On the Systematic Volatility of Unpriced Earnings
– Internet Appendix –

September 2013

1 Valuation of Unlevered Equity

The following matrix representation summarizes the dynamics of the three state variables in the economy.

\[
X_{t+1} = \begin{bmatrix} 0 \\ w_c \\ w_d \end{bmatrix} + \begin{bmatrix} \rho & 0 & 0 \\ 0 & \alpha_c & 0 \\ 0 & 0 & \alpha_d \end{bmatrix} X_t + \begin{bmatrix} \sigma_x \epsilon_{x,t+1} \\ s_c \eta_{c,t+1} \\ s_d \eta_{d,t+1} \end{bmatrix} = \mu + \Phi X_t + u_{t+1}
\]

where \( X_t \equiv [x_t \ v_{c,t} \ v_{d,t}]^\top \). Let \( \Omega \) denote the covariance matrix of the shocks, \( \Omega \equiv E_t [u_{t+1} u_{t+1}^\top] \).

The log pricing kernel of the Epstein–Zin preference is given as

\[
m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}
\]

where \( r_{c,t+1} \) denotes the log gross return on an asset that delivers aggregate consumption as its dividends each period. The parameter \( \theta \equiv \frac{1-\gamma}{1-1/\psi} \), with \( \gamma \geq 0 \) being the risk-aversion parameter and \( \psi \geq 0 \) the intertemporal elasticity of substitution (IES) parameter.

Following ?, \( r_{c,t+1} \) is log-linearly approximated as

\[
r_{c,t+1} = \kappa_0 + \kappa_1 z_{c,t+1} - z_{c,t} + \Delta c_{t+1}
\]
where \( z_{c,t} \) is the log price-dividend ratio of the claim to aggregate consumption, and \( \kappa_0 \) and \( \kappa_1 \) are approximating constants that both depend only on the average level of \( z^c \).

\[
\kappa_1 = \frac{e^{z^c}}{1 + e^{z^c}} \\
\kappa_0 = \log (1 + e^{z^c}) - \frac{e^{z^c}}{1 + e^{z^c}} z^c
\]

(5)

(6)

By plugging \( z_{c,t} = A_0 + A^\top X_t \) and the formulas above into the Euler equation, \( E_t [e^{m_{t+1}} e^{r_{c,t+1}}] = 1 \), we can derive

\[
A = \left( 1 - \frac{1}{\psi} \right) (I - \kappa_1 \Phi^\top)^{-1} \left[ \begin{array}{c} 1 \\ \frac{1}{2} (1 - \gamma) \\ 0 \end{array} \right] \\
A_0 = \frac{1}{1 - \kappa_1} \left\{ \log \delta + \kappa_0 + \kappa_1 A^\top \mu + \left( 1 - \frac{1}{\psi} \right) \mu_c + \frac{1}{2} \theta \kappa_1^2 A^\top \Omega A \right\}
\]

(7)

(8)

Similarly, the log gross return on a firm’s underlying asset (unlevered equity) can be approximated as

\[
r_{d,t+1} = \kappa_2 + \kappa_3 z_{d,t+1} - z_{d,t} + \Delta d_{t+1}
\]

(9)

where \( z_{d,t} \) represents the firm’s log price-dividend ratio, and \( \kappa_2 = \log (1 + e^{z^d}) - \frac{e^{z^d}}{1 + e^{z^d}} z^d \) and \( \kappa_3 = \frac{e^{z^d}}{1 + e^{z^d}} \). By plugging \( z_{d,t} = B_0 + B^\top X_t \) again into the Euler equation, its solutions can be derived as

\[
B = \{ I - \kappa_3 \Phi^\top \}^{-1} \left[ \begin{array}{c} \phi - \frac{1}{\psi} \\ \frac{1}{2} \left( (\gamma - \phi)^2 - (\gamma - 1) (\gamma - \frac{1}{\psi}) \right) \\ \frac{1}{2} \end{array} \right] \\
B_0 = \frac{1}{1 - \kappa_3} \left[ \theta \log \delta + (\phi - \gamma) \mu_c + (\theta - 1) \left\{ \kappa_0 + \kappa_1 (A_0 + A^\top \mu) - A_0 \right\} + \kappa_2 + \kappa_3 B^\top \mu + \frac{1}{2} \left\{ (\theta - 1) \kappa_1 A + \kappa_3 B \right\}^\top \Omega \left\{ (\theta - 1) \kappa_1 A + \kappa_3 B \right\} \right]
\]

