A Additional Literature

The study of agent belief heterogeneity begins with the market selection hypothesis of Alchian (1950) and Friedman (1953). By analogy with natural selection, the market selection hypothesis states that agents with systematically wrong beliefs will eventually be driven out of the market. The influence of agent heterogeneity on market outcomes under the standard assumption of discounted expected utility is well understood, and consistent with market selection. Sandroni (2000) and Blume and Easley (2006) find strong support for this hypothesis under the assumption of time-separable preferences in an economy without growth. Yan (2008) and Cvitanić, Jouini, Malamud, and Napp (2012) analyze the survival of investors in a continuous-time framework where there are not only differences in beliefs but also potentially differences in the utility parameters of the investors. They show that it is always the investor with the lowest survival index\footnote{Yan (2008) shows that the survival index increases with belief distortion, risk aversion, and the subjective time discount rate of the investor.} who survives in the long run. David (2008) considers a similar model setup, in which both agents have distorted estimates of the mean growth rate of the economy, showing that—as agents with lower risk aversion undertake more aggressive trading strategies—the equity premium increases the lower the risk aversion is. Chen, Joslin, and Tran (2012) analyze how differences in beliefs about the probability of disasters affect asset prices. They show that even if there is only a small fraction of investors who are optimistic about disasters, this fraction sells insurance for the disaster states and so eliminates most of the risk premium associated with disaster risk. Bhamra and Uppal (2014) consider the case of habit utility.

For recursive utility, this qualitative behavior changes fundamentally. However, there has
been less research in this area as solving such models is anything but trivial. Lucas and Stokey (1984) observe in the deterministic case that the problem of finding all Pareto-optimal allocations can be made recursive if we allow the weights in the social welfare function to be time varying. This approach is extended by Kan (1995) to the stochastic case with finite state spaces. Anderson (2005) develops an extensive theory for the special case of risk-sensitive preferences, no growth, and finite state spaces, and finds first-order conditions similar to those we use below. In particular, he shows how to characterize the equilibrium by a single value function instead of one value function for each agent. Collin-Dufresne, Johannes, and Lochstoer (2015) derive similar first-order conditions to ours for recursive utility by equating marginal utilities, but use a different procedure to solve for their allocations. Duffie, Geoffard, and Skiadas (1994) formulate the problem in continuous time, while Dumas, Uppal, and Wang (2000) reformulate it in terms of variational utility. Borovička (2019) uses this formulation to explore the question of the survival of agents with recursive utility in continuous time, and shows that agents with fundamentally wrong beliefs can survive or even dominate. So, inferences about market selection and equilibrium outcomes fundamentally differ under the assumption of general recursive utility compared to the special case of standard time-separable preferences. While Borovička (2019) concentrates on the special case of i.i.d. consumption growth, Branger, Dumitrescu, Ivanova, and Schlag (2011) generalize the results to a model with long-run risks as a state variable.

However, most papers with heterogeneous investors and recursive preferences consider only an i.i.d. process for consumption growth. For example, Gărleanu and Panageas (2015) analyze the influence of heterogeneity in the preference parameters on asset prices in a two-agent OLG economy. Roche (2011) considers a model in which the heterogeneous investors can only invest in a stock but there is no risk-free bond. Hence, as there is no savings trade-off, the impact of recursive preferences on equilibrium outcomes will be quite different.

Exceptions that relax the i.i.d. assumptions include the papers by Branger, Konermann, and Schlag (2019) and Collin-Dufresne, Johannes, and Lochstoer (2016). Both papers reexamine the influence of belief differences regarding disaster risk with Epstein–Zin instead of CRRA preferences as in Chen, Joslin, and Tran (2012). Branger, Konermann, and Schlag (2019) provide evidence that the influence of investors with more optimistic beliefs about disasters is less profound when the disaster occurs to the growth rate of consumption and show that the risk-sharing mechanism persists even when markets are incomplete. Collin-Dufresne, Johannes, and Lochstoer (2016) make a similar claim but for a different reason. They show that if the investors can learn about the probability of disasters and if they have recursive
preferences, the impact of the optimistic investor on asset prices decreases. Optimists are uncertain about the probability of disasters and hence will provide less insurance to pessimistic investors.

A different strand of the literature, which does not rely on the i.i.d. assumption, is comprised of papers on international asset pricing. This area (advanced by Riccardo Colacito and Mariano Croce in particular) considers models with Epstein–Zin preferences, two investors, and also two goods for which investors have different preferences (home bias). For example, Colacito and Croce (2013) argue that a model with highly correlated international long-run components in output can explain both the low correlation between consumption differentials and the tendency of high interest rate currencies to appreciate. The authors furthermore show that an increase in capital mobility can explain the structural break in the data for the pre- and post-1970 period. Colacito, Croce, and Liu (2018b) provide the theoretical foundation for the multiple good economy by providing results on equilibrium existence and agents’ survival; they also compare computational methods for solving the model. Furthermore, Colacito, Croce, Ho, and Howard (2018a) use a model with Epstein–Zin preferences and short- and long-run productivity shocks to study the effects of these shocks on international investment flows.

In a different direction, Epstein, Farhi, and Strzalecki (2014) argue that an Epstein–Zin investor dislikes long-run risk to the extent that he or she would pay a substantial premium to get rid of it. In a model with two agents, the agent who believes that risk is longer term than the other is willing to pay an insurance premium to the other agent to hedge against long-run risk.

B Proofs and Details

In this appendix, we provide proofs for the theoretical results presented in Section 2 of the paper. Along the way, we derive a system of first-order conditions for Epstein–Zin preferences. This system constitutes the foundation for our numerical solution method (see Appendix C).

B.1 Proofs for Section 2.1

Proof of Theorem 1. Let $\lambda = \{\lambda^1, \ldots, \lambda^H\}$ be a set of Negishi weights and let $\{C\}_0 = \{\{C^1\}_0, \ldots, \{C^H\}_0\}$ denote a vector of agents’ consumption processes. The optimal decision $\{C\}_0^*$ of the social planner in the initial period assigns consumption streams to all individual agents for all periods and possible states. Obviously, the optimal decisions must satisfy
the market-clearing condition (1) in all periods and states. For ease of notation we again abbreviate the state dependence; we use \( C^h_t \) for \( C^h(y^t) \) and \( U^h_{(t)} \) for \( U^h \{C^h\}_t \).

