Friends or Foes?
The Interrelationship between Angel and Venture Capital Markets

— Online Appendix —

Angel market: equilibrium equity shares and entrepreneur’s outside option.

According to the Nash product, \( \alpha^* \) is implicitly defined by

\[
\frac{dD_1^E(e_1^*)}{d\alpha} D_1^A(e_1^*) + (D_1^E(e_1^*) - U_1^E) \frac{dD_1^A(e_1^*)}{d\alpha} = 0.
\]  

(18)

Applying the Envelope Theorem we find that \( \frac{dD_1^E(e_1^*)}{d\alpha} < 0 \). We can then infer from Eq. (18) that \( \frac{dD_1^A(e_1^*)}{d\alpha} > 0 \) must hold for \( \alpha = \alpha^* \). Using Eq. (18) we can implicitly differentiate \( \alpha^* \) w.r.t. \( U_1^E \):

\[
\frac{d\alpha^*}{dU_1^E} = \frac{\frac{dD_1^A}{d\alpha}}{\frac{dD_1^E}{d\alpha} D_1^A + (D_1^E - U_1^E) \frac{dD_1^A}{d\alpha}}.
\]

(19)

Note that the denominator is strictly negative due to the second-order condition for \( \alpha^* \). Moreover, recall that \( \frac{dD_1^A}{d\alpha} > 0 \). Thus, \( \frac{d\alpha^*}{dU_1^E} < 0 \).

Angel market: optimal transfer payment.

Suppose the angel makes the transfer \( T \) to the entrepreneur in exchange for an additional equity stake \( \tilde{\alpha}(T) \). The angel’s new equity share is then given by \( \alpha(T) = \alpha^* + \tilde{\alpha}(T) \), with \( \alpha'(T) > 0 \), \( \alpha(T) \geq 0 \) \( \forall T \geq 0 \), and \( \alpha(T) < 0 \) \( \forall T < 0 \). Note that any post bargaining transfers aimed at adjusting the equity allocation, must improve joint efficiency to be implementable. The joint utility at the deal stage is

\[
D_1^A + D_1^E = \rho_1(e_1) \left[ g \left[ U_1^A + U_2^E \right] + (1 - g) y_1 \right] - k_1 - c(e_1),
\]

(20)

where \( e_1 \equiv e_1(\alpha(T)) \). Thus, the marginal effect of a transfer \( T \) on joint utility is given by

\[
\frac{d[D_1^A + D_1^E]}{dT} = \left[ \rho_1'(e_1) g \left[ U_2^A + U_2^E \right] - c'(e_1) \right] \frac{de_1}{d\alpha} \frac{d\alpha(T)}{dT}.
\]

(21)
Recall that \( d e_1 / d \alpha < 0 \), so that \( e_1 \) is maximized at \( \alpha = 0 \). Moreover, \( X \geq 0 \). Thus, \( d [D_1^A + D_1^E] / d T > 0 \) requires that \( d\alpha(T)/dT < 0 \), and therefore \( T < 0 \). However, because of his zero wealth, the entrepreneur cannot make a payment to the angel. Thus, \( T^* = 0 \).

**Derivation of angel market equilibrium.**

Using \( q_1^A = x_1 / M_1^A \) and \( x_1 = \phi_1 [M_1^E M_1^A]^{0.5} \), we can write Eq. (6) as

\[
\phi_1 D_1^A \left[ \frac{M_1^E}{M_1^A} \right]^{0.5} = \sigma_1^A. \tag{22}
\]

Using \( \theta_1 = M_1^A / M_1^E \) we then get the equilibrium degree of competition for the angel market: \( \theta_1^* = \left[ \phi_1 D_1^A / \sigma_1^A \right]^2 \). Next, note that we can write Eq. (22) as

\[
M_1^A = M_1^E \left[ \frac{\phi_1 \tilde{D}_1^A}{\sigma_1^A} \right]^2 = M_1^E \theta_1. \tag{23}
\]

Solving Eq. (8) for \( M_1^F \) and using \( q_1^E = \phi_1 [M_1^E M_1^A]^{0.5} / M_1^E = \phi_1 [M_1^A / M_1^E]^{0.5} \), we get the equilibrium stock of entrepreneurs in the early stage market:

\[
M_1^{E*} = \frac{F(U_1^E)}{\delta_1 + q_1^E} = \frac{F(U_1^F)}{\delta_1 + \phi_1 [M_1^A / M_1^E]^{0.5}} = \frac{F(U_1^F)}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \tag{24}
\]

Thus, the equilibrium stock of angels is given by

\[
M_1^{A*} = M_1^{E*} \theta_1^* = \frac{F(U_1^F) \theta_1^*}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \tag{25}
\]

Using \( M_1^{E*} = M_1^{A*} / \theta_1^* \) we can then write \( x_1^* \) as

\[
x_1^* = \phi_1 \left[ M_1^{A*} M_1^{E*} \right]^{0.5} = \frac{\phi_1 M_1^{A*}}{\sqrt{\theta_1^*}} = F(U_1^E) \frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \tag{26}
\]

Moreover, using Eq. (9) and \( q_1^A = x_1 / M_1^A \) we get \( m_1^{A*} = q_1^A M_1^{A*} = x_1^* \).

**Proof of Proposition 1.**

Recall that the equilibrium of the angel market is determined by the deal values \( D_1^E \) and \( D_1^A \), and therefore by the late stage utilities \( U_2^E \) and \( U_2^A \), as well as by the entrepreneur’s outside option \( U_1^E \) (through \( \alpha^* \)). We will show in Proof of Proposition 4 that \( U_2^E \) and \( U_2^A \) do not
depend on $\phi_1$, $\delta_1$, $\sigma^E_1$, $\sigma^A_1$, and $k_1$. Next we need to derive a condition which defines $U^E_1$. The
equilibrium condition (5) can be written as

$$U^E_1 [r + \delta_1] = -\sigma^E_1 + q^E_1 \left[ D^E_1 - U^E_1 \right]. \quad (27)$$

Using $q^E_1 = \phi_1 \left[ M^A_1 / M^E_1 \right]^{0.5} = \phi_1 \sqrt{\theta^*_1} = \phi_1^2 D^A_1 / \sigma^A_1$ we get the following condition which defines $U^E_1$:

$$U^E_1 [r + \delta_1] - \phi_1 D^A_1 \left[ D^E_1 - U^E_1 \right] + \sigma^E_1 = 0. \quad (28)$$

Now consider the equilibrium degree of competition $\theta^*_1$. Differentiating $\theta^*_1$ w.r.t. $\delta_1$ yields

$$\frac{d\theta^*_1}{d\delta_1} = 2 \frac{\phi_1^2 D^A_1 dD^A_1}{[\sigma^A_1]^2} \frac{d\delta_1}{d\alpha} = 2 \frac{\phi_1^2 D^A_1 dD^A_1}{[\sigma^A_1]^2} \frac{d\alpha}{dU^E_1} \frac{dU^E_1}{d\delta_1}. \quad (29)$$

Next we define $\Gamma \equiv D^A_1 \left[ D^E_1 - U^E_1 \right]$. We then get

$$\frac{dU^E_1}{d\delta_1} = - \frac{U^E_1}{r + \delta_1 - \phi_1^2} \frac{dU^E_1}{d\alpha} \frac{d\alpha}{dU^E_1} + \frac{d\Gamma}{dU^E_1}. \quad (30)$$

Note that $d\Gamma / d\alpha = 0$ due to the first-order condition for $\alpha^*$. Moreover, $\partial \Gamma / \partial U^E_1 = -D^A_1$. Consequently,

$$\frac{dU^E_1}{d\delta_1} = - \frac{U^E_1}{r + \delta_1 + \frac{\phi_1^2}{\sigma^A_1} D^A_1} < 0. \quad (31)$$

This in turn implies that $d\theta^*_1 / d\delta_1 > 0$. Likewise,

$$\frac{d\theta^*_1}{d\sigma^E_1} = 2 \frac{\phi_1^2 D^A_1 dD^A_1}{[\sigma^A_1]^2} \frac{d\alpha^*}{d\sigma^E_1} \frac{dU^E_1}{d\sigma^E_1}, \quad (32)$$

with $dD^A_1 / d\alpha > 0$, $d\alpha^* / dU^E_1 < 0$, and

$$\frac{dU^E_1}{d\sigma^E_1} = - \frac{1}{r + \delta_1 + \frac{\phi_1^2}{\sigma^A_1} D^A_1} < 0. \quad (33)$$

