Internet Appendix for

An Anatomy of the Market Return

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IA.A. Additional Examples

IA.A.1. Black-Scholes Model and Symmetry

Example IA.A.1 (Black-Scholes is Put-Call-Symmetric). In the Black-Scholes model, the log forward gross return is normally distributed

$$\log R \overset{\mathbb{Q}}{\sim} N\left(-\frac{1}{2}\sigma^2(T-t), \sigma\sqrt{T-t}\right).$$

From Theorem 2.5 in Carr and Lee (2009) an asset is put-call-symmetric if the distribution of its log return is symmetric under $\mathbb{H}$. For this purpose we investigate the characteristic function of $\log R$ for $u \in \mathbb{R}$

$$\mathbb{E}^{1/2}\left[e^{\mathrm{i}u\log R}\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{(1/2+\mathrm{i}u)\log R}\right]}{\mathbb{E}^{\mathbb{Q}}\left[R^{1/2}\right]} = \exp\left(-\frac{1}{2}\sigma^2u^2(T-t)\right) \tag{IA.A.1}$$

where i denotes the imaginary unit. The characteristic function is real and even, and hence the distribution of $\log R$ is symmetric under $\mathbb{H}$. Equivalently, the implied volatility surface of the Black-Scholes model as a constant is trivially symmetric. ♢

IA.A.2. Some Transform Applications in Economics and Finance

In this Section I contrast my approach with three existing applications using the same technique. The first is the Capital Asset Pricing Model (CAPM), the second is a closed-form expression for the transition density of a diffusion process, and the third are spectral methods in time series.

IA.A.2.1. Capital Asset Pricing Model

Denote as previously by $\mathcal{M} = \frac{d\mathbb{Q}}{d\mathbb{P}}$ the forward pricing kernel for a given time interval $[t, T]$. From no-arbitrage, the simply compounded forward return ξ of an arbitrary traded asset satisfies

$$\mathbb{E}^{\mathbb{P}}[\xi] = -Cov^{\mathbb{P}}(\mathcal{M}, \xi). \tag{IA.A.2}$$

Evaluating Eq. (IA.A.2) for the simply compounded forward market return and the forward return on some asset R_i yields

$$\mathbb{E}^{\mathbb{P}}[R_i - 1] = \frac{Cov^{\mathbb{P}}(\mathcal{M}, R_i)}{Cov^{\mathbb{P}}(\mathcal{M}, R)}\mathbb{E}^{\mathbb{P}}[R-1]. \tag{IA.A.3}$$

Linear asset pricing models start from assuming that the pricing kernel is linear in a set of factors $F := F_1, \ldots, F_j$. Alternatively, one could start from assuming that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{L}^2_{\mathbb{P}}$ and develop $\frac{d\mathbb{Q}}{d\mathbb{P}}$ against a multivariate polynomial in F. Take for simplicity $F = R - 1$, univariate in the forward market return. Stopping the polynomial expansion after the first order gives

$$\frac{d\mathbb{Q}}{d\mathbb{P}}^{(1)} = a_0 + a_1(R-1), \tag{IA.A.4}$$

in analogy to Eq. (9), where

$$a_1 \propto \left\langle \frac{d\mathbb{Q}}{d\mathbb{P}}, R\right\rangle_{\mathcal{L}^2_{\mathbb{P}}}.$$

The constant of proportionality arises from the orthonormalization of the canonical polynomial basis and likewise a_0 is proportional to constant parts of the orthogonalized polynomial in F. Plugging in the linear $\mathcal{L}^2_{\mathbb{P}}$ projection (IA.A.4) of $\mathcal{M}$ onto $R-1$ into Eq. (IA.A.2) gives

$$\mathbb{E}^{\mathbb{P}}[R_i - 1] = \frac{Cov^{\mathbb{P}}(R, R_i)}{\mathbb{V}^{\mathbb{P}}[R]}\mathbb{E}^{\mathbb{P}}[R-1],$$

the CAPM prediction. Methodologically identical to the framework outlined in Section 3.6, the above relation highlights a serious shortcoming arising from taking the unobservable $\mathbb{P}$ distribution as the $\mathcal{L}^2$ weight: the ratio $\frac{Cov^{\mathbb{P}}(R,R_i)}{\mathbb{V}^{\mathbb{P}}}$ is comprised of moments which need to be estimated unconditionally from sample averages. In contrast, the coefficients in Eq. (9) are functions of moments with respect to the observable forward-neutral density $\mathbb{Q}$ and as such fully conditional.