(10)

(11)

Note that \( B_3 \) is not zero because of Jensen’s inequality.
2 Analytic Solution of Debt and Levered Equity

Let’s begin with a lemma that is critical to derive the analytic solution of debt and levered equity.

**Lemma 1.** If $i$ is a $(2 \times 1)$ constant vector and $X = (X_1, X_2)^\top$ is a bivariate normal with mean vector $\mu$ and covariance matrix $\Sigma$, then

$$
\int_{a < X_2 < b} e^{X_i^\top i} dX = e^{\mu^\top i + \frac{1}{2} i^\top \Sigma i} P(a < X_2 < b; \mu^*, \Sigma)
$$

where $P(\cdot; \mu^*, \Sigma)$ is the bivariate density function with mean vector $\mu^* = \mu + \Sigma i$ and covariance matrix $\Sigma$.

In this internet appendix, we assume the traditional firm’s payoff functions.

\[
F_T = \begin{cases} 
R(V_T + D_T) & \text{if } V_T + D_T < K \\
K & \text{if } V_T + D_T \geq K 
\end{cases}
\]

\[
S_T = \begin{cases} 
0 & \text{if } V_T + D_T < K \\
V_T + D_T - K & \text{if } V_T + D_T \geq K 
\end{cases}
\]

where $F_T$ denotes the payoff to corporate debts at maturity, $S_T$ the payoff to levered equity, $V_T$ the value of underlying assets, $D_T$ dividends, $K$ the principal amount of debts, and $R$ the recovery rate in case of bankruptcy.

Suppose the maturity comes in one time period, $T = t + 1$. Today’s price of these assets can be derived as

\[
F_t = E_t [e^{m_{t+1} F_{t+1}}] = Ke^{-r_{f,t} G(\omega - \sigma_r)} + RV_t G(-\omega)
\]

\[
S_t = E_t [e^{m_{t+1} S_{t+1}}] = V_t G(\omega) - Ke^{-r_{f,t} G(\omega - \sigma_r)}
\]

where $\omega \equiv \{\ln (V_t/K) + r_{f,t} + \frac{1}{2}\sigma_r^2\} / \sigma_r$, $\sigma_r^2 \equiv \text{var}(r_{d,t+1}) = \phi^2 v_{c,t} + v_{d,t} + \kappa_3^2 B^\top \Omega B$ is the variance of unlevered asset returns, and $g(\cdot)$ and $G(\cdot)$ denote the pdf and cdf of standard normal random variable. Note that $\omega$ implies the firm’s leverage relative to its volatility. Its comparative statistics are

\[
\frac{\partial \omega}{\partial V_t} = \frac{1}{\sigma_r V_t}, \quad \frac{\partial \omega}{\partial \sigma_r} = 1 - \frac{\omega}{\sigma_r}, \quad \frac{\partial \omega}{\partial r_{f,t}} = \frac{1}{\sigma_r}
\]
In the rest of this section, it is assumed that the firm is not underwater, i.e., \( V_t \geq K e^{-r_{f,t}} \), thus \( \omega > 0 \).