Thus, $d\theta^*_1 / d\sigma^E_1 > 0$. Moreover, note that $dD^A_1 / dk_1 < 0$. Consequently, $d\theta^*_1 / dk_1 < 0$. For the remaining comparative statics it is useful to express the condition for $U^E_1$ in terms of $\theta^*_1$:

$$U^E_1 [r + \delta_1] - \phi_1 \sqrt{\theta^*_1} \left[ D^E_1 - U^E_1 \right] + \sigma^E_1 = 0, \quad (34)$$
so that
\[
\frac{dU_E^1}{d\theta_1^*} = \frac{\phi_1 \frac{1}{2\sqrt{\theta_1^*}} [D_E^1 - U_E^1]}{r + \delta_1 + \phi_1 \sqrt{\theta_1^*}} > 0. \tag{35}
\]
Moreover, using the definition of \(\theta_1^*\) we define
\[
G \equiv \theta_1^* - \left[ \frac{\phi_1}{\sigma_1^A} D_1^A \right]^2 = 0 \tag{36}
\]
where \(D_1^A = D_1^A(\alpha^*(U_E^1(\theta_1^*))). \)

We get
\[
\frac{d\theta_1^*}{d\phi_1} = \frac{2 \frac{\phi_1}{[\sigma_1^A]^2} [D_1^A]^2}{1 - 2 \left[ \frac{\phi_1}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{d\phi_1} \frac{dU_E^1}{d\theta_1^*} \right].} \tag{37}
\]
Recall that \(dD_1^A/\alpha > 0, d\alpha^*/dU_E^1 < 0, \) and \(dU_E^1/d\theta_1^* > 0. \) Thus, the denominator is positive, which implies that \(d\theta_1^*/d\phi_1 > 0. \) Likewise, using Eq. (36), we get
\[
\frac{d\theta_1^*}{d\sigma_1^A} = \frac{2 \frac{\phi_1}{[\sigma_1^A]^2} [D_1^A]^2}{1 - 2 \left[ \frac{\phi_1}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{d\sigma_1^A} \frac{dU_E^1}{d\theta_1^*} \right].} \tag{38}
\]
Again, the denominator is positive, which implies that \(d\theta_1^*/d\sigma_1^A < 0. \)

Next, note that \(dm_{E_E}/dU_E^1 = F'(U_E^1) > 0, \) and recall that \(dU_E^1/d\delta_1, dU_E^1/d\sigma_1^E < 0. \)

Moreover, using Eq. (28) we find
\[
\frac{dU_E^1}{d\phi_1} = \frac{2 \frac{\phi_1}{[\sigma_1^A]^2} D_1^A [D_E^1 - U_E^1]}{r + \delta_1 + \frac{\phi_1}{[\sigma_1^A]^2} D_1^A} > 0 \tag{39}
\]
\[
\frac{dU_E^1}{d\sigma_1^A} = \frac{- \frac{\phi_1^2}{[\sigma_1^A]} D_1^A [D_E^1 - U_E^1]}{r + \delta_1 + \frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A} < 0 \tag{40}
\]
Likewise, using \(\Gamma = D_1^A [D_E^1 - U_1^1], \)
\[
\frac{dU_E^1}{dk_1} = \frac{\frac{\phi_1^2}{[\sigma_1^A]} \Gamma}{r + \delta_1 + \frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A} \tag{41}
\]
with
\[
\frac{dT}{dk_1} = \frac{d\Gamma_1}{d\alpha} \frac{d\alpha^*}{dk_1} + \frac{\partial \Gamma}{\partial k_1} = - [D_1^E - U_1^E] < 0. \tag{42}
\]

Thus, \(dU_1^E/dk_1 < 0\). All this implies that \(m_1^E\) is increasing in \(\phi_1\), and decreasing in \(\delta_1, \sigma_1^E, \sigma_1^A, \) and \(k_1\).

Next, recall that \(m_1^A = x_1^*\) is given by
\[
m_1^A = x_1^* = F(U_1^E) \left( \frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}} \right). \tag{43}
\]

It is straightforward to show that \(dT/d(\phi_1 \sqrt{\theta_1^*}) > 0\). Because \(dU_1^E/d\phi_1 > 0\) and \(d\theta_1^*/d\phi_1 > 0\), we then have \(dm_1^A/d\phi_1 = dx_1^*/d\phi_1 > 0\). Likewise, we know that \(dU_1^E/d\sigma_1^A, dU_1^E/dk_1 < 0\) and \(d\theta_1^*/d\sigma_1^A, d\theta_1^*/dk_1 < 0\). Thus, \(dm_1^A/d\sigma_1^A < 0\) and \(dm_1^A/dk_1 = dx_1^*/dk_1 < 0\). Moreover, we have shown that \(dU_1^E/d\delta_1, dU_1^E/d\sigma_1^E < 0\), while \(d\theta_1^*/d\delta_1, d\theta_1^*/d\sigma_1^E > 0\). Thus, the effects of \(\delta_1\) and \(\sigma_1^E\) on \(m_1^A\) are ambiguous.

Now consider the equilibrium valuation \(V_1^*\). Note that \(V_1^*\) is decreasing in the angel’s equilibrium equity share \(\alpha^*\), which is defined by Eq. (18). Recall that \(d\alpha^*/dU_1^E < 0, dU_1^E/d\phi_1 > 0\) and \(dU_1^E/d\delta_1, dU_1^E/d\sigma_1^E, dU_1^E/d\sigma_1^A < 0\). Consequently, \(d\alpha^*/d\phi_1 < 0\) and \(d\alpha^*/d\delta_1, d\alpha^*/d\sigma_1^E, d\alpha^*/d\sigma_1^A > 0\). All this implies that \(V_1^*\) is increasing in \(\phi_1\), and decreasing in \(\delta_1, \sigma_1^E\) and \(\sigma_1^A\).

Furthermore, note that \(k_1\) affects \(D_1^A\) and \(U_1^A\). Using Eq. (18) we get
\[
\frac{d\alpha^*}{dk_1} = \frac{dD_1^E}{d\alpha} \frac{\partial D_1^A}{\partial k_1} - \frac{dU_1^E}{d\alpha} \frac{dD_1^A}{d\alpha} + (D_1^E - U_1^E) \frac{d^2 D_1^A}{d\alpha^2} \frac{dD_1^A}{d\alpha} \frac{d^2 D_1^A}{d\alpha^2}, \tag{44}
\]

where the denominator is strictly negative due to the second-order condition for \(\alpha^*\). Thus, to prove that \(d\alpha^*/dk_1 > 0\), we need to show that the numerator is positive. We know that \(dD_1^E/d\alpha < 0, dD_1^A/d\alpha > 0\), and \(dU_1^E/dk_1 < 0\). Moreover, \(\partial D_1^A/\partial k_1 = -1\) and \(d^2 D_1^A/(d\alpha dk_1) = 0\). Thus, the numerator is strictly positive, so that \(d\alpha^*/dk_1 > 0\). This in turn implies that the effect of \(k_1\) on \(V_1^* = k_1/\alpha^*\) is ambiguous.

Finally consider the equilibrium success probability \(\rho_1(e_1^*)\), with \(\rho_1'(e_1^*) > 0\). Using Eq. (3) we get
\[
\frac{de_1^*}{d\alpha} = \frac{d}{de_1} \left[ \rho_1'(e_1)[qU_2^A + (1-q)(1-\alpha)y_1] - c'(e_1) \right], \tag{45}
\]

5
where the denominator is strictly negative due to the second-order condition for \( e_1^* \). Thus, \( de_1^*/d\alpha < 0 \). Our comparative statics results for \( \alpha^* \) then imply that \( d\rho_1(e_1^*)/d\phi_1 > 0 \) and \( d\rho_1(e_1^*)/d\delta_1, d\rho_1(e_1^*)/d\sigma_1^E, d\rho_1(e_1^*)/d\sigma_1^A, d\rho_1(e_1^*)/dk_1 < 0 \).

\[ \square \]

**Early Stage Investment and Valuation.**

Consider first our base model with endogenous effort. Differentiating \( V_1^* \) w.r.t. \( k_1 \) yields

\[
\frac{dV_1^*}{dk_1} = d\left(\frac{k_1}{\alpha^*}\right) = \frac{\alpha^* - k_1 \frac{d\alpha^*}{dk_1}}{[\alpha^*]^2}. \tag{46}
\]

Note that \( dV_1^*/dk_1 > 0 \) when \( k_1 \to 0 \). Thus, the equilibrium valuation \( V_1^* \) is decreasing in \( k_1 \) when \( k_1 \) is sufficiently small.