IA.A.2.2. Density Approximations

Aït-Sahalia (2002) aims at developing a closed-form expression of the transition density $d\mathbb{Q}_X$ of $X_t \mid X_0 = x$, where X_t is a realization from a possibly nonlinear diffusion process. For this purpose he scales and shifts X into $\tilde{X}$ with corresponding transition law $d\tilde{\mathbb{Q}}_X$. He then takes an auxiliary standard normal density $d\mathbb{N}$ and expands the likelihood ratio in $\mathcal{L}^2_{\mathbb{N}}$. He shows, under technical conditions on the drift and diffusion functions of X, that

$$\frac{d\tilde{\mathbb{Q}}_X}{d\mathbb{N}} \in \mathcal{L}^2_{\mathbb{N}},$$

with the expectation inner product taken with respect to $\mathbb{N}$ and therefore analogously to Eq. (9)

$$\frac{d\tilde{\mathbb{Q}}_X}{d\mathbb{N}}^{(J)} = \sum_{i=0}^{J}\left\langle \frac{d\tilde{\mathbb{Q}}_X}{d\mathbb{N}}, \varphi_i\right\rangle_{\mathcal{L}^2_{\mathbb{N}}}\varphi_i,$$

with φ_i the i-th Hermite polynomial, an orthonormal basis of $\mathcal{L}^2_{\mathbb{N}}$. It is the i-th inner product $\left\langle \frac{d\tilde{\mathbb{Q}}_X}{d\mathbb{N}}, \varphi_i\right\rangle_{\mathcal{L}^2_{\mathbb{N}}}$ which he approximates using an additional expansion around the infinitesimal generator of $\tilde{X}$, similarly to Linetzky (2004a) and Linetzky (2004b). This second approximation step is avoided in Filipović et al. (2013), who exploit that polynomial moments of affine processes are known in closed form. To relate the method and goal of Aït-Sahalia (2002) to mine, his object of interest is the likelihood ratio $\frac{d\tilde{\mathbb{Q}}_X}{d\mathbb{N}}$, which multiplied with the standard normal density yields a J-order approximation of the unknown transition density $d\tilde{\mathbb{Q}}$, while in my approach the object of interest is the gross equity return R. As far as the involved probability distributions are concerned, both the original measure $d\mathbb{Q}_X$ and the standard normal distribution are obviously model-based, while the inverse Hellinger measure can be inferred from option prices without resorting to a model.

IA.A.2.3. Time Series Analysis

Another application in signal processing in finance and economics is concerned with time series, viewing an observed sample path as discrete evaluation points of a deterministic and periodic function of time, and working out the importance of frequencies in representing this sample path in terms of trigonometric basis functions.[22] Consider for simplicity a univariate process X observed at times $0 = t_1, \dots, t_n = T$ and write $x := (x_1, \dots, x_n)^\top$ for the discretely observed realizations from this process. For $y, z \in \mathbb{C}^n$, the complex n-dimensional vector space, define the inner product

$$\langle y, z \rangle_{\mathbb{C}^n} := \sum_{i=1}^{n} y_i \overline{z}_i,$$

where $\overline{z}$ denotes the complex conjugate of the complex number z. An orthonormal basis of this n-dimensional complex space weighted with a uniform density is $e := (e_1, \dots, e_n)^\top$, where