The partial derivatives of the levered claims can be derived like those of \( \phi \),

\[
\frac{\partial S_t}{\partial V_t} = G(\omega) \quad \frac{\partial F_t}{\partial V_t} = RG(-\omega) + \frac{1 - R}{\sigma_r} g(\omega) \quad (17)
\]

\[
\frac{\partial S_t}{\partial \sigma_r} = V_t g(\omega) \quad \frac{\partial F_t}{\partial \sigma_r} = -V_t g(\omega) \left\{ R + (1 - R) \frac{\omega}{\sigma_r} \right\} \quad (18)
\]

\[
\frac{\partial S_t}{\partial r_{f,t}} = Ke^{-r_{f,t}} G(\omega - \sigma_r) \quad \frac{\partial F_t}{\partial r_{f,t}} = -Ke^{-r_{f,t}} G(\omega - \sigma_r) + \frac{1}{\sigma_r} V_t g(\omega)(1 - R) \quad (19)
\]

And the comparative statistics of the Black-Scholes inputs with regard to our state variables are

\[
\frac{\partial V_t}{\partial x_t} = V_t B_{(1)} \quad \frac{\partial V_t}{\partial v_{c,t}} = V_t B_{(2)} \quad \frac{\partial V_t}{\partial v_{d,t}} = V_t B_{(3)} \quad (20)
\]

\[
\frac{\partial \sigma_r}{\partial x_t} = 0 \quad \frac{\partial \sigma_r}{\partial v_{c,t}} = \frac{\phi^2}{2 \sigma_r} \quad \frac{\partial \sigma_r}{\partial v_{d,t}} = \frac{1}{2 \sigma_r} \quad (21)
\]

\[
\frac{\partial r_{f,t}}{\partial x_t} = \frac{1}{\psi} \quad \frac{\partial r_{f,t}}{\partial v_{c,t}} = \frac{1 - \gamma (1 + \psi)}{2 \psi} \quad \frac{\partial r_{f,t}}{\partial v_{d,t}} = 0 \quad (22)
\]

where \( B_{(i)} \) denotes the \( i \)-th element of the column vector \( B \) in equation (20).

One interesting implications is that \( \frac{\partial V_t}{\partial v_{c,t}} = V_t B_{(2)} < 0 \) if \( 1 \leq \phi \leq 2\gamma - 1 \) and \( \psi > 1 \), that is, (i) \( (\phi \leq 2\gamma - 1) \) if Jensen’s inequality is not large enough to overcome the increase in risk premium and (ii) \( (\phi \geq 1) \) the claim to dividends is not a hedge to consumption risk. In other words, the unlevered asset value \( V_t \) can actually increase with \( v_{c,t} \) if \( \phi \) is either negative or excessively large. We will assume that \( \phi \) is within the given range so that \( B_{(2)} < 0 \).

Now, let’s study \( S_t \)’s comparative statistics with each of the state variables.

(1) \( S_t \) and \( x_t \)

\[
\frac{\partial S_t}{\partial x_t} = G(\omega) V_t B_{(1)} + Ke^{-r_{f,t}} G(\omega - \sigma_r) \frac{1}{\psi} > 0 \quad \text{always} \quad (23)
\]
(2) $S_t$ and $v_{c,t}$

$$
\frac{\partial S_t}{\partial v_{c,t}} = \frac{G(\omega)}{V_t} B(2) + V_t g(\omega) \frac{\phi^2}{2\sigma_r} + Ke^{-r_{f,t}} G(\omega - \sigma_r) \frac{1 - \gamma(1 + \psi)}{2\psi}
$$

$$
\leq V_t \left\{ G(\omega) B(2) + g(\omega) \frac{\phi^2}{2\sigma_r} \right\}
$$

$$
\leq V_t \left\{ G(\omega) B(2) + g(\omega) \frac{\phi^2}{2\kappa_3 \sqrt{B^\top \Omega B}} \right\}
$$

$$
\leq V_t \left\{ G(\omega) B(2) + g(\omega) \frac{\phi^2}{2\kappa_3 (-s_c B(2))} \right\}
$$

$$
\therefore \quad \frac{\partial S_t}{\partial v_{c,t}} < 0 \quad \text{if} \quad B(2) < -\frac{\phi}{\sqrt{2\kappa_3 s_c}}
$$