To derive the first-order conditions, we borrow a technique from the calculus of variations. For any function \( f_t \), we can vary the consumption of two agents by

\[
\begin{align*}
C^h_t &\rightarrow C^h_t + \epsilon f_t \\
C^u_t &\rightarrow C^u_t - \epsilon f_t.
\end{align*}
\]

(1)

It is sufficient to consider the variation with \( l = 1 \) and \( h \in \mathbb{H}^- \). For an optimal allocation it must be true that

\[
\frac{d\bar{S}_P(\{C\}_0; \lambda)}{d\epsilon} \bigg|_{\epsilon = 0} = 0.
\]

(2)

This gives us

\[
\check{\lambda}^h \check{U}^h_{0,t} = \check{\lambda}^l \check{U}^l_{0,t}, \quad h \in \mathbb{H}^-,
\]

(3)

where \( \check{U}^h_{t,t+k} \) is defined as

\[
\check{U}^h_{t,t+k} = \frac{dU^h(C^h_t, \ldots, C^h_{t+k} + f_{t+k}, \ldots)}{d\epsilon} \bigg|_{\epsilon = 0}.
\]

(4)

Using the expression given in Equation (2), the derivative \( \check{U}^h_{t,t+k} \) satisfies a recursive equation with the initial condition

\[
\hat{U}^h_{t,t} = \frac{dU^h(C^h_t + f_t, \ldots)}{d\epsilon} \bigg|_{\epsilon = 0} = F^h_1 \left( C^h_t, R_t[U^h_{(t+1)}] \right) \cdot f_t,
\]

where \( F^h_k \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \) denotes the derivative of \( F^h \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \) with respect to its \( k \)th argument. The recursive step is given by

\[
\begin{align*}
\hat{U}^h_{t,t+k} &= \frac{dF^h \left( C^h_t, R^h_t \left[ U^h(C^h_{t+1}, \ldots, C^h_{t+k} + f_{t+k}, \ldots) \right] \right)}{d\epsilon} \bigg|_{\epsilon = 0} \\
&= F^h_2 \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \cdot \frac{dR^h_t[U^h(\cdot)]}{d\epsilon} \bigg|_{\epsilon = 0} \\
&= F^h_2 \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \cdot \frac{dG^h_{(t)}(E^h G^h_{[U^h(\cdot)\]})}{dE^h G^h_{[U^h(\cdot)]}} \cdot \frac{dE^h G^h_{[U^h(\cdot)]}}{d\epsilon} \bigg|_{\epsilon = 0} \\
&= F^h_2 \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \cdot \frac{1}{G^h_{(t)}(E^h G^h_{[U^h(\cdot)]})} \cdot E^h \left( G^h_{(t)}(U^h_{(t+1)}) \cdot \hat{U}^h_{t+1, t+k} \right) \\
&= F^h_2 \left( C^h_t, R^h_t[U^h_{(t+1)}] \right) \cdot \frac{E^h \left( G^h_{(t)}(U^h_{(t+1)}) \cdot \hat{U}^h_{t+1, t+k} \right)}{G^h_{(t)}(R^h_t[U^h_{(t+1)}])},
\end{align*}
\]

(6)
where we use $\frac{\partial G^{-1}(x)}{\partial x} = \frac{1}{G'(G^{-1}(x))}$ and abbreviate $U^h(C^h_{t+1}, \ldots C^h_{t+k} + \epsilon f_{t+k}, \ldots)$ by $U^h(\cdot)$. We can recast this recursion into a useful form. For this purpose, we define a second recursion $U^h_{t,t+k}$ by

$$U^h_{t,t} = F^h_1 \left( C^h_t, R^h_t[U^h_{t+1}] \right)$$

and

$$U^h_{t,t+k} = \Pi^h_{t+1} \cdot U^h_{t+1,t+k},$$

where

$$\Pi^h_{t+1} = F^h_2 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \cdot \frac{G^h(\Pi^h_{t+1})}{G^h(R^h_t[U^h_{t+1}])} \frac{dP^h_{t,t+1}}{dP^h_{t,t+1}}.$$  

A simple induction shows that

$$\hat{U}^h_{t,t+k} = E_t(U^h_{t,t+k} f_t).$$  

Plugging (10) into the optimality condition (3) we obtain

$$E^0 \left( (\hat{\lambda}^h U^h_{0,t} - \hat{\lambda}^1 U^1_{0,t}) f_t \right) = 0, \quad h \in \mathbb{H}^-.$$  

Under a broad range of regularity conditions, this condition implies that

$$\hat{\lambda}^h U^h_{0,t} = \hat{\lambda}^1 U^1_{0,t}, \quad h \in \mathbb{H}^-.$$  

For example, if $\hat{\lambda}^h U^h_{0,t} - \hat{\lambda}^1 U^1_{0,t}$ has finite variance, then this holds for the Riesz Representation Theorem for $L^2$ random variables. We can then split Expression (12) into two parts. First define $\lambda^h_0 \equiv \hat{\lambda}^h$ to obtain

$$\frac{\lambda^h_0}{\lambda^1_0} = \frac{U^1_{0,t}}{U^h_{0,t}} = \frac{\Pi^1_1}{\Pi^h_1} \cdot \frac{U^1_{t,t+1}}{U^h_{t,t+1}} = \frac{\Pi^1_1}{\Pi^h_1} \cdot \frac{\lambda^h_1}{\lambda^1_1}, \quad h \in \mathbb{H}^-,$$

where $\lambda^h_1$ denotes the Negishi weight in the social planner’s optimal solution in $t = 1$. Generalizing this equation for any period $t$, we obtain the following dynamics for the optimal weightootnote{Note that we can either solve the model in terms of the ratio $\frac{\lambda^h_t}{\lambda^1_t}$ (this is equal to setting $\lambda^1_t = 1$ for all $t$ as done in Judd, Kubler, and Schmedders (2003)) or we can normalize the weights so that they remain bounded in $(0, 1)$. Our solution method uses the latter approach as it obtains better numerical properties.} $\lambda^h_{t+1}$:

$$\frac{\lambda^h_{t+1}}{\lambda^1_{t+1}} = \frac{\Pi^h_{t+1}}{\Pi^1_{t+1}} \cdot \frac{\lambda^h_t}{\lambda^1_t}, \quad h \in \mathbb{H}^-.$$  

Inserting the initial condition (7) into (12) for $t = 0$ and generalizing it for any social planner’s
optimal solution at time \( t \) yields

\[
\lambda^h_t F^h_1 \left( C^h_t, R^h_t[U^h_{t+1}] \right) = \lambda^1_t F^1_1 \left( C^1_t, R^1_t[U^1_{t+1}] \right), \quad h \in \mathbb{H}^-.
\] (14)

Equation (14) states the optimality conditions for the individual consumption choices at any time \( t \). This completes the proof of Theorem 1.

Note that for time-separable utility, \( F^h_1 \left( C^h_t, R^h_t[U^h_{t+1}] \right) \) is simply the marginal utility of agent \( h \) at time \( t \), and so we obtain the same optimality condition as, for example, Judd, Kubler, and Schmedders (2003) (see Equation (7) on page 2209). In this special case the Negishii weights can be pinned down in the initial period and thereafter remain constant. For general recursive preferences this is not true. The optimal weights vary over time following the law of motion described by Equation (13).