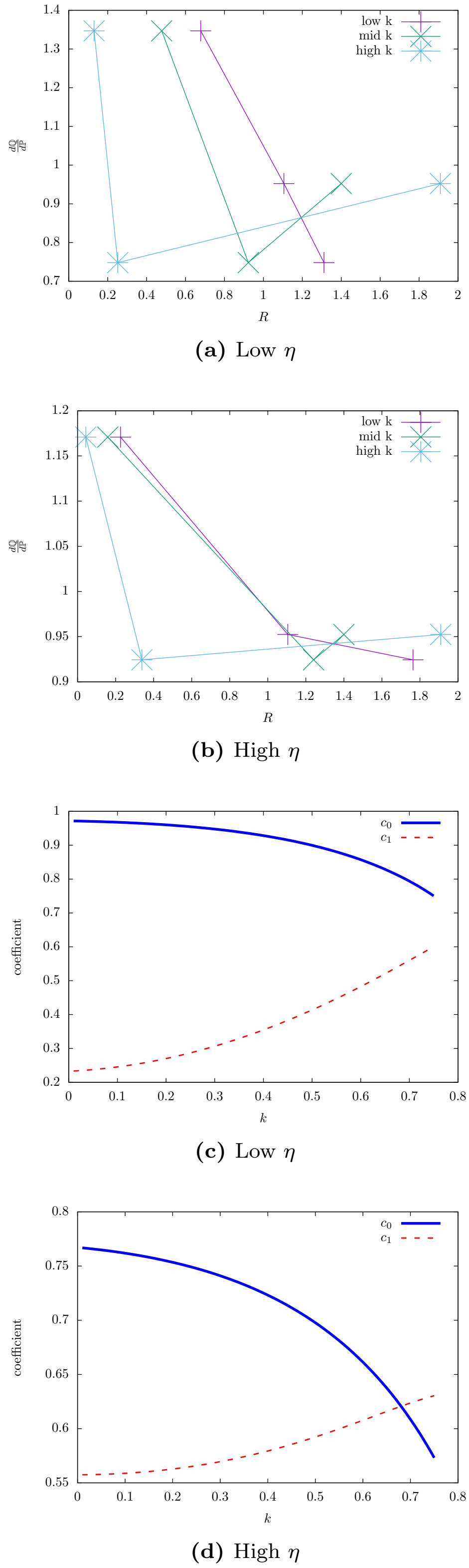
$$e_j := n^{-1/2}(e^{\mathrm{i}\omega_j}, e^{\mathrm{i}2\omega_j}, \dots, e^{\mathrm{i}n\omega_j})^\top, \\ j \in F_n := \{j \in \mathbb{Z} : -\pi < 2\pi j/n \le \pi\},$$

and $\omega_j := 2\pi j/n$. This basis is the canonical Fourier series, scaled with $n^{-1/2}$. Analogously to Eq. (9) we have the transform representation

$$x = \sum_{j \in F_n} \langle x, e_j \rangle_{\mathbb{C}^n} e_j. \tag{IA.A.5}$$

By construction, the energy $\|x\|^2_{\mathbb{C}^n}$ of the signal corresponds to n times the sample second moment of the time series.