Next, suppose the entrepreneur’s effort \( e_1 \) is exogenous, and define \( \rho_1 \equiv \rho_1(e_1) \). The early stage deal values are then given by

\[
D_E^1 = \rho_1 \left[ gU_2^E + (1 - g)(1 - \alpha)y_1 \right] - c \tag{47}
\]
\[
D_A^1 = \rho_1 \left[ gU_2^A + (1 - g)\alpha y_1 \right] - k_1, \tag{48}
\]

where \( c \) is the entrepreneur’s disutility of providing effort \( e_1 \). The optimal equity share for the angel, \( \alpha^* \), then satisfies the symmetric Nash bargaining solution, which accounts for the outside option of each party (\( U_1^E \) for the entrepreneur, and 0 for the angel because of free entry). Let \( \tilde{D}_1^E \) and \( \tilde{D}_1^A \) denote the deal values reflecting the Nash bargaining solution, which are given by

\[
\tilde{D}_1^E = \frac{1}{2} \left[ \rho_1 \left[ g \left(U_2^E + U_2^A\right) + (1 - g)y_1 \right] - k_1 - c + U_1^E \right] \tag{49}
\]
\[
\tilde{D}_1^A = \frac{1}{2} \left[ \rho_1 \left[ g \left(U_2^E + U_2^A\right) + (1 - g)y_1 \right] - k_1 - c - U_1^E \right]. \tag{50}
\]

The equilibrium equity share for the angel, \( \alpha^* \), then satisfies \( D_1^E(\alpha^*) = \tilde{D}_1^E \) and \( D_1^A(\alpha^*) = \tilde{D}_1^A \).

Recall that \( U_2^A = U_2^E \) in equilibrium. Thus,

\[
\alpha^* = \frac{1}{2} + \frac{k_1 - c - U_1^E}{2\rho_1(1 - g)y_1}. \tag{51}
\]
The equilibrium early stage valuation is $V_1^* = k_1/\alpha^*$. We get
\[ \frac{dV_1^*}{dk_1} = \frac{\alpha^* - k_1 \frac{d\alpha^*}{dk_1}}{[\alpha^*]^2}. \]  
(52)

The denominator is always non-negative. Moreover, note that $N \geq 0$ for $k_1 \to 0$, which implies that $dV_1^*/dk_1 \geq 0$ for $k_1 \to 0$. To show that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$, it is thus sufficient to verify that $dN/dk_1 > 0$:
\[ \frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left( \frac{d\alpha^*}{dk_1} + k_1 \frac{d^2\alpha^*}{dk_1^2} \right) = -k_1 \frac{d^2\alpha^*}{dk_1^2}. \]  
(53)

We need to find the sign of $d^2\alpha^*/dk_1^2$. We start by taking the first derivative of $\alpha^*$ w.r.t. $k_1$:
\[ \frac{d\alpha^*}{dk_1} = \frac{1}{2 \rho_1 (1 - g) y_1} \left[ 1 - \frac{dU_1^E}{dk_1} \right]. \]  
(54)

It is easy to see that $\tilde{D}_1^E - U_1^E = \tilde{D}_1^A$. Thus, the condition defining $U_1^E$ simplifies to
\[ U_1^E [r + \delta_1] - \frac{\phi_1^2}{\sigma_1^A} [\tilde{D}_1^A]^2 + \sigma_1^E = 0. \]  
(55)

Thus,
\[ \frac{dU_1^E}{dk_1} = -\frac{a_1 \tilde{D}_1^A}{r + \delta_1 + a_1 \tilde{D}_1^A}, \]  
(56)

where $a_1 = \phi_1^2/\sigma_1^A$. Consequently,
\[ \frac{d\alpha^*}{dk_1} = \frac{1}{2 \rho_1 (1 - g) y_1} \left[ 1 + \frac{1}{(r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-1} + 1} \right]. \]  
(57)

We then get
\[ \frac{d^2\alpha^*}{dk_1^2} = \frac{1}{2 \rho_1 (1 - g) y_1} \frac{-\frac{1}{2} a_1 (r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-2} \left[ 1 + \frac{dU_1^E}{dk_1} \right]}{(r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-1} + 1} \]  
(58)
Note that
\[ 1 + \frac{dU^E_1}{dk_1} = 1 - \frac{a_1\tilde{D}^A_1}{r + \delta_1 + a_1\tilde{D}^A_1} = \frac{r + \delta_1}{r + \delta_1 + a_1\tilde{D}^A_1} > 0. \] (59)
Thus, \(d^2\alpha*/dk_1^2 < 0\). This implies that \(dN/dk_1 > 0\), and therefore \(dV^*_1/dk_1 > 0\).

**Proof of Proposition 2.**

In equilibrium, \(U^E_2 = U^A_2\). Moreover, we will show in Proof of Proposition 3 that \(dU^E_2/d\phi_2 > 0\) and \(dU^E_2/d\delta_2, dU^E_2/d\sigma_2, dU^E_2/d\sigma^V_2, dU^E_2/dk_2 < 0\). Consider the equilibrium degree of competition \(\theta_1^*\). With \(U^E_2 = U^A_2\) note that
\[ \frac{d\theta^*_1}{dU^E_2} = 2\frac{\phi^2_1}{[\sigma^A_1]^2}D_1^A \frac{dD_1^A}{dU^A_2}. \] (60)

For a given \(\alpha\) we find that
\[ \frac{dD_1^A}{dU^E_2} = \rho_1(e_1^*) \frac{de_1^*}{U^E_2} \left[gU^E_2 + (1 - g)\alpha y_1\right] + \rho_1(e_1^*)g > 0. \] (61)

Moreover, applying the Envelope Theorem we get \(dD^E_1/dU^E_2 = g\rho_1(e_1^*) > 0\). Thus, the bargaining frontier shifts outwards, so that \(dD^E_1/dU^E_2 > 0\) and \(dD^A_1/dU^E_2 > 0\) at the equilibrium equity share \(\alpha^*\). This implies that \(d\theta^*_1/dU^E_2 > 0\), and consequently, \(d\theta^*_1/d\phi_2 > 0\) and \(d\theta^*_1/d\delta_2, d\theta^*_1/d\sigma_2, d\theta^*_1/d\sigma^V_2, d\theta^*_1/dk_2 < 0\).

Now consider the equilibrium inflow of entrepreneurs \(m^E_1^* = F(U^E_1)\), with \(F'(U^E_1) > 0\). Using Eq. (28) we get
\[ \frac{dU^E_1}{dU^E_2} = \frac{\phi^2_1}{\sigma^A_1} \frac{d\Gamma}{dU^E_2}, \] (62)
where \(\Gamma = D^A_1 [D^E_1 - U^E_1]\). Note that
\[ \frac{d\Gamma}{dU^E_2} = \frac{d\Gamma}{de_1 U^E_2} + \frac{d\Gamma}{d\alpha U^E_2} + \frac{\partial \Gamma}{\partial U^E_2}, \] (63)
where \( \frac{d\epsilon^*_1}{dU^E_2} > 0 \) and \( d\Gamma / d\alpha = 0 \) (see Eq. (18)). Moreover,

\[
\frac{d\Gamma}{de_1} = \frac{dD_1^A}{de_1} [D_1^E - U_1^E] + D_1^A \frac{D_1^E}{de_1} = \rho_1'(e_1^*) \frac{[gU_2^A + (1 - g)\alpha y_1] [D_1^E - U_1^E]}{\geq 0} > 0
\]

\[
\frac{\partial \Gamma}{\partial U_2^E} = \frac{\partial D_1^A}{\partial U_2^E} [D_1^E - U_1^E] + D_1^A \frac{\partial D_1^E}{\partial U_2^E} > 0
\]

(64)

This implies that \( dU_1^E / dU_2^E > 0 \), and therefore, \( dF(U_1^E) / dU_2^E > 0 \). Our comparative statics results for \( U_2^E \) (see Proof of Proposition 3) then imply that \( dm_1^{E*} / d\phi_2 > 0 \) and \( dm_1^{E*} / d\delta_2 \), \( dm_1^{E*} / d\sigma_2 \), \( dm_1^{E*} / d\sigma_Y \), \( dm_1^{E*} / dk_2 < 0 \).

Next consider the equilibrium inflow of angels, \( m_1^{A*} \), which is defined by

\[
m_1^{A*} = x_1^* = F(U_1^E) \left( \frac{\phi_1 \sqrt{\theta_1}}{\delta_1 + \phi_1 \sqrt{\theta_1}} \right) = m_1^{E*}.
\]

(66)

One can show that \( dT / d\sqrt{\theta_1} > 0 \). Our comparative statics results for \( m_1^{E*} \) and \( \theta_1^* \) then imply that \( dm_1^{A*} / d\phi_2 > 0 \) and \( dm_1^{A*} / d\delta_2 \), \( dm_1^{A*} / d\sigma_2 \), \( dm_1^{A*} / d\sigma_Y \), \( dm_1^{A*} / dk_2 < 0 \).