(3) $S_t$ and $v_{d,t}$

$$
\frac{\partial S_t}{\partial v_{d,t}} = G(\omega) V_t B(3) + V_t g(\omega) \frac{1}{2\sigma_r} > 0 \quad \text{always}
$$

Comparative statistics of corporate debts, $F_t$, are more complicated as they depend on the firm’s leverage. For example, the long-run growth rate, $x_t$, can move $F_t$ in either direction since it raises not only the underlying asset value ($V_t$) but also the riskfree interest rate ($r_{f,t}$). Let’s assume $R = 1$ for the sake of simplicity. $F_t$’s comparative statistics are derived as

(1) $F_t$ and $x_t$

$$
\frac{\partial F_t}{\partial x_t} = G(-\omega) V_t B(1) - Ke^{-r_{f,t}} G(\omega - \sigma_r) \frac{1}{\psi}
$$

$$
= \frac{1}{\psi} Ke^{-r_{f,t}} G(-\omega) \left\{ \frac{\phi \psi - 1}{1 - \kappa_3 \rho} \frac{e^{\sigma_r \omega - \frac{1}{2} \sigma^2_r}}{2\sigma_r^2} - \frac{G(\omega - \sigma_r)}{G(-\omega)} \right\}
$$

Since $\frac{G(\omega - \sigma_r)}{G(-\omega)} \in (0, \infty)$ is monotonically increasing in $\omega$ and rises faster than $e^{\sigma_r \omega}$, there exists $\omega^*$ such that $\frac{\partial F_t}{\partial x_t} (\omega^*) = 0$. Thus,

$$
\begin{cases}
\frac{\partial F_t}{\partial x_t} > 0 & \text{if leverage is high, i.e., } \ln(V_t/K) < \sigma_r \omega^* - r_{f,t} - \frac{1}{2} \sigma^2_r \\
\frac{\partial F_t}{\partial x_t} < 0 & \text{if leverage is low, i.e., } \ln(V_t/K) > \sigma_r \omega^* - r_{f,t} - \frac{1}{2} \sigma^2_r 
\end{cases}
$$
(2) \( F_t \) and \( v_{c,t} \)

\[
\frac{\partial F_t}{\partial v_{c,t}} = G(-\omega) V_t B(2) - V_t g(\omega) \frac{\phi^2}{2\sigma_r} \]

\[
- Ke^{-r_{f,t} G(\omega - \sigma_r)} \frac{1 - \gamma (1 + \psi)}{2\psi}
\]

(33)

\[
\lim_{\omega \to -\infty} \frac{\partial F_t}{\partial v_{c,t}} = V_t B(2) < 0
\]

(34)

\[
\lim_{\omega \to \infty} \frac{\partial F_t}{\partial v_{c,t}} = Ke^{-r_{f,t} \frac{\gamma (1 + \psi) - 1}{2\psi}} > 0
\]

(35)

Since \( \lim_{\omega \to -\infty} \frac{\partial F_t}{\partial v_{c,t}} < 0 \), \( \lim_{\omega \to \infty} \frac{\partial F_t}{\partial v_{c,t}} > 0 \) and \( \frac{\partial F_t}{\partial v_{c,t}} \) is continuous in \( \omega \), there exists \( \hat{\omega}^* \) such that \( \frac{\partial F_t}{\partial v_{c,t}}(\hat{\omega}^*) = 0 \). Therefore,

\[
\begin{cases} 
\frac{\partial F_t}{\partial v_{c,t}} < 0 & \text{if leverage is high, i.e., } \ln(V_t/K) < \sigma_r \hat{\omega}^* - r_{f,t} - \frac{1}{2} \sigma_r^2 \\
\frac{\partial F_t}{\partial v_{c,t}} > 0 & \text{if leverage is low, i.e., } \ln(V_t/K) > \sigma_r \hat{\omega}^* - r_{f,t} - \frac{1}{2} \sigma_r^2
\end{cases}
\]