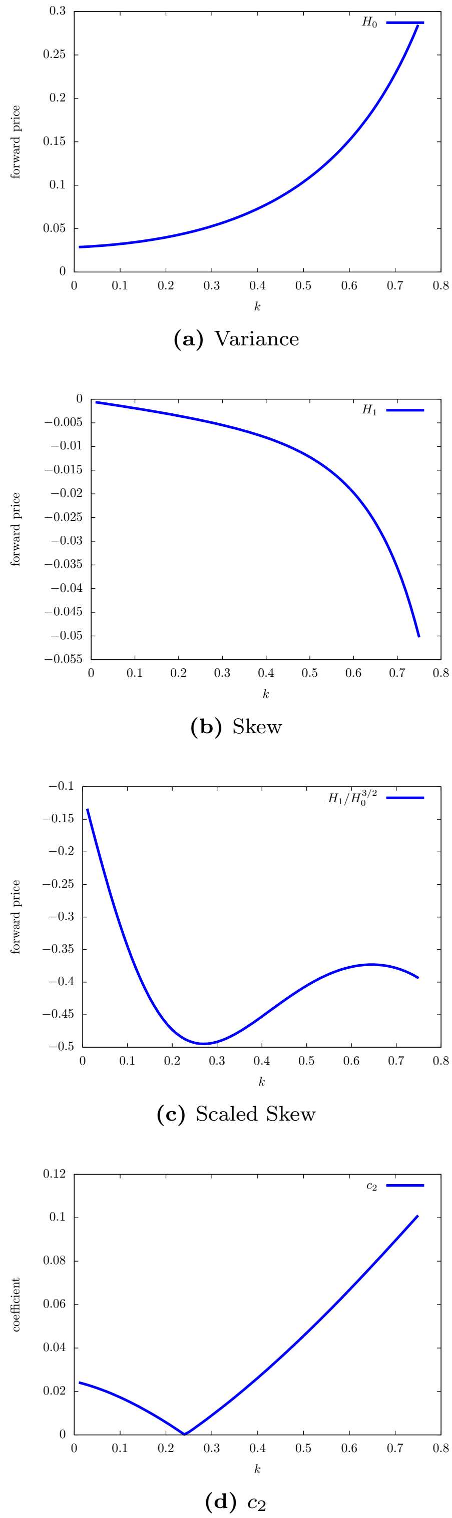
IA.B. Additional Figures



(a) Low η **(b)** High η **(c)** Low η **(d)** High η

Fig. IA.1. Lottery-Implied Pricing Kernels and Coefficients. Figures (a) and (b) show forward pricing kernels implied by the putting the lotteries A and B from Section 3.5 onto a joint tree implying forward probabilities given in the caption of Figure 3. Figure (a) is computed for η at one quarter of the admissible range, Figure (b) at three quarters. The parameter x is also set as a function of k to keep the forward equity premium constant at 5%. Panels (c) and (d) show the coefficients c_0 and c_1 from Eq. (9) for the same parameterization.

[22] See, for example, Brockwell and Davis (1993, Chapter 10) for an introduction. This Section follows his notation.



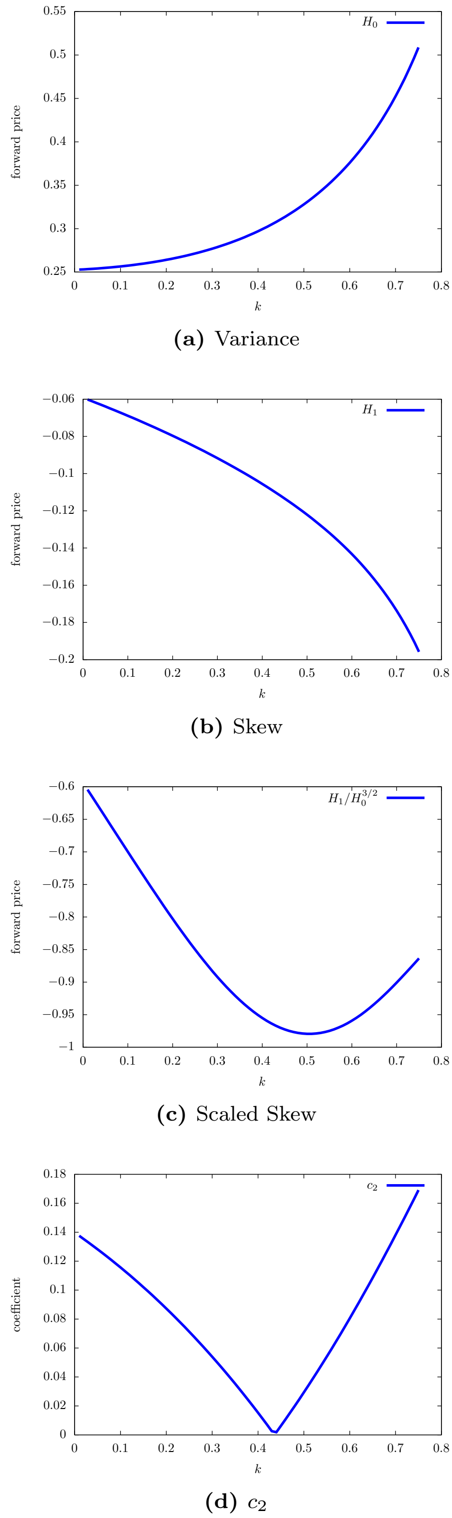
(a) Variance

(b) Skew

(c) Scaled Skew

(d) c_2

Fig. IA.2. Lottery-Implied Contracts and Coefficients (low zero mean- risk). The figures show forward prices of higher-order risk aversion and in an economy where the market is comprised of a cross section of lottery A and B with martingale probabilities displayed in Figure 3 with low zero mean risk η as a function of sure losses k. Figure (a) shows the price of variance, Figure (b) the price of skewness, and (c) a scaled version assimilating central skewness. The parameter x is also set as a function of k to keep the forward equity premium constant at 5%. Panels (d) shows the coefficient c_2 from Eq. (9) for the same parameterization.