Now consider the equilibrium equity share \( \alpha^* \) for angels. Recall that \( dD_1^E / dU_2^E > 0 \) and \( dD_1^A / dU_2^E > 0 \) at the equilibrium equity share \( \alpha^* \). Moreover, using the Envelope Theorem it is straightforward to show that \( dD_1^A / dU_2^E > dD_1^E / dU_2^E \). The Nash bargaining solution then implies that \( d\alpha^* / dU_2^E < 0 \). Thus, \( d\alpha^* / d\phi_2 < 0 \) and \( d\alpha^* / d\delta_2 \), \( d\alpha^* / d\sigma_2 \), \( d\alpha^* / d\sigma_Y \), \( d\alpha^* / dk_2 > 0 \). For the equilibrium valuation \( V_1^* = k_1 / \alpha^* \) we can then infer that \( dV_1^* / d\phi_2 > 0 \) and \( dV_1^* / d\delta_2 \), \( dV_1^* / d\sigma_2 \), \( dV_1^* / d\sigma_Y \), \( dV_1^* / dk_2 < 0 \).

Finally consider the equilibrium success rate \( \rho_1(e_1^*) \), with \( \rho_1'(e_1^*) > 0 \). Using Eq. (3) it is straightforward to show that \( \partial e_1^* / \partial U_2^E > 0 \) and \( \partial e_1^* / \partial \alpha < 0 \). Using our comparative statics results for \( U_2^E \) and \( \alpha^* \) we can then infer that \( de_1^* / d\phi_2 > 0 \) and \( de_1^* / d\delta_2 \), \( de_1^* / d\sigma_2 \), \( de_1^* / d\sigma_Y \), \( de_1^* / dk_2 < 0 \). Consequently, \( d\rho_1(e_1^*) / d\phi_2 > 0 \) and \( d\rho_1(e_1^*) / d\delta_2 \), \( d\rho_1(e_1^*) / d\sigma_2 \), \( d\rho_1(e_1^*) / d\sigma_Y \), \( d\rho_1(e_1^*) / dk_2 < 0 \).
VC market: derivation of deal values and equity shares.

Let $CV_i$ denote the value generated by the coalition $i = EAV, EV, EA, AV, E, A, V$. Using the Shapley value we get the following general deal values from the tripartite bargaining game:

$$D^E_2 = \frac{1}{3} [CV_{EAV} - CV_{AV}] + \frac{1}{6} [CV_{EA} - CV_A] + \frac{1}{6} [CV_{EV} - CV_V] + \frac{1}{3} CV_E \quad (67)$$

$$D^A_2 = \frac{1}{3} [CV_{EAV} - CV_{EV}] + \frac{1}{6} [CV_{EA} - CV_E] + \frac{1}{6} [CV_{AV} - CV_V] + \frac{1}{3} CV_A \quad (68)$$

$$D^V_2 = \frac{1}{3} [CV_{EAV} - CV_{EA}] + \frac{1}{6} [CV_{EV} - CV_E] + \frac{1}{6} [CV_{AV} - CV_A] + \frac{1}{3} CV_V \quad (69)$$

We note that $CV_{EAV} = \pi$ and $CV_{AV} = CV_{EV} = CV_E = CV_A = CV_V = 0$. Moreover, by assumption we have $U^E_2 + U^A_2 > y_1$, so that $CV_{EA} = U^E_2 + U^A_2$. Thus,

$$D^E_2 = \frac{1}{3} \pi + \frac{1}{6} [U^E_2 + U^A_2] \quad (70)$$

$$D^A_2 = \frac{1}{3} \pi + \frac{1}{6} [U^E_2 + U^A_2] \quad (71)$$

$$D^V_2 = \frac{1}{3} \pi - \frac{1}{3} [U^E_2 + U^A_2] \quad (72)$$

The deal values then allow us to derive the equilibrium equity shares $\beta^{E*}$, $\beta^{A*}$, and $\beta^{V*}$. The equilibrium equity share for entrepreneurs, $\beta^{E*}$, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*} y_2 = D^E_2$. Solving this for $\beta^{E*}$ yields

$$\beta^{E*} = \frac{D^E_2}{y_2} = \frac{1}{6y_2} [2\pi + U^E_2 + U^A_2]. \quad (73)$$

Likewise we get

$$\beta^{A*} = \frac{D^A_2}{y_2} = \frac{1}{6y_2} [2\pi + U^E_2 + U^A_2] \quad (74)$$

$$\beta^{V*} = \frac{k_2 + D^V_2}{y_2} = \frac{1}{3y_2} [3k_2 + \pi - (U^E_2 + U^A_2)]. \quad (75)$$
Derivation of VC market equilibrium.

The first part of the derivation follows along the lines of the derivation of the angel market equilibrium: Using Eq. (13) we get

$$\theta^* = \phi_2 D^V / \sigma^V.$$ 

Moreover, using Eq. (14) and the relationship $M^V_2 = M^E_2 \theta^*_2$ we find

$$M^V_2 = g \rho_1 (e_1^*) x_1^* \phi_2^* \frac{1}{\phi_2 \sqrt{\theta^*_2}}.$$ 

(76)

Using $M^E_2 = M^V_2 / \theta^*_2$ and the definition of $M^V_2$, we can write $x_2^*$ as

$$x_2^* = \phi_2 [M^V_2 M^E_2]^{0.5} = \phi_2 M^V_2 \frac{1}{\phi_2 \theta^*_2} = m^E_2 \frac{\phi_2 \sqrt{\theta^*_2}}{\phi_2 \theta^*_2}.$$ 

(77)

where $m^E_2 = g \rho_1 (e_1^*) x_1^*$. Furthermore, using Eq. (15) and $q^V_2 = x_2 / M^V_2$ we find that $m^V_2 = q^V_2 M^V_2 = x_2^*$.

Finally, using the equilibrium equity share $\beta^V_2$ for VCs we can write $V_2^*$ as follows:

$$V_2^* = \frac{k_2}{\beta^V_2} = \frac{k_2 y_2}{k_2 + D^V_2} = \left(\frac{3 k_2}{3 k_2 + \pi - (U^E_2 + U^A_2)}\right) y_2.$$ 

(78)

Proof of Proposition 3.

First we need to derive a condition which defines $U^E_2$. We can write Eq. (12) as

$$U^E_2 [r + \delta_2] = -\sigma_2 + q^E_2 [D^E_2 - U^E_2].$$ 

(79)

Note that $D^E_2 - U^E_2 = \pi / 3 - 2U^E_2 / 3 = D^V_2$. Using $q^E_2 = \phi_2 [M^V_2 / M^E_2]^{0.5} = \phi_2 \sqrt{\theta^*_2} = \phi_2^2 D^V_2 / \sigma^V_2$, we get the following condition which defines $U^E_2$:

$$U^E_2 [r + \delta_2] - \frac{\phi_2^2}{\sigma^V_2} [D^V_2]^2 = 0.$$ 

(80)

Consider the equilibrium degree of competition $\theta^*_2$. Recall that $U^A_2 = U^E_2$ in equilibrium; thus,

$$\frac{d \theta^*_2}{d U^A_2} = \frac{d \theta^*_2}{d U^E_2} = 2 \phi_2^2 D^V_2 [\sigma^V_2]^2 d U^E_2 = -\frac{4 \phi_2^2 D^V_2}{3 [\sigma^V_2]^2} < 0.$$ 

(81)

Note that $\delta_2$ only affects $U^E_2$ in the definition of $\theta^*_2$. Implicitly differentiating $U^E_2$ w.r.t. $\delta_2$ yields

$$\frac{d U^E_2}{d \delta_2} = -\frac{U^E_2}{r + \delta_2 + \frac{4 \phi_2^3}{3 \sigma^V_2} D^V_2} < 0,$$ 

(82)
which implies that $d\theta_2^*/d\sigma_2 > 0$. Likewise, $\sigma_2$ only affects $U_2^E$ in the definition of $\theta_2^*$. We get

$$\frac{dU_2^E}{d\sigma_2} = -\frac{1}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} < 0. \quad (83)$$