We can use Equations (13) and (14) together with the market-clearing condition (1) to compute the social planner’s optimal solution. We therefore define \( \lambda_t^- = \{ \lambda^B_t, \lambda^3_t, \ldots, \lambda^H_t \} \) and let \( V^h \) denote the value function of agent \( h \in \mathbb{H} \). We are looking for model solutions of the form \( V^h(\lambda_t^-, y^t) \). So the model solution depends on both the exogenous state \( y^t \) and the time-varying Negishii weights \( \lambda_t^- \). An optimal allocation is then characterized by the following four equations:

- the market-clearing condition (1)

\[
\sum_{h=1}^H C^h(\lambda_t^-, y^t) = C(y^t); \quad (15)
\]

- the value functions (2) of the individual agents

\[
V^h(\lambda_t^-, y^t) = F^h \left( C^h(\lambda_t^-, y^t), R^h_t[V^h(\lambda_{t+1}^-, y^{t+1})] \right), \quad h \in \mathbb{H}; \quad (16)
\]

- the optimality conditions (14) for the individual consumption decisions for \( h \in \mathbb{H}^- \)

\[
\lambda^h_t F^h_1 \left( C^h(\lambda_t^-, y^t), R^h_t[V^h(\lambda_{t+1}^-, y^{t+1})] \right) = \lambda^1_t F^1_1 \left( C^1(\lambda_t^-, y^t), R^1_t[V^1(\lambda_{t+1}^-, y^{t+1})] \right); \quad (17)
\]

- the equations (13) for the dynamics of \( \lambda_t^- \)

\[
\frac{\lambda_{t+1}^-}{\lambda_t^-} = \frac{\Pi_{t+1}^h}{\Pi_t^h} \lambda^h_t, \quad h \in \mathbb{H}^-; \quad (18)
\]
with
\[
\Pi_{t+1}^h = F_{2}^h \left( C_h^t(\lambda_{-t}^i, y_t^i), R_t^{h_i}[V^h(\lambda_{t+1}^i, y_{t+1}^i)] \right) \cdot \frac{G_h'(V^h(\lambda_{t+1}^i, y_{t+1}^i))}{G_h'(R_t^{h_i}[V^h(\lambda_{t+1}^i, y_{t+1}^i)])} \frac{dP_{t+1}^h}{dP_{t+1}}. \tag{19}
\]

This concludes the general description of the equilibrium obtained from the social planner’s optimization problem.

To prove Theorem 2, we first derive a variant of Lemma 1 in Blume and Easley (2006).

**Lemma 1.** Let \( X_i^t, i = 1, 2, \ldots, H, \) be a family of positive random variables for each \( t = 0, 1, 2, \ldots, \) such that \( A \leq \sum_i X_i^t \leq B \) with \( B \in \mathbb{R}_{++} \). Let \( f^i : \mathbb{R}_{++} \to \mathbb{R}_{++}, \ i = 1, 2, \ldots, H, \) be a family of decreasing functions such that \( f^i(x) \to \infty \) as \( x \to 0 \). If \( f^i(X_i^t)/f^j(X_i^j) \to \infty, \) then \( X_i^t \to 0 \) for \( t \to \infty \). If \( X_i^t \to 0, \) then for at least one \( j, \limsup_t f^i(X_i^t)/f^j(X_i^j) = \infty. \)

**Proof.** Since \( X_i^t \) is positive, \( X_i^t \leq B \) for all \( i, t. \) By assumption, \( 0 < f^j(B) \leq f^j(X_i^j). \) Thus, \( f^i(X_i^t)/f^j(X_i^j) \to \infty \) if and only if \( f^i(X_i^t) \to \infty, \) which happens when \( X_i^t \to 0 \) as \( t \to \infty. \)

Conversely, assume \( X_i^t \to 0. \) Every period, for at least one \( j, X_i^j \geq A/H \) (otherwise \( \sum_{i=1}^H X_i^j < A \)). Since there are only finitely many random variables, for at least one \( j \) we have \( X_i^j \geq A/H \) infinitely often. Then, by assumption, \( f^j(X_i^j) \leq f^j(A/H) \) infinitely often, and so \( \limsup_t f^i(X_i^t)/f^j(X_i^j) = \infty. \)

**Proof of Theorem 2.** By the first-order condition (5), \( \lambda_{i}/\lambda_{i}^j = F_i^j(C_i^j, R_i^j)/F_i^j(C_i^j, R_i^j). \) Since \( F^h \) is additively separable, \( F_i^h \) is a function of consumption alone. Let \( f^i = F_i^i, f^j = F_i^j, \) \( A = \underline{C}, \) and \( B = \overline{C}, \) and apply Lemma 1.

**B.2 Proofs for Section 2.2**

In this section we provide the specific expressions for \( V^h, F_i^h, F_2^h, \) and \( \Pi^h \) when the heterogeneous investors have recursive preferences as in Epstein and Zin (1989) and Weil (1989). The value function for Epstein–Zin (EZ) preferences is given by\(^3\)

\[
V_i^h = \left[ (1 - \delta^h)(C_i^h)^{\rho^h} + \delta^h R_i^h (V_{t+1}^h)^{\rho^h} \right]^{1/\rho^h} \tag{20}
\]

with
\[
R_i^h (V_{t+1}^h) = G_h^{-1} (E_t^h \left[ G_h (V_{t+1}^h) \right]) \quad G_h (V_{t+1}^h) = (V_{t+1}^h)^{\alpha^h}.
\]

\(^3\)For ease of notation, we again abbreviate the dependence on the exogenous state \( y_t \) and the endogenous state \( \lambda_t^i. \) Hence we write \( V_i^h \) for \( V^h(\lambda_t^i, y_t) \) or \( C_i^h \) for \( C^h(\lambda_t^i, y_t). \)
Recall that the parameter $\delta^h$ is the discount factor, $\rho^h = 1 - \frac{1}{\psi^h}$ determines the EIS, $\psi^h$, and $\alpha^h = 1 - \gamma^h$ determines the relative risk aversion $\gamma^h$ of agent $h$. The derivatives of $F^h(C^h_t, R^h_t[V^h_{t+1}]) = V^h_t$ with respect to its first and second argument are then given by

$$F^h_{1,t} = (1 - \delta^h)(C^h_t)^{\rho^h-1}(V^h_t)^{1-\rho^h}$$

and

$$F^h_{2,t} = \delta^h R^h_t(V^h_{t+1})^{\rho^h-1}(V^h_t)^{1-\rho^h}.$$  

In this paper we focus on growth economies. Therefore, we introduce the following normalization to obtain a stationary formulation of the model. We define the consumption share of agent $h$ by $s^h_t = \frac{C^h_t}{C_t}$ and the normalized value functions, $v^h_t = \frac{V^h_t}{C_t}$. Recall that $\Delta c_{t+1} = c_{t+1} - c_t$ with $c_t = \log(C_t)$. The value function (20) is then given by

$$v^h_t = \left[ (1 - \delta^h)(s^h_t)^{\rho^h} + \delta^h R^h_t (v^h_{t+1} e^{\Delta c_t})^{\rho^h} \right]^{\frac{1}{\rho^h}}.$$  

By inserting (21) into (17) we obtain the optimality condition for the individual consumption decisions

$$\lambda^h_t F^h_{1} \left( C^h(\lambda^h_t^{-1}, y^t), R^h_t [V^h(\lambda^h_{t+1}^{-1}, y^{t+1})] \right) = \lambda^1_t F^1_{1} \left( C^1(\lambda^1_t^{-1}, y^t), R^1_t [V^1(\lambda^1_{t+1}^{-1}, y^{t+1})] \right),$$

which simplifies to

$$\lambda^h_t (1 - \delta^h)(C^h_t)^{\rho^h-1}(V^h_t)^{1-\rho^h} = \lambda^1_t (1 - \delta^1)(C^1_t)^{\rho^1-1}(V^1_t)^{1-\rho^1}. \quad (24)$$