(36)

(3) \( F_t \) and \( v_{d,t} \)

Two opposite channels are effective: (i) \( v_{d,t} \uparrow \Rightarrow V_t \uparrow \) (Jensen’s inequality) \( \Rightarrow F_t \uparrow \) and (ii) \( v_{d,t} \uparrow \Rightarrow \sigma_r \uparrow \Rightarrow F_t \downarrow \).

\[
\frac{\partial F_t}{\partial v_{d,t}} = G(-\omega) V_t B(3) - V_t g(\omega) \frac{1}{2\sigma_r}
\]

\[
= V_t G(-\omega) \left\{ B(3) - \frac{g(\omega)}{1 - G(\omega)} \frac{1}{2\sigma_r} \right\}
\]

(37)

(38)

Since \( \frac{g(\omega)}{1 - G(\omega)} \in (0, \infty) \) is monotonically increasing in \( \omega \), there exists \( \bar{\omega}^* \) such that \( \frac{\partial F_t}{\partial v_{d,t}}(\bar{\omega}^*) = 0 \). However, since \( \frac{g(\omega)}{1 - G(\omega)} > 0.79 \) for \( \omega > 0 \) as assumed previously,

\[
\therefore \frac{\partial F_t}{\partial v_{d,t}} < 0 \quad \text{if the Jensen’s inequality is not dominant, i.e., } B(3) \leq \frac{0.79}{2\sigma_r}
\]

(39)

In contrast to the previous two cases, the comparative statics of credit spreads, \( cr \equiv -\ln(F_t/K) - r_{f,t} \), are much simpler.
Now let’s move on to expected excess returns. The excess returns of unlevered assets are determined by the covariance of its return with the pricing kernel, 

$$- \sigma_{mr} \equiv - \text{cov}_t (m_{t+1}, r_{d,t+1}) = \phi \gamma v_{c,t} + \text{constant}$$ (43)

Those of levered equity can be derived as

$$eer_S \equiv \ln \left( \frac{E_t[S_{t+1}]}{S_t} \right) - r_{f,t}$$ (44)

$$\exp(eer_S) = \frac{V_t e^{-\sigma_{mr}} G(\omega - \frac{\sigma_{mr}}{\sigma_r}) - Ke^{-r_{f,t} G(\omega - \frac{\sigma_{mr}}{\sigma_r})}}{V_t G(\omega) - Ke^{-r_{f,t} G(\omega - \sigma_r)}}$$ (45)

\[= \frac{V_t G(\omega)}{S_t G(\omega)} \approx -\sigma_{mr} V_t G(\omega) \text{ by the Taylor approximation} \] (46)
And below are the excess returns of corporate bonds.

\[ eer_F \equiv \ln \left( \frac{E_t[F_{t+1}]}{F_t} \right) - r_{f,t} \quad (47) \]

\[ \exp(eer_F) = \frac{V_t e^{-\sigma_r G(\omega + \frac{\sigma_r}{\sigma_r})} + Ke^{-r_{f,t} G(\omega - \sigma_r - \frac{\sigma_r}{\sigma_r})}}{V_t G(\omega) + Ke^{-r_{f,t} G(\omega - \sigma_r)}} \quad (48) \]

\[ \therefore \quad eer_F \approx -\sigma_{mr} \frac{V_t}{F_t} G(\omega) \quad (49) \]

Note that the value-weighted average of stock and corporate bond excess returns is equal to the risk premium of the unlevered asset.

\[ \frac{S_t}{V_t} eer_S + \frac{F_t}{V_t} eer_F = -\sigma_{mr} \quad (50) \]

Let’s define the leverage factor of stocks as follows.

\[ L_S(\omega) \equiv \frac{V_t}{S_t} G(\omega) = \frac{1}{1 - L_S(\omega)} \quad (51) \]

\[ l_S(\omega) \equiv e^{-\sigma_r \omega + \frac{1}{2} \sigma_r^2} \frac{G(\omega - \sigma_r)}{G(\omega)} \quad (52) \]

Both \( L_S(\omega) \) and \( l_S(\omega) \) are monotonically decreasing in \( \omega \). They will turn useful in the derivation of the comparative statistics of \( eer_S \) that follow.