Thus, $d\theta_2^*/d\sigma_2 > 0$. Next, differentiating $U_2^E$ w.r.t. $\phi_2$ yields

$$\frac{d\theta_2^*}{d\phi_2} = \frac{2 \phi_2 D_2^V}{[\sigma_2^V]^2} \left[ D_2^V + \phi_2 \frac{dD_2^Y}{dU_2^E} \frac{dU_2^E}{d\phi_2} \right] = \frac{2 \phi_2 D_2^V}{[\sigma_2^V]^2} \left[ D_2^V - \frac{2}{3} \frac{\phi_2}{dU_2^E} \frac{dU_2^E}{d\phi_2} \right], \quad (84)$$

with

$$\frac{dU_2^E}{d\phi_2} = \frac{2 \phi_2 \sigma_2^V [D_2^V]^2}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} > 0. \quad (85)$$

Therefore,

$$\frac{d\theta_2^*}{d\phi_2} = \frac{2 \phi_2 D_2^V}{[\sigma_2^V]^2} \frac{(r + \delta_2) D_2^V}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} > 0. \quad (86)$$

Likewise,

$$\frac{d\theta_2^*}{d\sigma_2^V} = \frac{2 \phi_2^2 D_2^V}{\sigma_2^V [\sigma_2^V]^2} \left[ -\frac{2}{3} \frac{dU_2^E}{d\sigma_2^V} \sigma_2^V D_2^V - D_2^V \right], \quad \text{with} \quad \frac{dU_2^E}{d\sigma_2^V} = -\frac{\phi_2^2 D_2^Y D_2^V}{\sigma_2^V [\sigma_2^V]^2} \frac{1}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} < 0. \quad (87)$$

Consequently,

$$\frac{d\theta_2^*}{d\sigma_2^V} = -\frac{2 \phi_2^2 D_2^V}{\sigma_2^V [\sigma_2^V]^2} \frac{1}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} \left[ D_2^V + (r + \delta_2) D_2^V \right] < 0. \quad (88)$$

Moreover, we get

$$\frac{d\theta_2^*}{dk_2} = \frac{2 \phi_2^2 D_2^V}{[\sigma_2^V]^2} \frac{dD_2^Y}{dk_2} = \frac{2 \phi_2^2 D_2^V}{[\sigma_2^V]^2} \left[ -\frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} \right], \quad \text{with} \quad \frac{dU_2^E}{dk_2} = -\frac{\phi_2^2 D_2^Y D_2^V}{\sigma_2^V [\sigma_2^V]^2} \frac{1}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} < 0. \quad (89)$$

Thus,

$$\frac{d\theta_2^*}{dk_2} = -\frac{2 \phi_2^2 D_2^V}{3 [\sigma_2^V]^2} \frac{r + \delta_2}{r + \delta_2 + \frac{4}{3} \frac{\phi_2^2}{\sigma_2^2} D_2^V} < 0. \quad (90)$$
Next, recall that $m_2^{V*} = x_2^*$ is given by

$$m_2^{V*} = x_2^* = g\rho_1(e_1^*)x_2^* = \frac{\phi_2\sqrt{\theta_2^*}}{\delta + \phi_2\sqrt{\theta_2^*}}.$$  

(91)

We have shown in Proof of Proposition 2 that $dx_1^*/d\phi_2 > 0$ and $dx_1^*/d\delta_2$, $dx_1^*/d\sigma_2$, $dx_1^*/d\sigma_2^Y$, $dx_1^*/dk_2 < 0$. Likewise, we have shown that $d\rho_1(e_1^*)/d\phi_2 > 0$ and $d\rho_1(e_1^*)/d\delta_2$, $d\rho_1(e_1^*)/d\sigma_2$, $d\rho_1(e_1^*)/d\sigma_2^Y$, $d\rho_1(e_1^*)/dk_2 < 0$. Moreover, it is straightforward to verify that $dT/d(\phi_2\sqrt{\theta_2^*}) > 0$. Using our comparative statics results for $\theta_2$, we can infer that $dT/d\phi_2$, $dT/d\delta_2$, $dT/d\sigma_2 > 0$, and $dT/d\sigma_2^Y$, $dT/dk_2 < 0$. All this implies that $dm_2^{V*}/d\phi_2 > 0$ and $dm_2^{V*}/d\sigma_2^Y$, $dm_2^{V*}/dk_2 < 0$, while the effects of $\delta_2$ and $\sigma_2$ on $m_2^{V*}$ are ambiguous.

Now consider the equilibrium late stage valuation $V_2^*$:

$$V_2^* = \left(\frac{3k_2}{3k_2 + \pi - 2U_2^E}\right) y_2.$$  

(92)

Recall that $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\sigma_2^Y$, $dU_2^E/d\delta_2 < 0$. Thus, $dV_2^*/d\phi_2 > 0$ and $dV_2^*/d\sigma_2$, $dV_2^*/d\sigma_2^Y$, $dV_2^*/d\delta_2 < 0$. Furthermore, recall that $V_2^*$ can also be written as $V_2^* = k_2 y_2/(k_2 + D_2^Y)$. Taking the first derivative of $V_2^*$ w.r.t. $k_2$ yields

$$\frac{dV_2^*}{dk_2} = \frac{k_2 + D_2^Y - k_2 \left[ 1 - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} \right] y_2}{[k_2 + D_2^Y]^2} = \frac{1}{3} k_2 + D_2^Y + \frac{2}{3} \frac{dU_2^E}{dk_2} y_2.$$  

(93)

Note that the denominator is always positive. Moreover, we have $N > 0$ for $k_2 \to 0$. Thus, $dV_2^*/dk_2 > 0$ for $k_2 \to 0$. To verify that $dV_2^*/dk_2 > 0$ for all $k_2 > 0$, it is sufficient to show that $dN/dk_2 > 0$:

$$\frac{dN}{dk_2} = \frac{1}{3} - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} + \frac{2}{3} \left[ \frac{dU_2^E}{dk_2} + k_2 \frac{d^2U_2^E}{dk_2^2} \right] = \frac{2}{3} k_2 \frac{d^2U_2^E}{dk_2^2}.$$  

(94)

It remains to identify the sign of $d^2U_2^E/dk_2^2$. Using $a_2 \equiv \phi_2^2/\sigma_2^Y$ we can write $dU_2^E/dk_2$ as

$$\frac{dU_2^E}{dk_2} = -\frac{\frac{2}{3} a_2 D_2^Y}{r + \delta_2 + \frac{4}{3} a_2 D_2^Y} = -\frac{\frac{2}{3}}{\left( r + \delta_2 \right) \left[ a_2 D_2^Y \right]^{-1} + \frac{4}{3}}.$$  

(95)
Thus,
\[
\frac{d^2U_2^E}{dk_2^2} = \frac{8}{9}a_2(r + \delta_2) \left[ a_2 D_2^V \right]^{-2} \left[ r + 2\frac{dU_2^E}{dk_2} \right]. \tag{96}
\]

Note that
\[
1 + 2\frac{dU_2^E}{dk_2} = 1 - \frac{4}{3}a_2 D_2^V = \frac{r + \delta_2}{r + \delta_2 + \frac{4}{3}a_2 D_2^V} > 0. \tag{97}
\]

Hence, \(d^2U_2^E/dk_2^2 > 0\), so that \(dN/dk_2 > 0\). Consequently, \(dV_2^*/dk_2 > 0\).

**Proof of Proposition 4.**

We can see from Eq. (80) that \(U_2^E\) (and therefore \(U_2^A\)) does not depend on the early stage parameters \(\phi_1, \delta_1, \sigma_1^E, \sigma_1^A,\) and \(k_1\). This also implies that \(D_2^V\), and therefore \(\theta_2^*\) and \(V_2^*\), do not depend on these parameters.

Now consider the equilibrium inflow of start-ups \(m_2^{E*}\). Recall from Proposition 1 that \(dx_1^*/d\phi_1 > 0\) and \(dx_1^*/d\sigma_1^A, dx_1^*/dk_1 < 0\), while the effects of \(\delta_1\) and \(\sigma_1^E\) are ambiguous. Moreover, we know from Proposition 1 that \(d\rho_1(e_1^*)/d\phi_1 > 0\) and \(d\rho_1(e_1^*)/d\delta_1, d\rho_1(e_1^*)/d\sigma_1^E, d\rho_1(e_1^*)/d\sigma_1^A, d\rho_1(e_1^*)/dk_1 < 0\). This implies that \(dm_2^{E*}/d\phi_1 > 0\) and \(dm_2^{E*}/d\sigma_1^A, dm_2^{E*}/dk_1 < 0\), while the effects of \(\delta_1\) and \(\sigma_1^E\) are ambiguous.

Finally consider the equilibrium inflow of VCs \(m_2^{V*}\), as defined by
\[
m_2^{V*} = x_2^* = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}. \tag{98}
\]

Recall that \(\theta_2^*\) does not depend on the early stage parameters. Our comparative statics results for \(m_2^{E*}\) then imply that \(dm_2^{V*}/d\phi_1 > 0\) and \(dm_2^{V*}/d\sigma_1^A, dm_2^{V*}/dk_1 < 0\), while the effects of \(\delta_1\) and \(\sigma_1^E\) are ambiguous.