Recall the definition of the normalized Negishi weights, $\lambda^h_t = \frac{\lambda^h_t}{(v^h_t)^{\rho^h-1}}$. From Equation (24) we obtain

$$\lambda^h_t (1 - \delta^h)(s^h_t)^{\rho^h-1} = \lambda^1_t (1 - \delta^1)(s^1_t)^{\rho^1-1}. \quad (25)$$

This equation is the optimality condition for the individual consumption decisions we employ for solving for the model with Epstein–Zin preferences. Inserting the de-trended weight $\lambda^h_t$ into the dynamics for the weights (18), we obtain

$$\frac{\lambda^h_{t+1}}{\lambda^h_{t+1}} = \frac{\lambda^h_{t+1} (v^h_{t+1})^{\rho^h-1}}{\lambda^1_{t+1} (v^1_{t+1})^{\rho^1-1}} = \frac{\lambda^h_t (v^h_t)^{\rho^h-1} \Pi^h_{t+1}}{\lambda^1_t (v^1_t)^{\rho^1-1} \Pi^1_{t+1}}, \quad h \in \mathbb{H}^{-}. \quad (26)$$

Plugging the expressions for Epstein–Zin preferences (20)–(22) into Equation (19), we obtain
the following expression for $\Pi_{t+1}^h$:

$$
\Pi_{t+1}^h = \delta^h R_t^h (V_{t+1}^h)^{c_{t+1} - 1} \left( V_t^h \right)^{c_t - 1 - \rho^h} \left( V_{t+1}^h \right)^{\alpha^h - 1} \frac{dP_{t,t+1}^h}{R_t^h (V_{t+1}^h)^{\alpha^h - \rho^h} dP_{t,t+1}}.
$$

(27)

Using the normalized value function $v_t^h = \frac{V_t^h}{c_t}$, we have

$$
\Pi_{t+1}^h = \delta^h (v_t^h)^{1 - \rho^h} \left( v_{t+1}^h e^{\Delta c_{t+1}} \right)^{c_{t+1} - 1} \frac{dP_{t,t+1}^h}{R_t^h (v_{t+1}^h e^{\Delta c_{t+1}})^{\alpha^h - \rho^h} dP_{t,t+1}}.
$$

(28)

Equation (26) can then be written as

$$
\frac{\lambda_{t+1}^h}{\lambda_{t+1}^l} = \frac{\lambda_t^h \Pi_{t+1}^h}{\lambda_t^l \Pi_{t+1}^l}, \quad h \in \mathbb{H}^-,
$$

(29)

where

$$
\Pi_{t+1}^h = \delta^h e^{\rho^h \Delta c_{t+1}} dP_{t,t+1}^h \left( v_{t+1}^h e^{\Delta c_{t+1}} \right)^{c_{t+1} - 1 - \rho^h} \left( v_{t+1}^h e^{\Delta c_{t+1}} \right)^{\alpha^h - \rho^h}.
$$

(30)

For $\alpha^h = \rho^h$, we obtain the standard term for CRRA preferences; the dynamics of $\lambda_{t+1}^h$ only depend on the subjective discount factor, the EIS, and the subjective beliefs of the investors. For Epstein–Zin preferences, we obtain an extra term that reflects the time trade-off. Using the normalization $\sum_{h=1}^H \lambda_t^h = 1$, the dynamics for $\lambda_{t+1}^h$ are then given by

$$
\lambda_{t+1}^h = \frac{\lambda_t^h \Pi_{t+1}^h}{\sum_{h=1}^H \lambda_t^h \Pi_{t+1}^h}.
$$

(31)

Hence, for Epstein–Zin preferences we obtain the following system for the first-order conditions (15)–(19):

<table>
<thead>
<tr>
<th>The market-clearing condition:</th>
<th>$\sum_{h=1}^H s_t^h = 1$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The optimality condition for the individual consumption decisions:</td>
<td></td>
</tr>
</tbody>
</table>

9
\[ \Delta^h_t (1 - \delta^h)(s^h_t)^{\rho^h - 1} = \Delta^i_t (1 - \delta^i)(s^i_t)^{\rho^i - 1}, \quad h \in \mathbb{H}, \]  \hspace{1cm} (CD) \]

with \( \sum_{h=1}^{H} \Delta^h_t = 1. \)

The value functions of the individual agents:

\[ v^h_t = \left[ (1 - \delta^h)(s^h_t)^{\rho^h} + \delta^h R^h_t \left( v^h_{t+1} e^{\Delta c_{t+1}} \right)^{\rho^h} \right]^{\frac{1}{\rho^h}}, \quad h \in \mathbb{H}. \]  \hspace{1cm} (VF) \]

The equation for the dynamics of \( \Delta^h_t: \)

\[ \Delta^h_{t+1} = \frac{\Delta^h_t \Pi^h_{t+1}}{\sum_{h=1}^{H} \Delta^h_t \Pi^h_{t+1}} \]

\[ \Pi^h_{t+1} = \delta^h e^{\rho^h \Delta c_{t+1}} \frac{dP^h_{t+1}}{dP_t} R^h_t \left( v^h_{t+1} e^{\Delta c_{t+1}} \right)^{\rho^h - \rho^h}, \quad h \in \mathbb{H}. \]  \hspace{1cm} (D\lambda) \]

Note that the conditions (MC, CD, VF, D\lambda) are just the equilibrium conditions (11)–(15) stated in Section 2.2 of the paper. We observe that Equation (CD) and hence the individual consumption decisions \( s^h_t \) only depend on time \( t \) information and that there is no intertemporal dependence. This feature allows us to first solve for \( s^h_t \) given the current state of the economy, and in a second step to solve for the dynamics of the Negishi weights. Hence, we can separate solving the optimality conditions (11)–(15) into two steps in order to reduce the computational complexity. In Appendix C we describe this approach in detail.

Using condition (CD) we can prove Theorem 3. Recall that \( \rho^h = 1 - \frac{1}{\psi^h} < 1 \) for all possible values of an agent’s EIS, \( \psi^h > 0. \)

**Proof of Theorem 3.** Condition (CD) implies

\[ \frac{\lambda^j_i}{\lambda^i} = \frac{(1 - \delta^i)(s^i_t)^{\rho^i - 1}}{(1 - \delta^j)(s^j_i)^{\rho^j - 1}}. \]

Now let \( f^i(s) = s^{\rho^i - 1}, \ f^j(s) = s^{\rho^j - 1}, \) and \( A = B = 1, \) and apply Lemma 1. \( \square \)

## C Solution Method

We describe our solution method for asset-pricing models with heterogeneous agents and recursive preferences.
C.1 Computational Procedure—A Two-Step Approach

For ease of notation the following procedures are described for $H = 2$ agents and a single state variable $y_t \in \mathbb{R}^1$. However, the approach can analogously be extended to the general case of $H > 2$ agents and multiple states. We solve the social planner’s problem using a collocation projection. For this we perform the usual transformation from an equilibrium described by the infinite sequences (with a time index $t$) to the equilibrium described by functions of some state variable(s) $x$ on a state space $X$. We denote the current exogenous state of the economy by $y$ and the subsequent state in the next period by $y'$ with the state space $Y \in \mathbb{R}$. The term $\Lambda_2$ denotes the current endogenous state of the Negishi weight and $\Lambda'_2$ denotes the corresponding state in the subsequent period with $\Lambda_2 \in (0, 1)$.