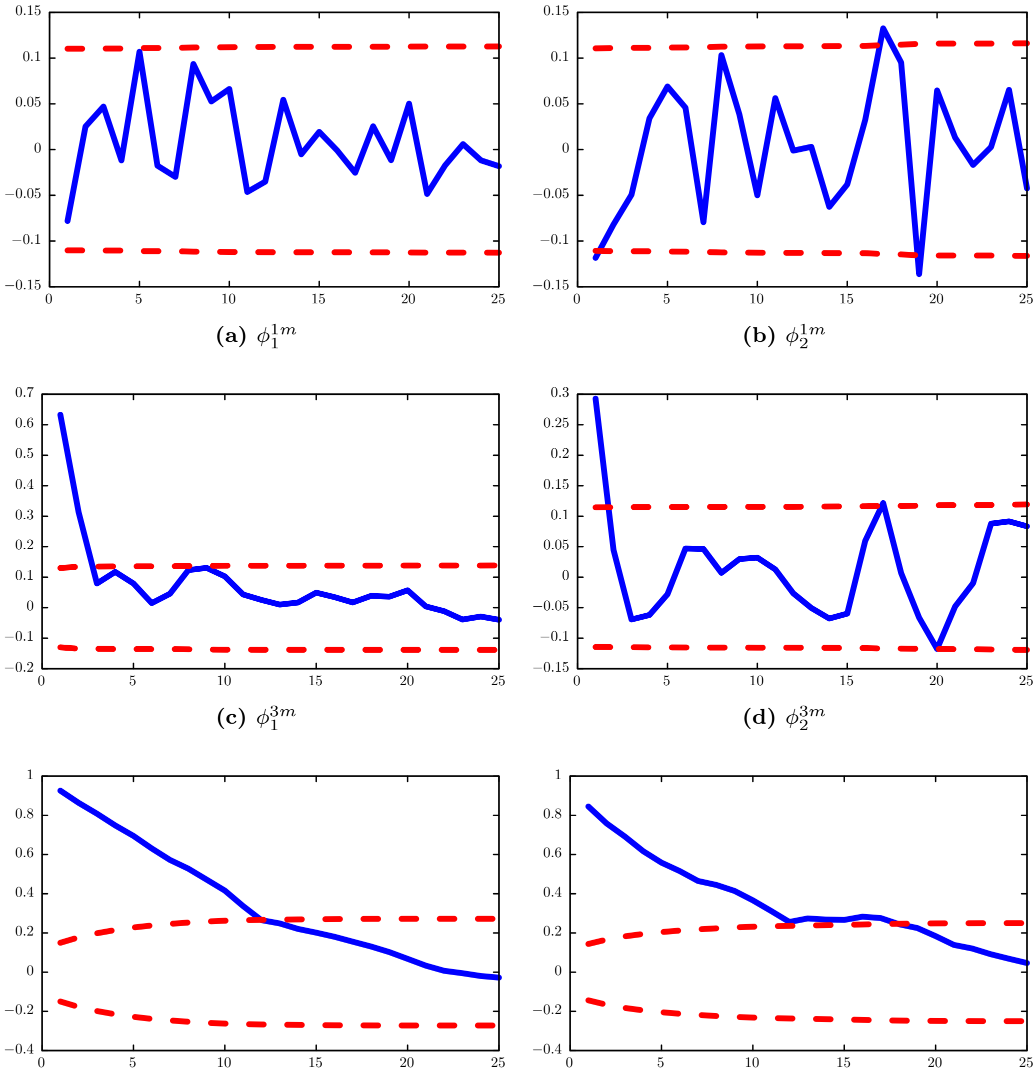
(a) Variance

(b) Skew

(c) Scaled Skew

(d) c_2

Fig. IA.3. Lottery-Implied Contracts and Coefficients (high zero mean- risk). The figures show forward prices of higher-order risk aversion and in an economy where the market is comprised of a cross section of lottery A and B with martingale probabilities displayed in Figure 3 with high zero mean risk η as a function of sure losses k. Figure (a) shows the price of variance, Figure (b) the price of skewness, and (c) a scaled version assimilating central skewness. The parameter x is also set as a function of k to keep the forward equity premium constant at 5%. Panels (d) shows the coefficient c_2 from Eq. (9) for the same parameterization.

