the current middle-aged cohort to have a significantly underlevered portfolio after a good shock leading into their youth, but only slightly overlevered portfolio after a bad shock leading into their youth.

To summarize the history dependence of prices, we present the key price statistics conditional on the current shock, the past shock, and the two past shocks in Table A1. The kind of reasoning above can be extended to interpret the consequences of a longer history of shocks, but the most recent shocks have the most powerful effect on the model economy. One reason we include the effects of a three-shock history is the identity relating the realized tree return to the history of the price-dividend ratio:

$$R_t^{\text{Tree}} = \frac{P_t + D_t}{P_{t-1}} = \frac{D_t}{D_{t-1}} \left(\frac{1 + (P_t/D_t)}{P_{t-1}/D_{t-1}} \right) = G_t \left(\frac{1 + p_t}{p_{t-1}} \right).$$

Thus, the realized tree return depends on the values of the price-dividend ratio in two different periods. Among three-shock histories, the realized tree return is highest after (Good, Good, Good) since the two good shocks at $t-2$ and $t-1$ depress p_{t-1}, while the good shock at t gives a high G_t.[11] The realized tree return is lowest after the three-shock history (Good, Bad, Bad) because the combination of a good shock at $t-2$ followed by a bad shock at $t-1$ elevates p_{t-1}, while the bad shock at t gives a low G_t.

Table A1: History Dependence of Key Variables

	P_t/D_t		$\log R_t^{\text{Tree}}$		$\log R_t$		$E(\log R_t^{\text{Tree}})$		$E(\log Z_t)$	
G_t	Bad	Good	Bad	Good	Bad	Good	Bad	Good	Bad	Good
Mean	19.20	18.69	1.61	5.60	1.75	2.10	3.91	4.08	2.16	1.98
G_{t-1}										
Bad	18.99	19.23	1.51	5.60	1.81	1.96	3.96	3.95	2.15	1.98
Good	19.41	19.17	1.71	5.60	1.68	2.23	3.86	4.20	2.18	1.97
G_{t-2}, G_{t-1}										
Bad, Bad	19.07	19.32	1.58	5.67	1.79	1.94	3.94	3.93	2.15	1.99
Good, Bad	18.90	19.15	1.45	5.54	1.84	1.99	3.98	3.97	2.14	1.98
Bad, Good	19.21	17.81	1.53	5.41	1.74	2.33	3.91	4.29	2.17	1.95
Good, Good	19.59	18.52	1.87	5.79	1.62	2.12	3.81	4.11	2.19	1.99

Note: P_t/D_t denotes the price-dividend ratio, $\log R_t$ denotes the log risk free interest rate, $\log R_t^{\text{Tree}}$ denotes the log realized tree return, given by $\log(D_t + P_t) - \log P_{t-1}$, $E(\log R_t^{\text{Tree}})$ denotes the expected log tree return, and $E(\log Z_t)$ denotes the expected log excess return. All prices are annualized. All prices are in percentage, except the price-dividend ratio. G_t, G_{t-1} and G_{t-2} denote the dividend growth rate shock in period t, $t-1$ and $t-2$, respectively.

Table A1 also shows that the risk free rate is higher after a good shock than after a bad shock in period t. This pattern becomes particularly pronounced after a good shock in period $t-1$. The expected tree return is higher after a good shock in period t. However, the pattern does not hold when we consider a two-period history of shocks. After a good shock in period $t-1$, the expected tree return is higher after a good shock. In contrast, after a bad shock in period $t-1$, the expected tree return is higher after a bad shock. The expected excess return is lower after a good shock in period t, given the high aggregate risk tolerance after a good shock in period t. The expected excess return is especially low after a history of the bad and good shocks.

[11] The realized tree return is second-highest after the three-shock history (Bad, Bad, Good) not only because two bad shocks at $t-2$ and $t-1$ depress p_{t-1} somewhat, but also because the tail end of a bad shock at $t-1$ followed by a good shock at t puts p_t at a reasonably high level as well as giving a high G_t.

Appendix C: Alternative Parameterizations

We solve the model with different sets of numerical assumptions to determine how robust our primary findings are. First, we recalculate the risk tolerances of cautious and daring agents setting the standard deviation of risk tolerance equal to 0.085, which is half of the standard deviation of the risk tolerances listed in Table 1. We then solve the model with the other assumptions in Table 1 and report results analogous to Table 2 below. Next we we double the standard deviation of risk tolerances. Finally, we both halve and double the standard deviations of economic shocks and report the results below. In each case we find results that are qualitatively similar to those obtained with our main parameterization in the text.

Table A2: Low Dispersion in Tolerance

	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
G_t	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	72	76	69	92	88	70	61
Young	72	76	69	60	61	46	42
Middle	73	76	70	32	27	24	19
Daring	359	388	330	8	12	30	39
Young	360	389	330	5	5	20	18
Middle	359	388	330	2	7	10	22
Total	95	101	90	100.0	100.0	100	100
Young	95	101	90	66	66	66	59
Middle	96	101	90	34	34	34	41

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A3: High Dispersion in Tolerance

	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta\mathcal{W}_i}{\mathcal{W}}$	
G_t	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	23	27	18	94	81	25	14
Young	21	27	15	60	62	16	10
Middle	24	28	21	34	19	9	4
Daring	860	1226	495	6	19	75	86
Young	853	1213	492	5	5	63	27
Middle	868	1238	499	1	14	12	59
Total	90	123	56	100.0	100.0	100	100
Young	88	122	54	65	67	79	37
Middle	92	124	59	35	33	22	63

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A4: Low Dispersion in Growth

	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta\mathcal{W}_i}{\mathcal{W}}$	
G_t	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	66	69	63	93	89	64	56
Young	66	69	62	60	61	42	38
Middle	66	69	63	32	29	22	18
Daring	454	489	418	7	11	36	44
Young	454	490	418	5	5	26	22
Middle	453	489	418	2	5	10	22
Total	97	103	91	100.0	100.0	100	100
Young	97	103	90	66	66	68	60
Middle	97	103	91	34	34	32	40

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Table A5: High Dispersion in Growth

	Portfolio Share θ			Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$		Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$	
G_t	Average	Bad	Good	Bad	Good	Bad	Good
Cautious	19	22	17	93	75	20	12
Young	19	22	15	59	65	13	10
Middle	20	22	18	34	10	7	2
Daring	746	1091	401	7	25	80	88
Young	750	1100	401	5	6	57	24
Middle	742	1083	401	2	19	23	64
Total	77	108	47	100.0	100.0	100	100
Young	77	108	46	64	70	70	34
Middle	78	107	48	36	30	31	66

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. G_t denotes the dividend growth rate shock in period t.

Appendix D: Key Model Variables

We list all of the important model variables with a short description in this table. In the table $j \in \{H, L\}$ and $g \in \{Y, M, O\}$. Note that key model parameters are defined in Table 1.

Table A6: Key Model Variables

Variable	Per D_t	Description
D_t		Total dividends and level of technology
P_t	p_t	Price of one unit of tree
Y_t	y_t	Aggregate output
R_t		Gross risk free rate
Z_t		Excess tree return
C^j_{gt}	c^j_{gt}	Consumption for age g and tolerance j
W^j_{gt}	w^j_{gt}	Wealth for age g and tolerance j
S^j_{gt}	s^j_{gt}	Tree holdings for age g and tolerance j
B^j_{gt}	b^j_{gt}	Bond holdings for age g and tolerance j
θ^j_{gt}		Risky asset share for age g and tolerance j
ϕ^j_{gt}		Certainty equivalent total return for age g and tolerance j