We approximate the value functions of the two agents, $v^h(\lambda_2, y), h = \{1, 2\}$, by two-dimensional cubic splines and we denote the approximated value functions by $\hat{v}^h(\lambda_2, y)$. For the collocation projection we have to choose a set of collocation nodes $\{\lambda_{2k}\}_{k=1}^n$ and $\{y_i\}_{i=1}^m$ at which we evaluate $\hat{v}^h(\lambda_2, y)$. The individual consumption shares only depend on the endogenous state $\lambda_{2k}$. So in the following we show how to first solve for the individual consumption shares at the collocation nodes $s^h_k = s^h(\lambda_{2k})$, which are then used to solve for the value functions $v^h$ and the dynamics of the endogenous state $\lambda_2$.

**Step 1: Computing Optimal Consumption Allocations**

Equation (13) has to hold at each collocation node $\{\lambda_{2k}\}_{k=1}^n$:

$$\lambda_{2k} (1 - \delta^2) (s^2_k)^{\rho^2 - 1} = (1 - \lambda_{2k})(1 - \delta^1)(s^1_k)^{\rho^1 - 1}.$$  

Together with the market-clearing condition (11) we get

$$\lambda_{2k} (1 - \delta^2) (s^2_k)^{\rho^2 - 1} = (1 - \lambda_{2k})(1 - \delta^1)(1 - s^2_k)^{\rho^1 - 1}. \quad (32)$$

So for each node $\{\lambda_{2k}\}_{k=1}^n$ the optimal consumption choice $s^2_k$ can be computed by solving Equation (32) and $s^1_k$ is obtained by the market-clearing condition (11). For the special case of $\rho^2 = \rho^1$ we can solve for $s^2$ as a function of $\lambda_2$ analytically, and hence we do not have to solve the system of equations for each node.

4Note that in the case of $H$ agents we have to solve a system of $H - 1$ equations that pin down the $H - 1$ individual consumption choices $s^h \in \mathbb{R}^-$.
Step 2: Solving for the Value Function and the Dynamics of the Negishi Weights

Solving for the value function is not as straightforward as it depends on the dynamics of the endogenous state $\lambda_2$, which are unknown and follow Equation (15). We compute the expectation over the exogenous state by a Gaussian quadrature with $Q$ quadrature nodes. This implies that the values for $y'$ at which we evaluate $v^h$ are given by the quadrature rule.

We denote the corresponding quadrature nodes by $\{y_{l,g}'\}_{l=1,g=1}^{m,Q}$ and the weights by $\{\omega_g\}_{g=1}^Q$.5 We can then solve Equation (15) for a given pair of collocation nodes $\{\lambda_{2k}, y_l\}_{k=1,l=1}^{n,m}$ and the corresponding quadrature nodes $\{y_{l,g}'\}_{l=1,g=1}^{m,Q}$ to compute a vector $\lambda_{2l}^'$ of size $n \times m \times Q$ that consists of the corresponding values $\lambda_{2l,1,g}^'$ for each node. For each $\lambda_{2l,1,g}^'$, Equation (15) then reads

$$
\lambda_{2l,1,g}^' = \frac{\lambda_{2l}^2 \Pi^2}{(1 - \lambda_{2l}^2) \Pi_1^l + \lambda_{2l}^2 \Pi^2}
$$

$$
\Pi^h = \delta^h e^{\rho^h \Delta c(y_{l,g}')} \left( \frac{v^h(\lambda_{2l,1,g}^', y_{l,g}' \cdot e^{\Delta c(y_{l,g}')})}{R^h \left[ v^h(\lambda_{2l}^, y') e^{\Delta c(y') | \lambda_{2l}^, y_l} \right]} \right)^{\alpha^h - \rho^h} dP^h(y_{l,g}' | y_l),
$$

where

$$
R^h \left[ v^h(\lambda_{2l}^, y') e^{\Delta c(y')} | \lambda_{2l}^, y_l \right] = G_h^{-1} \left( E \left[ G_h \left( v^h(\lambda_{2l}^', y') e^{\Delta c(y')} \right) \frac{dP^h(y')}{dP(y')} \lambda_{2l}^, y_l \right] \right).
$$

Note that $\lambda_{2l,1,g}^'$ depends on the full distribution of $\lambda_{2l}^$ through the expectation operator. By applying the Gaussian quadrature to compute the expectation we get

$$
E \left[ G_h \left( v^h(\lambda_{2l}^, y') e^{\Delta c(y')} \right) \frac{dP^h(y')}{dP(y')} \lambda_{2l}^, y_l \right] \approx \sum_{g=1}^Q G_h \left( v^h(\lambda_{2l,1,g}^', y_{l,g}' \cdot e^{\Delta c(y_{l,g})}) \right) \cdot \omega_g.
$$

By computing the expectation with the quadrature rule, we do not need the full distribution of $\lambda_{2l}^$; instead, we only have to evaluate $v^h$ at those values $\lambda_{2l,1,g}^'$ that can be obtained by solving (33) for each pair of collocation nodes $\{\lambda_{2l}, y_l\}_{k=1,l=1}^{n,m}$ and the corresponding quadrature nodes $\{y_{l,g}'\}_{l=1,g=1}^{m,Q}$. So at the end we have a square system of equations with $n \times m \times Q$ unknowns, $\lambda_{2l,1,g}^'$, and as many equations (33) for each $\{k, l, g\}$.

The value function is in general not known so we have to compute it simultaneously when

---

5Note that the quadrature nodes $\{y_{l,g}'\}_{l=1,g=1}^{m,Q}$ depend on the state today, $\{y_l\}_{l=1}^m$. 
solving for $\lambda_{k,l,g}$. Plugging the approximation $\hat{v}^h(\lambda_2, y)$ into the value function (12) yields

$$
\hat{v}^h(\lambda_{2k}^l, y_l) = \left[ (1 - \delta^h) (s^h_k)^{\rho^h} + \delta^h R^h \left( \hat{v}^h(\lambda_2^l, y') e^{\Delta \alpha(y')} \right) \right]^{1/\rho_h}.
$$

(34)

The collocation projection conditions require that the equation has to hold at each collocation node $\{\lambda_{2k}^l, y_l\}_{k=1,m=1}^{n,m}$. So we obtain a square system of equations with $n \times m \times 2$ equations (34) and as many unknowns for the spline interpolation at each collocation node, which we solve simultaneously with the system for $\lambda_{k,l,g}$ described above.