Angel protection: derivation of deal values and equity shares.

The new coalition values are given by \(CV_{EAV} = \pi, CV_{E_A} = U_2^E + U_2^A, CV_{EV} = \lambda\pi,\) and \(CV_{AV} = CV_{E} = CV_{A} = CV_{V} = 0\). Using the general deal values (67), (68), and (69), we get
\[ D^E_2 = \frac{1}{6} [2 + \lambda] \pi + \frac{1}{6} [U^E_2 + U^A_2] \]  
(99)

\[ D^A_2 = \frac{1}{3} [1 - \lambda] \pi + \frac{1}{6} [U^E_2 + U^A_2] \]  
(100)

\[ D^V_2 = \frac{1}{6} [2 + \lambda] \pi - \frac{1}{3} [U^E_2 + U^A_2] \]  
(101)

The new equilibrium equity share for entrepreneurs, \( \beta^E^* \), ensures that their actual net payoff equals their deal value from the bargaining game: \( \beta^E^* y_2 = D^E_2 \). Solving this for \( \beta^E^* \) yields

\[ \beta^E^* = \frac{D^E_2}{y_2} = \frac{1}{6y_2} [(2 + \lambda) \pi + U^E_2 + U^A_2] . \]  
(102)

Likewise we get

\[ \beta^A^* = \frac{D^A_2}{y_2} = \frac{1}{6y_2} [2 (1 - \lambda) \pi + U^E_2 + U^A_2] \]  
(103)

\[ \beta^V^* = \frac{k_2 + D^V_2}{y_2} = \frac{1}{6y_2} [6k_2 + (2 + \lambda) \pi - 2 (U^E_2 + U^A_2)] . \]  
(104)

**Proof of Proposition 5.**

We first show that \( \frac{dU^A_2}{d\lambda} < 0 \). Note that \( D^A_2 \neq D^E_2 \) for \( \lambda > 0 \), and recall that \( q^E_2 = \phi_2 [M^V_2/M^E_2]^{0.5} = \phi_2 D^V_2 / \sigma^V_2 \). Thus, using Eq. (12) we define

\[ F \equiv U^E_2 (r + \delta_2) + \sigma - a_2 D^V_2 \left[ D^E_2 - U^E_2 \right] = 0 \]  
(105)

\[ G \equiv U^A_2 (r + \delta_2) + \sigma - a_2 D^V_2 \left[ D^A_2 - U^A_2 \right] = 0, \]  
(106)

where \( a_2 = \phi_2^2 / \sigma^V_2 \). Using Cramer's rule we get

\[
\frac{dU^A_2}{d\lambda} = \begin{vmatrix}
-\frac{\partial F}{\partial \lambda} & -\frac{\partial F}{\partial U^E_2} \\
-\frac{\partial G}{\partial \lambda} & -\frac{\partial G}{\partial U^E_2} \\
\frac{\partial F}{\partial U^A_2} & \frac{\partial F}{\partial U^A_2} \\
\frac{\partial G}{\partial U^A_2} & \frac{\partial G}{\partial U^A_2}
\end{vmatrix}
= -\frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U^E_2} + \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U^A_2} - \frac{\partial F}{\partial U^A_2} \frac{\partial G}{\partial U^E_2} .
\]  
(107)
The denominator is negative if
\[
\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} < \frac{\partial G}{\partial U_2^A} \frac{\partial F}{\partial U_2^E},
\] (108)
which is equivalent to
\[
\frac{1}{6} a_2 \left[ 2 [D_2^E - U_2^E] - D_2^V \right] \frac{1}{6} a_2 \left[ 2 [D_2^A - U_2^A] - D_2^V \right] < \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 [D_2^E - U_2^E] + 5D_2^V \right] \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 [D_2^E - U_2^E] + 5D_2^V \right] \right].
\] (109)
If this condition holds for \( r + \delta_2 = 0 \), then it also holds for all \( r + \delta_2 > 0 \). Setting \( r + \delta_2 = 0 \) we get
\[
\] (110)
This condition is satisfied as \( D_2^E > U_2^E \) and \( D_2^A > U_2^A \). Thus, the denominator of \( dU_2^A/d\lambda \) is strictly negative. Likewise, the numerator is positive if
\[
\frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial U_2^E} > \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E},
\] (111)
which is equivalent to
\[
\frac{1}{6} \pi a_2 \left[ [D_2^A - U_2^A] - 2D_2^V \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 [D_2^E - U_2^E] + 5D_2^V \right] \right] < \frac{1}{6} \pi a_2 \left[ [D_2^E - U_2^E] + D_2^V \right] \frac{1}{6} a_2 \left[ 2 [D_2^A - U_2^A] - D_2^V \right].
\] (112)
This condition can be written as
\[
\frac{2}{a_2} (r + \delta_2) \left[ [D_2^A - U_2^A] - 2D_2^V \right] + [D_2^A - U_2^A] D_2^V - D_2^V [D_2^E - U_2^E] < 3 [D_2^V]^2.
\] (113)
From \( F \) and \( G \) we know that
\[
D_2^V [D_2^E - U_2^E] = \frac{U_2^E (r + \delta_2) + \sigma}{a_2} \quad \text{and} \quad D_2^V [D_2^A - U_2^A] = \frac{U_2^A (r + \delta_2) + \sigma}{a_2},
\] (114)
so that we can write condition (113) as follows:

\[
\frac{2}{a_2} (r + \delta_2) \left[ [D_2^A - U_2^A] - 2D_2^V \right] + \frac{U_2^A (r + \delta_2) + \sigma}{a_2} - \frac{U_2^E (r + \delta_2) + \sigma}{a_2} < 3 [D_2^V]^2 \quad (115)
\]

\[
\iff (r + \delta_2) \left[ 2D_2^A - U_2^A - 4D_2^V - U_2^E \right] < 3 [D_2^V]^2 a_2. \quad (116)
\]

We now show that \( T < 0 \). Using the definitions of \( D_2^A \) and \( D_2^V \) we can write \( T < 0 \) as

\[
\frac{2}{3} [1 - \lambda] \pi + \frac{1}{3} [U_2^E + U_2^A] - U_2^A - \frac{2}{3} [2 + \lambda] \pi + \frac{4}{3} [U_2^E + U_2^A] - U_2^E < 0 \quad (117)
\]

\[
\iff U_2^E + U_2^A < [1 + 2\lambda] \pi. \quad (118)
\]

This condition is satisfied for all \( \lambda \geq 0 \) because \( \pi > U_2^E + U_2^A \). Thus, the numerator of \( dU_2^A / d\lambda \) is strictly positive. Consequently, \( dU_2^A / d\lambda < 0 \). Finally note that \( \partial D_2^E / \partial \lambda = \pi / 6 < \left| \partial D_2^A / \partial \lambda \right| = \pi / 3 \). Thus, \( d \left[ U_2^E + U_2^A \right] / d\lambda < 0 \), which implies that \( dD_2^V / d\lambda > 0 \).

Next we analyze the effects of \( \lambda \) on the early stage equilibrium variables. Consider the equilibrium degree of competition \( \theta_1^* \). We get

\[
\frac{d\theta_1^*}{d\lambda} = 2 \frac{\phi_1^3}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{d\lambda}. \quad (119)
\]

Recall that

\[
\frac{d}{d\lambda} \left( U_2^A + U_2^E \right) = \frac{dU_2^A}{d\lambda} + \frac{dU_2^E}{d\lambda} < 0. \quad (120)
\]

This implies

\[
\frac{dD_1^A}{d\lambda} + \frac{dD_1^E}{d\lambda} < 0 \quad \iff \quad \frac{dD_1^A}{d\lambda} < 0. \quad (121)
\]

Thus, \( d\theta_1^*/d\lambda < 0 \).