1. \( eer_S \) and \( x_t \)

\[ \frac{\partial l_S(\omega)}{\partial x_t} = \frac{\partial l_S(\omega)}{\partial \omega} \cdot \frac{1}{\sigma_r} \left\{ B(1) + \frac{1}{\psi} \right\} < 0 \quad (53) \]

Thus,

\[ \frac{\partial eer_S}{\partial x_t} < 0 \quad (54) \]

2. \( eer_S \) and \( v_{c,t} \)

it is difficult to tell the sign of \( \frac{\partial eer_S}{\partial v_{c,t}} \) as it not only raises risk premium \((-\sigma_{mr})\) but also affects the leverage.

3. \( eer_S \) and \( v_{d,t} \)

\[ \frac{\partial eer_S}{\partial v_{d,t}} < 0 \quad \text{if } \omega > \sigma_r \quad (55) \]
Similarly, let’s define the leverage factor of corporate debts as follows.

\[ L_F(\omega) \equiv \frac{V_t}{F_t} G(-\omega) = \frac{1}{1 + l_F(\omega)} \quad (56) \]

\[ l_F(\omega) \equiv e^{-\sigma_r \omega + \frac{1}{2} \sigma_r^2} \frac{G(\omega - \sigma_r)}{G(-\omega)} \quad (57) \]

\( l_F(\omega) \) is monotonically increasing in \( \omega \) and \( L_F(\omega) \) monotonically decreasing.

(1) \( e_{er_F} \) and \( x_t \)

\[ \frac{\partial l_F(\omega)}{\partial x_t} = \frac{\partial l_F(\omega)}{\partial \omega} \cdot \frac{1}{\sigma_r} \left\{ B_{(1)} + \frac{1}{\psi} \right\} > 0 \quad (58) \]

Thus,

\[ \frac{\partial e_{er_F}}{\partial x_t} < 0 \quad (59) \]

At first glance, this result may look counter-intuitive since \( \frac{\partial}{\partial x_t} e_{er_S} + \frac{\partial}{\partial x_t} e_{er_F} = -\sigma_{mr} \), \( \frac{\partial e_{er_S}}{\partial x_t} < 0 \), \( \frac{\partial e_{er_F}}{\partial x_t} < 0 \) but \( \frac{\partial \sigma_{mr}}{\partial x_t} = 0 \). Note that, however, the value-weights themselves, \( \frac{S_t}{V_t} \) and \( \frac{F_t}{V_t} \), are also determined by \( x_t \). In other words, \( x_t \) is related to the spread between \( e_{er_S} \) and \( e_{er_F} \).

(2) \( e_{er_F} \) and \( v_{c,t} \)

\[ \frac{\partial l_F}{\partial v_{c,t}} < 0 \quad \text{if } \omega > \sigma_r \quad (60) \]

Thus,

\[ \frac{\partial e_{er_F}}{\partial v_{c,t}} = \phi \gamma L_F(\omega) - \sigma_{mr} \frac{\partial L_F}{\partial v_{c,t}} > 0 \quad (61) \]

(3) \( e_{er_F} \) and \( v_{d,t} \)

\[ \frac{\partial l_F}{\partial v_{d,t}} < 0 \quad \text{if } \omega > \sigma_r \text{ and } B_{(3)} (\text{Jensen’s inequality) is not strong} \quad (62) \]

Thus,

\[ \frac{\partial e_{er_F}}{\partial v_{d,t}} > 0 \quad (63) \]