For all results presented in the paper, we choose an approximation interval for $x_t$ that covers $\pm 4$ (unconditional) standard deviations around the unconditional mean of the process. For $\lambda_{t}^2$, the minimum and maximum values are given by 0 and 1 so the full state space is included.

We approximate the value functions using two-dimensional cubic splines with not-a-knot end conditions. We provide the solver with additional information that we can formally derive for the limiting cases. For example, we know that for $\lambda_{t}^2 = 1$ ($\lambda_{t}^2 = 0$) agent 2 (1) consumes everything, so this corresponds to the representative-agent economy populated only by agent 2 (1). Hence, we require that the value function for these cases equals the value function for the corresponding representative-agent economy. We also know that for $\lambda_{t}^2 = 0$ ($\lambda_{t}^2 = 1$) the consumption of agent 2 (1) is 0, and hence the value function is also 0. As the shocks in the model are normally distributed, we compute the expectations over the exogenous states by Gauss–Hermite quadrature using five nodes for the shock in $x_{t+1}$ and three for the shock in $\Delta c_{t+1}$.

### C.2 Accuracy of the Solution Method

In the following we provide details of the accuracy of the solution method. We report numerical errors in the fixed-point equation (12), which determines the value functions of the two agents. In addition, we report the errors in the equilibrium conditions (15) for the Negishih weights. For the models with Epstein–Zin preferences, we compute these errors on a $200 \times 200$ uniform grid for the two states of the model—the exogenous long-run risk state, $x_t$, and the endogenous Negishih weight, $\lambda_{t}^B$.

As a benchmark for our analysis, we also report numerical errors for a model with CRRA preferences. In the CRRA case, the dynamics of $\lambda_t^B$ are exogenous and given in closed form; see equation (15), which shows that for $\rho^h = a^h$, $\lambda_{t+1}^B$ does not depend on the value functions. Thus, we only have to approximate the value functions in the fixed-point equation (12) for the exogenous process $x_t$ specified in equation (16) and the exogenous weights $\lambda_t^B$ given by
(15) using the probability ratio specified in equation (17). This benchmark case gives us a first indication of the adequateness of the projection approach to precisely approximating the value functions of the agents.

Table 1 reports the root-mean-square error (RMSE) as well as the maximum absolute error (MAE) for different numbers of collocation nodes using a uniform grid with \( n \) nodes for the \( \lambda_t^B \) dimension and \( m \) nodes for the \( x_t \) dimension. The first panel reports these errors for a CRRA model with \( \psi^h = \frac{1}{x^h} = 1.5 \) and persistence parameter values \( \rho^A_x = 0.985 \) and \( \rho^B_x = 0.975 \). We observe that for \( n = 16 \) and \( m = 8 \), errors are already small with a maximum error in the value function of agent A of 5.2e-4. For \( n = 20 \) the MAE can be decreased to 2.8e-5 and for \( n = 50 \) all errors are smaller then 1.0e-5. So, we observe a high accuracy of the projection approach for the approximation of the agents’ value functions.

The second panel of Table 1 reports errors for the same calibration but with Epstein–Zin preferences (\( \gamma^h = 10, \psi^h = 1.5 \)) as used in Sections 3 and 5 of the paper. Now the Negishi weights \( \lambda_{t+1}^B \) depend on the value functions and, therefore, we must solve for the value functions in equation (12) jointly with the dynamics for the Negishi weights; see equation (15). For the small belief difference, the numerical errors are already small for \( n = 16 \) and \( m = 12 \) nodes with a maximum absolute error of 3.1e-5 in the equation for the Negishi weights. For \( n = 50 \) and \( m = 22 \) the maximum error in the Negishi weights reduces to 8.6e-6 and the root-mean-square errors are below 5e-7.

The bottom panel of Table 1 reports errors for the main economy with \( \rho^A_x = 0.99 \) and \( \rho^B_x = 0.96 \), which we discussed extensively in Section 4 of the paper. We observe that the numerical errors are large for the approximation with \( n = 16 \) and \( m = 12 \) with an RMSE of 0.0016 for both \( v_t^B \) and \( \lambda_t^B \) and maximum absolute errors of 0.0229 and 0.0450, respectively. Figure 1 plots the numerical errors in the value functions. We observe that the errors are especially large for \( \lambda_t^B \) close to 0. Therefore, we adjust the uniform collocation grid in order to obtain higher accuracy close to the boundary. In particular, we choose the grid such that half the nodes are uniformly distributed between 0 and 0.1 and the other half are uniformly distributed between 0.1 and 1. Figure 2 shows that the errors in \( v_t^B \) close to the boundary can be reduced by using this adjusted grid. In Table 1 the results for the adjusted grid are marked with an asterisk (*). We observe that using the same number of nodes, the maximum error in \( v_t^B \) can almost be halved to 0.0127 and the maximum error in \( \lambda_t^B \) can be reduced by more than a factor of 3 to 0.0135. Increasing the number of collocation points reduces the errors further. For \( n = 50 \) and \( m = 22 \) the largest RMSEs is 2.3e-4 for \( v_t^B \) with a corresponding maximum error of 0.0100. Figure 3 shows that the errors are still large, especially close to the
The table shows root-mean-square errors (RMSEs) as well as maximum absolute errors (MAEs) in the value functions (12) as well as the equilibrium conditions for the Negishi weights (15) for different model calibrations and numbers of collocation nodes. Errors are reported for a state grid of ±4 standard deviations around the unconditional mean of $x_t$. For $\lambda_t^B$ the full grid between 0 and 1 is used. We denote the number of collocation nodes for the $x_t$-dimension by $n$ and the number of collocation nodes for the $x_t$-dimension by $m$. An asterisk (*) denotes cases where we used an adjusted grid—instead of a uniform grid—to account for the nonlinearity close to the boundary at $\lambda_t^B = 0$. 