Now consider the equilibrium entry of entrepreneurs \( m_1^{E*} \). Using Eq. (28), we get

\[
\frac{dU_1^E}{d\lambda} = \frac{\phi_1^2}{\sigma_1^E} \left[ \frac{dD_1^A}{d\lambda} [D_1^E - U_1^E] + D_1^A \frac{dD_1^E}{d\lambda} \right] + \left[ \frac{d\Gamma}{dU_1^E} \frac{d\alpha^*}{dU_1^E} + \frac{\partial \Gamma}{\partial U_1^E} \right], \quad (122)
\]
where \( \Gamma = D^A_1 \left[ D^E_1 - U^E_1 \right] \). Note that \( d\Gamma /d\alpha = 0 \); see Eq. (18). Thus,

\[
\frac{dU^E_1}{d\lambda} = \frac{\delta_1^2 \left[ \frac{dD^A_1}{d\lambda} [D^E_1 - U^E_1] + D^A_1 \frac{dD^E_1}{d\lambda} \right]}{r + \delta_1 + \frac{\delta_1^2}{\sigma_1^2} D^A_1},
\]

(123)

where the denominator is positive. Consequently, \( dU^E_1 /d\lambda \) < 0 if

\[
\frac{dD^A_1}{d\lambda} [D^E_1 - U^E_1] + D^A_1 \frac{dD^E_1}{d\lambda} < 0.
\]

(124)

Using Eq. (18) we can derive the following expression for \( D^E_1 - U^E_1 \):

\[
D^E_1 - U^E_1 = -\frac{dD^E_1}{d\alpha} D^A_1,
\]

(125)

so that Eq. (124) can be written as

\[
\frac{dD^A_1}{d\lambda} \left( -\frac{dD^E_1}{d\alpha} \right) + \frac{dD^E_1}{d\lambda} < 0.
\]

(126)

Recall that \( d(D^A_1 + D^E) /d\lambda \) < 0, with \( dD^A_1 /d\lambda \) < 0; thus, this condition is satisfied when \( X \geq 1 \). Note that \( dD^E_1 /d\alpha \) < 0 and \( dD^A_1 /d\alpha \) > 0. Hence, \( X \geq 1 \) if

\[
0 \geq \frac{dD^A_1}{d\alpha} + \frac{dD^E_1}{d\alpha} = \frac{d}{d\alpha} \left[ D^A_1 + D^E_1 \right].
\]

(127)

It is easy to show that the joint surplus is maximized when \( \alpha = 0 \) (which maximizes effort incentives for the entrepreneur); thus

\[
\left. \frac{d \left[ D^A_1 + D^E_1 \right]}{d\alpha} \right|_{\alpha=\alpha^*>0} < 0,
\]

(128)

so that \( X \geq 1 \). Consequently, \( dU^E_1 /d\lambda \) < 0, and therefore \( dm^E_1 /d\lambda = dF(U^E_1) /d\lambda < 0 \).

Next consider the equilibrium inflow of angels, \( m^A_1^{*} \), which is defined by

\[
m^A_1^{*} = x^*_1 = \frac{F(U^E_1)}{\phi_1 \sqrt{\theta^*_1}} \frac{\phi_1 \sqrt{\theta^*_1}}{\delta_1 + \phi_1 \sqrt{\theta^*_1}} \equiv T.
\]

(129)
Note that $dT/d(\sqrt{\theta_1^*}) > 0$. Our comparative statics results for $m_1^{E*}$ and $\theta_1^*$ then imply that $dm_1^{A*}/d\lambda = dx_1^*/d\lambda < 0$.

Now consider the angel’s equilibrium equity share $\alpha^*$, which is defined by Eq. (18). We get

$$ \frac{d\alpha^*}{d\lambda} = \frac{d\alpha^*}{dU_2^E} \frac{dU_2^E}{d\lambda} + \frac{d\alpha^*}{dU_2^A} \frac{dU_2^A}{d\lambda}, $$

(130)

where $dU_2^E/d\lambda > 0$ and $dU_2^A/d\lambda < 0$. Moreover, the Nash bargaining solution implies that $d\alpha^*/dU_2^E > 0$ and $d\alpha^*/dU_2^A < 0$. Thus, $d\alpha^*/d\lambda > 0$. For the equilibrium valuation $V_1^* = k_1/\alpha^*$ this concurrently implies that $dV_1^*/d\lambda < 0$. Finally we know that $dD_1^E/d\lambda > 0$ in equilibrium. Using the Envelope Theorem we get

$$ \frac{dD_1^E}{d\lambda} = \rho_1(e_1) \frac{d}{d\lambda} [gU_2^E + (1 - g)(1 - \alpha^*)y_1] > 0, $$

(131)

which implies that $T > 0$. Using Eq. (3) we find

$$ \frac{de_1^*}{d\lambda} = -\frac{\rho_1(e_1)}{de_1} \frac{d}{d\lambda} [gU_2^E + (1 - g)(1 - \alpha^*)y_1] \equiv T $$

(132)

where $T > 0$, and the denominator is negative due to the second-order condition for $e_1^*$. Thus, $de_1^*/d\lambda > 0$. This in turn implies that $d\rho_1(e_1^*)/d\lambda > 0$.

Finally we analyze the effects of $\lambda$ on the late stage equilibrium variables. Note that $d(U_2^E + U_2^A)/d\lambda < 0$ also implies that $dD_2^E/d\lambda > 0$. Using the definitions of $\theta_2^*$, $\beta V^*$ and $V_2^*$, we can then infer that $d\theta_2^*/d\lambda > 0$, $d\beta V^*/d\lambda > 0$ and $dV_2^*/d\lambda < 0$. Moreover,

$$ \frac{dm_2^{E*}}{d\lambda} = \frac{d}{d\lambda} [g\rho_1(e_1^*)x_1^*] = g \left[ \rho_1'(e_1^*) \frac{de_1^*}{d\lambda} x_1^* + \rho_1(e_1^*) \frac{dx_1^*}{d\lambda} \right]. $$

(133)

In general, the effect on $m_2^{E*}$ is ambiguous as $de_1^*/d\lambda > 0$ and $dx_1^*/d\lambda < 0$. However, we can see that $dm_2^{E*}/d\lambda < 0$ when $\rho_1'(e_1^*) \rightarrow 0$. Moreover, for $\delta_1 \rightarrow 0$ we have $m_1^{A*} = m_1^{E*}$; with $m_1^{E*}$ being sufficiently inelastic, we have $dx_1^*/d\lambda \rightarrow 0$, so that $dm_2^{E*}/d\lambda > 0$. Next, recall that $m_2^{V*}$ is defined by

$$ m_2^{V*} = x_2^* = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}. $$

(134)
One can show that $dT/d\sqrt{\theta} > 0$, so that $dT/d\lambda > 0$. Recall, however, that the sign of $dm^E/d\lambda$ is ambiguous. Thus, the effect of $\lambda$ on $m^*_V = x^*_2$ is also ambiguous.

Angel protection – angel not required for VC search.

Suppose the entrepreneur can search for a VC without the angel. The entrepreneur then incurs the search cost $\gamma \sigma$, with $\gamma > 2$. Using Nash bargaining, the deal values for the VC ($\hat{D}^V_2$) and the entrepreneur ($\hat{D}^E_2$) are then given by

$$
\hat{D}^V_2 = \frac{1}{2} \left[ \lambda \pi - \hat{U}^E_2 \right] \quad \hat{D}^E_2 = \frac{1}{2} \left[ \lambda \pi + \hat{U}^E_2 \right],
$$

where $\hat{U}^E_2$ denotes the entrepreneur’s outside option.

Now consider the bargaining problem at the late stage between entrepreneur, angel, and VC. The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U^E_2 + U^A_2$, $CV_{EV} = \lambda \pi$, $CV_E = \hat{U}^E_2$, and $CV_{AV} = CV_A = CV_V = 0$. Using the Shapley value we then get the following deal values:

$$
D^E_2 = \frac{1}{6} \left[ 2 + \lambda \right] \pi + \frac{1}{6} \left[ U^E_2 + U^A_2 \right] + \frac{1}{3} \hat{U}^E_2
$$

$$
D^A_2 = \frac{1}{3} \left[ 1 - \lambda \right] \pi + \frac{1}{6} \left[ U^E_2 + U^A_2 \right] - \frac{1}{6} \hat{U}^E_2
$$

$$
D^V_2 = \frac{1}{6} \left[ 2 + \lambda \right] \pi - \frac{1}{3} \left[ U^E_2 + U^A_2 \right] - \frac{1}{6} \hat{U}^E_2
$$

The expected utilities from search, $U^A_2$, $U^E_2$, and $\hat{U}^E_2$, are then defined by

$$
F \equiv U^A_2 (r + \delta_2) + \sigma - a_2 D^V_2 \left[ D^A_2 - U^A_2 \right] = 0
$$

$$
G \equiv U^E_2 (r + \delta_2) + \sigma - a_2 D^V_2 \left[ D^E_2 - U^E_2 \right] = 0
$$

$$
H \equiv \hat{U}^E_2 (r + \delta_2) + \gamma \sigma - a_2 \hat{D}^V_2 \left[ \hat{D}^E_2 - \hat{U}^E_2 \right] = 0,
$$
where \( a_2 = \phi_2^2 / \sigma_2' \). Using \( H \) we find that

\[
\frac{d\hat{U}_2^E}{d\lambda} = \frac{1/2 a_2 \pi \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]}{r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]} > 0 \tag{142}
\]

\[
\frac{d\hat{U}_2^E}{d\gamma} = -\frac{\sigma}{r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]} < 0 \tag{143}
\]