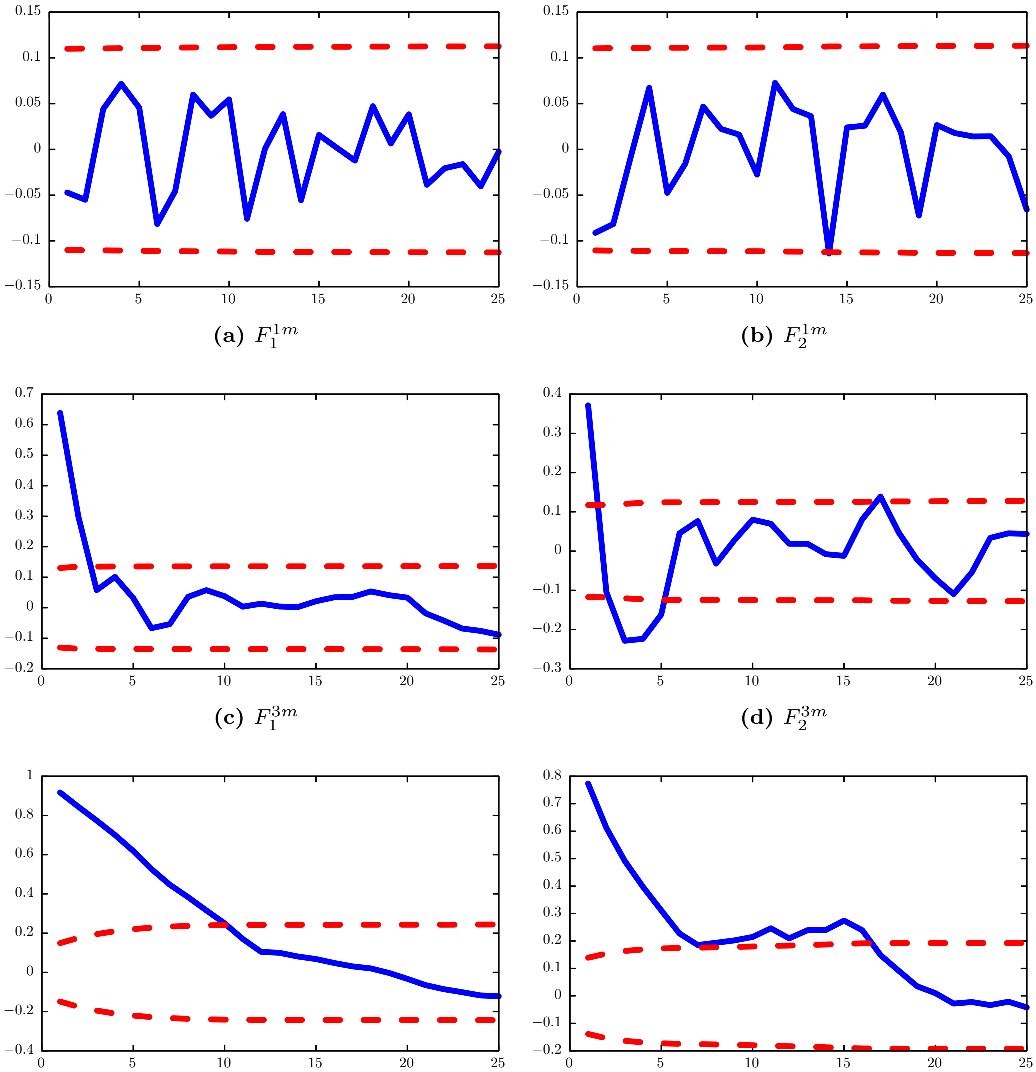


(a) ϕ_1^{1m} **(b)** ϕ_2^{1m}

(c) ϕ_1^{3m} **(d)** ϕ_2^{3m}

(e) ϕ_1^{12m} **(f)** ϕ_2^{12m}

Fig. IA.4. Basis Function Autocorrelations. The panels show autocorrelations of basis functions ϕ_i from Eq. (19) for orders $i = 1$, and 2 (solid blue). The dotted red lines represent 95% confidence intervals. The data producing the panels are European options on the S&P 500 from 1990 to 2016 with maturities 1, 3, and 12 months.



(a) F_1^{1m} (b) F_2^{1m}

(c) F_1^{3m} (d) F_2^{3m}

(e) F_1^{12m} (f) F_2^{12m}

Fig. IA.5. Factor Autocorrelations. The panels show autocorrelations of factors F_i from Eq. (19) for orders $i = 1$, and 2 (solid blue). The dotted red lines represent 95% confidence intervals. The data producing the panels are European options on the S&P 500 from 1990 to 2016 with maturities 1, 3, and 12 months.