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$v_t^A$ RMSE</th>
<th>$v_t^A$ MAE</th>
<th>$v_t^B$ RMSE</th>
<th>$v_t^B$ MAE</th>
<th>$\lambda_t^B$ RMSE</th>
<th>$\lambda_t^B$ MAE</th>
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<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>9.9e-6</td>
<td>1.7e-4</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>20</td>
<td>8</td>
<td>2.9e-4</td>
<td>2.8e-4</td>
<td>5.2e-6</td>
<td>9.0e-5</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>50</td>
<td>8</td>
<td>1.9e-6</td>
<td>1.0e-5</td>
<td>1.8e-6</td>
<td>7.7e-6</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\rho_x^A = 0.985, \rho_x^B = 0.975$, EZ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>12</td>
<td>1.4e-7</td>
<td>1.3e-6</td>
<td>7.8e-7</td>
<td>6.5e-6</td>
<td>2.7e-6</td>
<td>3.1e-5</td>
</tr>
<tr>
<td>50</td>
<td>22</td>
<td>1.1e-8</td>
<td>1.2e-7</td>
<td>7.7e-8</td>
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<td>4.3e-7</td>
<td>8.6e-6</td>
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<td></td>
<td></td>
<td>$\rho_x^A = 0.99, \rho_x^B = 0.96$, EZ</td>
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<td></td>
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</tr>
<tr>
<td>16</td>
<td>12</td>
<td>4.7e-5</td>
<td>6.6e-4</td>
<td>0.0016</td>
<td>0.0229</td>
<td>0.0016</td>
<td>0.0450</td>
</tr>
<tr>
<td>16*</td>
<td>12</td>
<td>9.1e-5</td>
<td>6.3e-4</td>
<td>4.5e-4</td>
<td>0.0127</td>
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<tr>
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<td>2.0e-4</td>
<td>4.7e-6</td>
<td>2.6e-4</td>
</tr>
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</table>
boundary of $\lambda^B_t = 0$. By increasing the number of nodes in the $\lambda^B$ dimension further to 80, the errors can be reduced significantly with root-mean-square errors between $1.9e-7$ and $6.5e-6$ and maximum errors in the order of $1e-4$. Note that errors are computed on a very large grid covering $\pm 4$ unconditional standard deviations of the $x_t$-process. To verify that the errors are not only numerically small but also small in economic terms, we conduct two exercises. First we ask the hypothetical question by how much one needs to change the consumption share of the respective agent in order that the errors are exactly zero. For the high degree approximation with $n = 80$ we find maximum values for agent 1 of $0.0023$ and agent 2 of $0.0038$. This implies that the consumption shares need to be adjusted by a maximum of only $0.0038$ in order to exactly satisfy the equilibrium condition. As a second test for economic significance, we analyze the influence of the errors on asset prices. We find that increasing the approximation degree does not change the model outcomes. This is demonstrated in Table 2 which shows the annualized asset-pricing moments as reported in Table 2 for the different approximation degrees. While the moments change slightly for the degree 16 compared to the degree 50 approximation, there is hardly a difference between the degree 50 and degree 80 solutions. Hence, the small errors near the boundary of the approximation space do not affect the model outcomes and the quantitative conclusions drawn in the paper.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$E (p_t - d_t)$</th>
<th>$\sigma (p_t - d_t)$</th>
<th>$AC1 (p_t - d_t)$</th>
<th>$E \left( R^m_t - R^f_t \right)$</th>
<th>$E \left( R^f_t \right)$</th>
<th>$\sigma (R^m_t)$</th>
<th>$\sigma (R^f_t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>12</td>
<td>3.5264</td>
<td>0.3823</td>
<td>0.8007</td>
<td>5.3815</td>
<td>2.2911</td>
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<td>1.9147</td>
</tr>
<tr>
<td>16*</td>
<td>12</td>
<td>3.5304</td>
<td>0.3866</td>
<td>0.8005</td>
<td>5.4512</td>
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<td>21.896</td>
<td>1.9342</td>
</tr>
<tr>
<td>50*</td>
<td>22</td>
<td>3.5304</td>
<td>0.3845</td>
<td>0.8002</td>
<td>5.4203</td>
<td>2.2856</td>
<td>21.806</td>
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</tr>
<tr>
<td>80*</td>
<td>22</td>
<td>3.5305</td>
<td>0.3846</td>
<td>0.8002</td>
<td>5.4204</td>
<td>2.2856</td>
<td>21.807</td>
<td>1.9246</td>
</tr>
</tbody>
</table>

The table shows selected annualized moments as in Table 2 for the two-agent economy for different approximation degrees. Agent A has the correct beliefs with $\rho^A_x = \rho^A_t = 0.99$; agent B has the belief $\rho^B_x = 0.96$. $n$ denotes the number of collocation nodes for the $\lambda^B$-dimension and $m$ denotes the number of collocation nodes for the $x_t$-dimension. An asterisk (*) denotes cases where we used an adjusted grid—instead of a uniform grid—to account for the nonlinearity close to the boundary at $\lambda^B_t = 0$.

### D Closed-Form Solutions for the CRRA Case

In this section we derive closed-form solutions for the long-run risk model (16) with two investors and CRRA utility. Assume that the two agents A and B have CRRA utility with
The figure plots numerical errors in the value functions (12) for $\rho^A_x = 0.99$ and $\rho^B_x = 0.96$. A state grid of ±4 standard deviations around the unconditional mean of the $x_t$-process is used. For $\lambda^B_t$ the full grid between 0 and 1 is used. The projection method uses $n = 16$ collocation nodes for the $\lambda^B_t$-dimension and $m = 12$ nodes for the $x_t$-dimension.

the same degree of risk aversion. The equilibrium conditions (29) and (25) then simplify to

$$\left( \frac{s_{t+1}^A}{s_t^A} \right)^\gamma = \left( \frac{s_{t+1}^B}{s_t^B} \right)^\gamma \frac{dP_{t+1}^A}{dP_{t+1}^B}. \quad (35)$$

Taking logs and using the market-clearing condition yields

$$\log \left( \frac{s_{t+1}^A}{1 - s_{t+1}^A} \right) = \log \left( \frac{s_t^A}{1 - s_t^A} \right) + \frac{1}{\gamma} \log \left( \frac{dP_{t+1}^A}{dP_{t+1}^B} \right). \quad (36)$$

We log-linearize $\log(1 - s_{t+1}^A)$ around $\log(s_{t+1}^A) = \log(s_t^A)$. This step gives us

$$\log \left( 1 - e^{\log(s_{t+1}^A)} \right) \approx \log (1 - s_t^A) - \frac{s_t^A}{1 - s_t^A} (\log (s_{t+1}^A) - \log (s_t^A))$$

$$= \log (1 - s_t^A) + \frac{s_t^A}{1 - s_t^A} \log (s_t^A) - \frac{s_t^A}{1 - s_t^A} a - \frac{s_t^A}{1 - s_t^A} b n_{x,t+1}. \quad (37)$$
Figure 2: Numerical Errors for $\rho^A_x = 0.99, \rho^B_x = 0.96, n = 16, \text{ and } m = 12$ with the Adjusted Grid

The figure plots numerical errors in the value functions (12) for $\rho^A_x = 0.99$ and $\rho^B_x = 0.96$. A state grid of $\pm 4$ standard deviations around the unconditional mean of the $x_t$-process is used. For $\lambda^B_t$ the full grid between 0 and 1 is used. The projection method uses $n = 16$ collocation nodes for the $\lambda^B_t$-dimension and $m = 12$ nodes for the $x_t$-dimension. Results are shown for the adjusted grid to account for the nonlinearity close to $\lambda^B_t = 0$. 
The figure plots numerical errors in the value functions (12) for $\rho^A_x = 0.99$ and $\rho^B_x = 0.96$. A state grid of $\pm4$ standard deviations around the unconditional mean of the $x_t$-process is used. For $\lambda^B_t$ the full grid between 0 and 1 is used. The projection method used $n = 50$ collocation nodes for the $\lambda^B_t$-dimension and $m = 22$ nodes for the $x_t$-dimension. Results are shown for the adjusted grid to account for the non-linearity close to $\lambda^B_t = 0$.