Next we show that \( dU_2^A / d\lambda < 0 \). Using Cramer's rule we get \( dU_2^A / d\lambda = A / B \), where

\[
A = \begin{vmatrix}
-\frac{\partial F}{\partial \lambda} & -\frac{\partial F}{\partial U_2^A} & -\frac{\partial F}{\partial U_2^E} \\
-\frac{\partial G}{\partial \lambda} & -\frac{\partial G}{\partial U_2^A} & -\frac{\partial G}{\partial U_2^E} \\
-\frac{\partial H}{\partial \lambda} & -\frac{\partial H}{\partial U_2^A} & -\frac{\partial H}{\partial U_2^E}
\end{vmatrix}
\]

\[
B = \begin{vmatrix}
\frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial U_2^V} \\
\frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial U_2^V} \\
\frac{\partial H}{\partial U_2^A} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial U_2^V}
\end{vmatrix}
\]

Consider first the denominator \( B \). Since \( \partial H / \partial U_2^A = 0 \) and \( \partial H / \partial U_2^E = 0 \), we can write \( B \) as

\[
B = \frac{\partial H}{\partial \hat{U}_2^E} \left[ \frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} - \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial U_2^A} \right], \tag{145}
\]

where

\[
\frac{\partial H}{\partial \hat{U}_2^E} = r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right] > 0. \tag{146}
\]

Thus, \( B > 0 \) if

\[
\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} > \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial U_2^A}, \tag{147}
\]

which can be written as

\[
\left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left[ D_2^A - U_2^A \right] + 5 D_2^V \right] \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left[ D_2^E - U_2^E \right] + 5 D_2^V \right] \right] > \frac{1}{6} a_2 \left[ 2 \left[ D_2^A - U_2^A \right] - D_2^V \right] \frac{1}{6} a_2 \left[ 2 \left[ D_2^E - U_2^E \right] - D_2^V \right]. \tag{148}
\]

Note that this condition holds for all \( r + \delta_2 > 0 \) if it holds for \( r + \delta_2 = 0 \). Setting \( r + \delta_2 = 0 \) we get

\[
12 \left[ D_2^A - U_2^A \right] D_2^V + 12 D_2^V \left[ D_2^E - U_2^E \right] + 24 \left[ D_2^V \right]^2 > 0. \tag{149}
\]
Note that $D^E_2 > U^E_2$ and $D^A_2 > U^A_2$. Thus, this condition is satisfied, so that $B > 0$. Next consider the numerator $A$. With $\partial H / \partial U^E_2 = 0$ we can write $A$ as

$$A = \left[\frac{-\partial F}{\partial \lambda} \frac{\partial G}{\partial U^E_2} + \frac{\partial F}{\partial U^E_2} \frac{\partial G}{\partial \lambda}\right] \frac{\partial H}{\partial U^E_2} = \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U^E_2} - \frac{\partial F}{\partial U^E_2} \frac{\partial G}{\partial \lambda} \equiv X_1$$

Recall that $\partial H / \partial \hat{U}^E_2 > 0$. Moreover,

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{2} \pi a_2 \left[ \hat{D}^E_2 - \hat{U}^E_2 + \hat{D}^V_2 \right] < 0.$$ \hspace{1cm} (151)

Thus, $A < 0$ when $X_1 < 0$ and $X_2 < 0$. Note that $X_1 < 0$ if

$$\frac{\partial F}{\partial U^E_2} \frac{\partial G}{\partial \lambda} < \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U^E_2},$$ \hspace{1cm} (152)

which can be written as

$$\frac{1}{6} a_2 \left[ 2 \left[ D^A_2 - U^A_2 \right] - D^V_2 \right] \frac{1}{6} \pi a_2 \left[ D^E_2 - U^E_2 + D^V_2 \right]$$

$$> \frac{1}{6} \pi a_2 \left[ D^A_2 - U^A_2 - 2 D^V_2 \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left[ D^E_2 - U^E_2 \right] + 5 D^V_2 \right] \right].$$ \hspace{1cm} (153)

Simplifying yields

$$\frac{2}{a_2} (r + \delta_2) \left[ \left[ D^A_2 - U^A_2 \right] - 2 D^V_2 \right] + \left[ D^A_2 - U^A_2 \right] D^V_2 - D^V_2 \left[ D^E_2 - U^E_2 \right] < 3 \left[ D^V_2 \right]^2.$$ \hspace{1cm} (154)

From $F$ and $G$ we know that

$$D^V_2 \left[ D^A_2 - U^A_2 \right] = \frac{U^A_2 (r + \delta_2) + \sigma}{a_2} \text{ and } D^V_2 \left[ D^E_2 - U^E_2 \right] = \frac{U^E_2 (r + \delta_2) + \sigma}{a_2},$$ \hspace{1cm} (155)

so that condition (154) can be written as

$$\left( r + \delta_2 \right) \frac{2 D^A_2 - U^A_2 - 4 D^V_2 - U^E_2}{a_2} < 3 \left[ D^V_2 \right]^2 a_2.$$ \hspace{1cm} (156)

It remains to prove that $T < 0$. Using the definitions of $D^A_2$ and $D^V_2$ we can write $T < 0$ as

$$U^E_2 + U^A_2 < \left[ 1 + 2 \lambda \right] \pi.$$ \hspace{1cm} (157)
This condition is satisfied for all $\lambda \geq 0$ as $\pi > U_2^E + U_2^A$. Thus, $X_1 < 0$. Moreover, $X_2 < 0$ if
\[
\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \hat{U}_2^E} < \frac{\partial F}{\partial \hat{U}_2^E} \frac{\partial G}{\partial U_2^E},
\]
which is equivalent to
\[
\frac{1}{6} a_2 \left[ 2 \left[ D_2^A - U_2^A \right] - D_2^Y \right] \frac{1}{6} a_2 \left[ D_2^E - U_2^E - 2D_2^V \right] < \frac{1}{6} a_2 \left[ D_2^A - U_2^A + D_2^Y \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left[ D_2^E - U_2^E \right] + 5D_2^Y \right] \right].
\]
Again, $D_2^A > U_2^A$ and $D_2^E > U_2^E$. Thus, if this condition holds for $r + \delta_2 = 0$, then it also holds for all $r + \delta_2 > 0$. Setting $r + \delta_2 = 0$ we get
\[
0 < 3D_2^V \left[ D_2^E - U_2^E \right] + 9D_2^Y \left[ D_2^A - U_2^A \right] + 3D_2^Y D_2^V.
\]
Hence, $X_2 < 0$, so that $A < 0$. Consequently, $dU_2^A / d\lambda < 0$. Moreover, note that $\partial D_2^E / \partial \lambda = \pi / 6 < \partial D_2^A / \partial \lambda = \pi / 3$. Thus, $d \left[ U_2^E + U_2^A \right] / d\lambda < 0$. Finally, using $H$ we get
\[
\frac{d\hat{U}_2^E}{d\lambda} = \frac{\pi}{r + \delta_2 + \frac{1}{6} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^Y \right]}\left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^Y \right].
\]
where $Z \in (0, 1)$. Thus,
\[
\frac{dD_2^Y}{d\lambda} = \frac{1}{6} \pi \left[ 1 - Z \right] - \frac{1}{3} \frac{d}{d\lambda} \left[ U_2^E + U_2^A \right].
\]
Consequently, $dD_2^Y / d\lambda > 0$.

All this implies that the results from Proposition 5 continue to hold when the entrepreneur can search for a VC without the angel.

**Proof of Proposition 6.**

Recall that $U_2^E = U_2^A$ in equilibrium. Moreover, as shown in Proof of Proposition 3, $dU_2^E / d\phi_2 > 0$, and $dU_2^E / d\sigma_2$, $dU_2^E / d\delta_2$, $dU_2^E / d\sigma_Y$, $dU_2^E / dk_2 < 0$. Consequently, $d\gamma^* / d\phi_2 <
0, and \( d\gamma^*/d\sigma_2, d\gamma^*/d\delta_2, d\gamma^*/d\sigma_2^y, d\gamma^*/dk_2 > 0. \) \( \square \)

**Proof of Proposition 7.**

Recall from Proof of Proposition 5 that \( d \left[ U_2^E + U_2^A \right] / d\lambda < 0. \) Thus, \( d\gamma^*/d\lambda > 0. \) \( \square \)