Since $x_{t+1} \sim N(\rho_x x_t, \sigma^2_x)$, the probability ratio is given by

$$
\log \left( \frac{dP^A_{t,t+1}}{dP^B_{t,t+1}} \right) = \log \left( e^{-\frac{0.5(\rho_x x_t + \eta_{x,t+1} - \rho^A_x x_t)^2}{\sigma^2_x} + 0.5(\rho_x x_t + \eta_{x,t+1} - \rho^B_x x_t)^2}{\sigma^2_x}} \right) 
= \frac{x_t^2}{2\sigma^2_x} \left( (\rho_x - \rho^B_x)^2 - (\rho_x - \rho^A_x)^2 \right) + \frac{x_t}{\sigma_x} (\rho^A_x - \rho^B_x) \eta_{x,t+1}. 
$$

Hence, we find that the consumption share in $t+1$ is a linear function of $\eta_{x,t+1}$,

$$
\log \left( s^A_{t+1} \right) = a^{\text{CRA}} + b^{\text{CRA}} \eta_{x,t+1} 
$$

and the coefficients are given by

$$
b^{\text{CRA}} = \frac{(1 - s^A_0) x_t (\rho^A_x - \rho^B_x)}{\sigma^2_x} 
$$

$$
a^{\text{CRA}} = \log \left( s^A_0 \right) + \frac{(1 - s^A_t) x_t^2}{2\sigma^2_x} \left[ (\rho_x - \rho^B_x)^2 - (\rho_x - \rho^A_x)^2 \right]. 
$$

The slope $b^{\text{CRA}}$ determines how the consumption share of investor A changes in response to shocks to $x_{t+1}$. Assume that $\rho^A_x > \rho^B_x$. The sign of $b^{\text{CRA}}$ depends on the sign of $x_t$. If
The figure plots numerical errors in the value functions (12) for $\rho^A_x = 0.99$ and $\rho^B_x = 0.96$. A state grid of $\pm 4$ standard deviations around the unconditional mean of the $x_t$-process is used. For $\lambda_t^B$ the full grid between 0 and 1 is used. The projection method uses $n = 80$ collocation nodes for the $\lambda_t^B$-dimension and $m = 22$ nodes for the $x_t$-dimension. Results are shown for the adjusted grid to account for the nonlinearity close to $\lambda_t^B = 0$. 

$x_t$ is positive (negative), $b^{CRRA}$ is positive (respectively, negative); this sign implies that the larger the shock to $x_{t+1}$, the larger (smaller) will be $\log s_{t+1}^A$ and, hence, the larger (smaller) the consumption share of agent A. The intuition is that investor B believes in faster mean reversion and hence puts more probability weight on states where $x_{t+1}$ moves toward its long-run mean of 0. So investors bet on states depending on the subjective probabilities they assign to those states. We call this motivation for investments the "speculation motive" of the investors. This motive increases with $|\rho^A_x - \rho^B_x|$ and $|x_t|$ and decreases with the risk aversion $\gamma$.

The larger the difference in the beliefs, $|\rho^A_x - \rho^B_x|$, the larger is the difference in the probabilities that the investors assign to different states. For $x_t = 0$ investors share the same beliefs but the larger $|x_t|$ is, the more important becomes the difference in the beliefs about the speed of mean reversion. Finally, the more risk averse investors are, the less they are willing to speculate on future outcomes.

We observe that this speculation motive is independent of the true persistence $\rho_x$. However, the true persistence does influence the average change in the consumption share. Assume that investor A has the correct beliefs, $\rho_x = \rho^A_x$. The average change in the log consumption share
is then given by

\[
\begin{align*}
E_t \left( \log \left( s_{t+1}^A \right) \right) - \log \left( s_t^A \right) &= a^{CRRA} - \log \left( s_t^A \right) \\
&= \frac{(1 - s_t^A) x_t^2}{2 \sigma_{x,t}^2} (\rho_x^A - \rho_x^B)^2 \geq 0.
\end{align*}
\] (40)

We observe that—indeed of the states \( s_t^A \) and \( x_t \) and whether \( \rho_x^A \) is larger or smaller than \( \rho_x^B \)—the consumption share of investor A, who has the correct beliefs, will always increase on average. So for CRRA utility, the only thing that matters for the average change in the consumption shares is which investor has the correct beliefs. The speed at which he or she accumulates wealth depends on the risk aversion of the investor. So the more risk averse the investor, the less he or she will be willing to speculate on future outcomes and, hence, the slower will be the wealth accumulation.

E Additional Details for the Model with Time-Varying Persistence

Below we provide the state vectors as well as the Markov transition probabilities for the model with time-varying persistence used in Section 4.5 of the paper. For the two-state economy, the state vector is given by \( \rho_{x,t} = [0.9686, 0.9798] \) and the Markov transition probabilities are given by

\[
P = \begin{bmatrix}
0.9963 & 0.0037 \\
0.0037 & 0.9963
\end{bmatrix}.
\]

For the nine-state economy we have

\[
\rho_{x,t} = [0.9584, 0.9623, 0.9663, 0.9702, 0.9742, 0.9782, 0.9821, 0.9861, 0.9900]
\]
with

$$P = \begin{bmatrix}
0.9708 & 0.0288 & 0.0004 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0036 & 0.9709 & 0.0252 & 0.0003 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0072 & 0.9709 & 0.0216 & 0.0002 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0108 & 0.9710 & 0.0180 & 0.0001 & 0 & 0 & 0 \\
0 & 0 & 0.0001 & 0.0144 & 0.9710 & 0.0144 & 0.0001 & 0 & 0 \\
0 & 0 & 0 & 0.0001 & 0.0180 & 0.9710 & 0.0108 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0002 & 0.0216 & 0.9709 & 0.0072 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0003 & 0.0252 & 0.9709 & 0.0036 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0004 & 0.0288 & 0.9708
\end{bmatrix}.$$ 

**F Additional Results**

This section presents additional results, which have been referenced in the main body of the paper. Figure 5 shows the conditional mean and variance of continuation utilities for the discrete state economy in the good state; see Section 3.2 of the paper.

In Section 3.1 of the paper, we examine the distribution of the persistence of forecaster predictions using a panel regression. An alternative approach is to consider the regression coefficients of individual forecasters in the following regression,

$$\Delta c'_{t,i} = A_i + \beta_i \Delta c_{t,i} + \epsilon_{t,i}. \quad (41)$$

Figure 6 shows the distribution of $\beta_i$ coefficients.
The figure plots the conditional mean and variance of continuation utilities as a function of the consumption share of investor B for the discrete state economy. The top panel shows the difference in expected utility between an economy in which the agents are allowed to trade and a no-trade assumption. The lower panel shows the conditional variance for both the trade and no-trade case. The left panel shows the results for investor A, who believes in the higher persistence and the right panel for investor B, who believes in lower persistence. Results are shown for economy being in the good state.
The figure shows a histogram of the AR(1) coefficients ($\beta_i$) for real-consumption-growth forecasts, see model (41), for a group of forecasters from the U.S. Survey of Professional Forecasters. Only forecasters with predictions in at least eight surveys are included in the sample. For the presentation of the histogram, the two highest and the two lowest coefficients have been removed.

References


